Miskolc Mathematical Notes
Vol. 27 (2026), No. 1, pp. 329–344

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2026.5039

ON FRACTAL BERNOULLI DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

DANIEL ALFONSO SANTIESTEBAN, RICARDO ABREU BLAYA, YAMILET QUINTANA, AND JOSÉ MARÍA SIGARRETA ALMIRA

Received 26 April, 2024

Abstract. In this paper we study a Bernoulli-type differential equation that replace the usual derivative by a fractal derivative. We show the goodness-of-fit of this model for real data comparing it with classic models.

2010 *Mathematics Subject Classification:* 26A24; 28A80; 26A27; 26A33

Keywords: fractal derivatives, Bernoulli differential equation, real data fitting

1. INTRODUCTION

Fractal calculus (also called local fractional calculus) is utilized to handle various nondifferentiable problems that appear in complex systems of the real-world phenomena. This new calculus was first proposed by Kolwankar and Gangal in 1996 through renormalization of Riemann–Liouville definition [23]. Fractal derivatives (or local fractional derivative) are defined as a non-Newtonian generalization of the derivative dealing with the measurement of fractals and play an important role in the study of anomalous diffusion. There are many definitions of fractal derivatives and local fractional integrals (see for instance [3, 4, 6, 8, 21]). In this paper we used a particular version of the definitions given in [12, 13, 31]: for $\alpha, \beta > 0$, let us define the (β, α)-*fractal derivative* of a function f at the point t_0 by

$$\frac{d^\beta f}{dt^\alpha}(t_0) = \lim_{t \to t_0} \frac{f^\beta(t) - f^\beta(t_0)}{t^\alpha - t_0^\alpha} \tag{1.1}$$

if there exists this limit and it is finite. In this case, we say that f is (β, α)-*fractal differentiable* at t_0. Here, the function t^α is defined for $t \in \mathbb{R}$ as $t^\alpha := t|t|^{\alpha-1}$ and so,

The first author was supported in part by the financial support of the Postgraduate Study Fellowship of the Secretaría de Ciencia, Humanidades, Tecnología e Innovación (SECIHTI) (grant number 1043969).

$(t^\alpha)' = \alpha|t|^{\alpha-1}$; also, $(|t|^\alpha)' = \alpha t^{\alpha-1}$. When $\beta = 1$, this derivative (1.1) can be viewed as a fractional derivatives via fractional differences, which are very useful for solving numerical problems of fractional differential equations (see e.g. [2, 16, 20, 22, 29]). However, for $\beta \neq 1$, this fractal derivative is not a linear operator and so, it does not enjoy some desirable properties.

These derivatives engender a new kind of differential equations, referred as fractal differential equations essentially different from the well-known fractional differential equations. As of today, many applications of these fractal derivatives are studied, e.g. the Fokker-Planck equation in modelling phenomena involving fractal time [25] and the time-space fabric underlying anomalous diffusion [12]. Moreover, in fractal cosmology it has proved to be a nice derivative to treat certain universe models considering fractal space-time. Good references on this topic are the books [18, 29–31] and the papers [7, 8, 11, 14, 21, 24, 32].

The classic Bernoulli differential equation is a non-linear differential equation of the form

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = a(t)y(t) + b(t)y^n(t), \quad n \in \mathbb{Z}_+,$$

where $a(t)$ and $b(t)$ are continuous functions. The first discussion of this equation goes back to Jacob Bernoulli in his work from 1695 [9]. Modern physics indeed uses Bernoulli differential equations for modelling the dynamics behind certain circuit elements, known as Bernoulli memristors [17]. The Bernoulli differential equation also show up in some economic utility maximization problems (see for example [26]).

In the present paper we study Bernoulli-type differential equations of the form

$$\frac{\mathrm{d}^\beta y(t)}{\mathrm{d}t^\alpha} = a(t)\big(y(t)\big)^\beta + b(t)\big|y(t)\big|^{\beta\gamma}, \tag{1.2}$$

where $\gamma \in \mathbb{R} \setminus \{1\}$. We solve an initial value problem for the equation (1.2). Finally, it is shown the effectiveness of a fractal Bernoulli differential model to fit real data associated with tuberculosis in Mexico.

2. Basic properties

This section summarizes a series of preliminary notions and tools that are required in the main part of this work (for more details we refer to [5]).

Proposition 1. *Let f be a differentiable function at t_0. Assume that $t_0 \neq 0$ if $\alpha > 1$, and $f(t_0) \neq 0$ if $0 < \beta < 1$. Then f is (β,α)-fractal differentiable at t_0, and the following formula holds*

$$\frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0) = \frac{\beta}{\alpha}|t_0|^{1-\alpha}|f(t_0)|^{\beta-1}f'(t_0).$$

In particular,

$$\frac{\mathrm{d}^1 f}{\mathrm{d}t^\alpha}(t_0) = \frac{1}{\alpha}|t_0|^{1-\alpha}f'(t_0).$$

Proposition 2. *Let $c \in \mathbb{R}$ and let f be a (β,α)-fractal differentiable function at t_0. Then the following statements hold:*

i. *We have*

$$\frac{\mathrm{d}^1 f}{\mathrm{d}t^1}(t_0) = f'(t_0), \qquad \frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0) = \frac{\mathrm{d}^1 (f^\beta)}{\mathrm{d}t^\alpha}(t_0).$$

ii. *The function c is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta c}{\mathrm{d}t^\alpha}(t_0) = 0.$$

iii. *The function cf is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta (cf)}{\mathrm{d}t^\alpha}(t_0) = c^\beta \frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0).$$

iv. $\mathrm{d}^\beta/\mathrm{d}t^\alpha$ *is a linear operator if and only if* $\beta = 1$.

The following result shows how to compute the fractal derivative of the product and the quotient of two functions.

Proposition 3 (Leibniz rule)**.** *Let f, g be (β,α)-differentiable functions at t_0. Then the following statements hold:*

i. *fg is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta (fg)}{\mathrm{d}t^\alpha}(t_0) = \frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0)\, g^\beta(t_0) + f^\beta(t_0)\, \frac{\mathrm{d}^\beta g}{\mathrm{d}t^\alpha}(t_0).$$

ii. *If $g(t_0) \neq 0$, then $1/g$ is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\alpha}\left(\frac{1}{g}\right)(t_0) = \frac{-\frac{\mathrm{d}^\beta g}{\mathrm{d}t^\alpha}(t_0)}{|g(t_0)|^{2\beta}}.$$

iii. *If $g(t_0) \neq 0$, then f/g is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\alpha}\left(\frac{f}{g}\right)(t_0) = \frac{\frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0)\, g^\beta(t_0) - f^\beta(t_0)\, \frac{\mathrm{d}^\beta g}{\mathrm{d}t^\alpha}(t_0)}{|g(t_0)|^{2\beta}}.$$

The next result is a kind of chain rule for the fractal derivatives.

Proposition 4 (Chain rule)**.** *Let g be a continuous and (α,α)-differentiable function at t_0 and let f be a (β,α)-differentiable function at $g(t_0)$. Then $f \circ g$ is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\alpha}(f \circ g)(t_0) = \frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(g(t_0))\frac{\mathrm{d}^\alpha g}{\mathrm{d}t^\alpha}(t_0).$$

Let I be an interval with $t_0 \in I$ and $\alpha, \beta > 0$. If f is a locally integrable function on I with respect to the measure $|s|^{\alpha-1}\mathrm{d}s$, let us define the operators

$$\begin{aligned} L^{\alpha}_{t_0}(f)(t) &= \int_{t_0}^{t} \alpha\,|s|^{\alpha-1} f(s)\,\mathrm{d}s, \\ K^{\alpha,\beta}_{t_0}(f)(t) &= \Big(\int_{t_0}^{t} \alpha\,|s|^{\alpha-1} f(s)\,\mathrm{d}s\Big)^{1/\beta} = \big(L^{\alpha}_{t_0}(f)(t)\big)^{1/\beta}, \end{aligned} \tag{2.1}$$

for $t \in I$. These operators behave as inverse operators of $\mathrm{d}^1/\mathrm{d}t^{\alpha}$ and $\mathrm{d}^{\beta}/\mathrm{d}t^{\alpha}$, respectively,

$$\frac{\mathrm{d}^1}{\mathrm{d}t^{\alpha}}\big(L^{\alpha}_{t_0}(f)\big)(t) = \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\alpha}}\big(K^{\alpha,\beta}_{t_0}(f)\big)(t) = f(t).$$

3. Fractal Bernoulli differential equation

The Bernoulli differential equation is a fundamental theoretical element in the solution of problems involving nonlinear differential equations. From the practical point of view, its importance lies in its ability to describe and analyze problems in Physics, Biology and Engineering, fundamentally, it allows the study of the behavior of flows in systems where pressure, velocity and height phenomena intervene. It also serves as a basis for the study of system dynamics, enzyme kinetics and population dynamics (see [10, 15, 28]). This equation has recently been studied in the context of the global fractional operators [27]. The previous research has been complemented by studying this Bernoulli differential equation by means of the local operators (conformable and non-conformable). Likewise, the research leaves open the possibility of comparing the scope of both approaches (global and local) in practice (see for example [20]). One of the advantages of the local differential operators studied, as well as the global differential operators (Caputo, Fabricio, etc.) for $\alpha \in (0,1)$, is that they allow to use, generalize and extend many classic results.

Fractal calculus enjoys opening up new lines of research that also generalize many classic results to distant and multiple areas [19]. The linear operator $L^{\alpha}_{t_0}$ allows to solve Bernoulli-type differential equation. The following theorem describes an initial value problem for (1.2).

Theorem 1. *Let $\gamma \in \mathbb{R} \setminus \{1\}$, $y_0 \in \mathbb{R}$, I an interval with $t_0 \in I$, and let a, b be continuous functions on I. The function*

$$y(t) = \begin{cases} \exp\Big(\frac{1}{\beta} L^{\alpha}_{t_0}(a)(t)\Big)\Big((1-\gamma)L^{\alpha}_{t_0}\Big(b(s)\exp\big((\gamma-1)L^{\alpha}_{t_0}(a)(s)\big)\Big)(t) + y_0^{\beta-\beta\gamma}\Big)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma < 1, \\ y_0 \exp\Big(\frac{1}{\beta} L^{\alpha}_{t_0}(a)(t)\Big)\Big(y_0^{\beta\gamma-\beta}(1-\gamma)L^{\alpha}_{t_0}\Big(b(s)\exp\big((\gamma-1)L^{\alpha}_{t_0}(a)(s)\big)\Big)(t) + 1\Big)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma > 1, \end{cases} \tag{3.1}$$

will be a solution of the following initial value problem:

$$\frac{\mathrm{d}^{\beta}y}{\mathrm{d}t^{\alpha}}(t) = a(t)\big(y(t)\big)^{\beta} + b(t)\big|y(t)\big|^{\beta\gamma}, \qquad y(t_0) = y_0,$$

with $t \in I$*, and*

$$(1-\gamma)L^{\alpha}_{t_0}\Big(b(s)\exp\big((\gamma-1)L^{\alpha}_{t_0}(a)(s)\big)\Big)(t) + y_0^{\beta-\beta\gamma} \neq 0,$$

provided that $\gamma < 0$*, whereas*

$$y_0^{\beta\gamma-\beta}(1-\gamma)L^{\alpha}_{t_0}\Big(b(s)\exp\big((\gamma-1)L^{\alpha}_{t_0}(a)(s)\big)\Big)(t) + 1 \neq 0,$$

as long as $\gamma > 1$.

Proof. Firstly, let us collect here the equalities that we will use:

$$\frac{\mathrm{d}^1(fg)}{\mathrm{d}t^{\alpha}}(t) = \frac{\mathrm{d}^1 f}{\mathrm{d}t^{\alpha}}(t)\,g(t) + f(t)\,\frac{\mathrm{d}^1 g}{\mathrm{d}t^{\alpha}}(t), \tag{3.2}$$

$$\frac{\mathrm{d}^1}{\mathrm{d}t^{\alpha}}\big(f^a\big)(t) = a\,|f(t)|^{a-1}\,\frac{\mathrm{d}^1 f}{\mathrm{d}t^{\alpha}}(t), \tag{3.3}$$

$$\frac{\mathrm{d}^1}{\mathrm{d}t^{\alpha}}\big(e^g\big)(t) = e^{g(t)}\,\frac{\mathrm{d}^1 g}{\mathrm{d}t^{\alpha}}(t), \tag{3.4}$$

$$\frac{\mathrm{d}^1}{\mathrm{d}t^{\alpha}}\big(L^{\alpha}_t(f)\big)(t) = f(t). \tag{3.5}$$

Since the case $\gamma > 1$ is similar, we can assume $\gamma < 1$. Let us define

$$\begin{aligned} z(t) &= y(t)^{\beta} \\ &= \exp\big(L^{\alpha}_{t_0}(a)(t)\big)\Big((1-\gamma)L^{\alpha}_{t_0}\Big(b(s)\exp\big((\gamma-1)L^{\alpha}_{t_0}(a)(s)\big)\Big)(t) + y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)}. \end{aligned}$$

Proposition 2 gives that it suffices to prove that $z(t)$ satisfies the initial value problem

$$\frac{\mathrm{d}^1 z}{\mathrm{d}t^{\alpha}}(t) = a(t)\,z(t) + b(t)\big|z(t)\big|^{\gamma}, \qquad z(t_0) = y_0^{\beta}.$$

We have

$$\begin{aligned} z(t_0) &= \exp\big(L^{\alpha}_{t_0}(a)(t_0)\big)\Big((1-\gamma)L^{\alpha}_{t_0}\Big(b(s)\exp\big((\gamma-1)L^{\alpha}_{t_0}(a)(s)\big)\Big)(t_0) + y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)} \\ &= \exp(0)\Big(0 + y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)} \\ &= y_0^{\beta}. \end{aligned}$$

Note that $1/(1-\gamma) \geq 0$ and $\gamma/(1-\gamma) \geq 0$ if and only if $\gamma \in [0,1)$; if $\gamma < 0$, we have the hypothesis $(1-\gamma)L^{\alpha}_{t_0}\Big(b(s)\exp\big((\gamma-1)L^{\alpha}_{t_0}(a)(s)\big)\Big)(t) + y_0^{\beta-\beta\gamma} \neq 0$.

Since a and b are continuous functions on I, (3.5) implies

$$\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\big(L_{t_0}^\alpha(a)\big)(t) = a(t),$$

$$\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\Big(L_{t_0}^\alpha\Big(b(s)\exp\big((\gamma-1)L_{t_0}^\alpha(a)(s)\big)\Big)\Big)(t) = b(t)\exp\big((\gamma-1)L_{t_0}^\alpha(a)(t)\big).$$

By the above relations, in view of (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
\frac{\mathrm{d}^1 z}{\mathrm{d}t^\alpha}(t) &= \frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\Big(\exp\big(L_{t_0}^\alpha(a)(t)\big)\Big)\Big((1-\gamma)L_{t_0}^\alpha\Big(b(s)\exp\big((\gamma-1)L_{t_0}^\alpha(a)(s)\big)\Big)(t)+y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)} \\
&\quad + \exp\big(L_{t_0}^\alpha(a)(t)\big)\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\Big(\Big((1-\gamma)L_{t_0}^\alpha\Big(b(s)\exp\big((\gamma-1)L_{t_0}^\alpha(a)(s)\big)\Big)(t)+y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)}\Big) \\
&= a(t)\exp\big(L_{t_0}^\alpha(a)(t)\big)\Big((1-\gamma)L_{t_0}^\alpha\Big(b(s)\exp\big((\gamma-1)L_{t_0}^\alpha(a)(s)\big)\Big)(t)+y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)} \\
&\quad + \exp\big(L_{t_0}^\alpha(a)(t)\big)\frac{1}{1-\gamma}\Big|(1-\gamma)L_{t_0}^\alpha\Big(b(s)\exp\big((\gamma-1)L_{t_0}^\alpha(a)(s)\big)\Big)(t)+y_0^{\beta-\beta\gamma}\Big|^{1/(1-\gamma)-1} \\
&\quad \times (1-\gamma)\,b(t)\exp\big((\gamma-1)L_{t_0}^\alpha(a)(t)\big) \\
&= a(t)\exp\big(L_{t_0}^\alpha(a)(t)\big)\Big((1-\gamma)L_{t_0}^\alpha\Big(b(s)\exp\big((\gamma-1)L_{t_0}^\alpha(a)(s)\big)\Big)(t)+y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)} \\
&\quad + b(t)\exp\big(\gamma L_{t_0}^\alpha(a)(t)\big)\Big|(1-\gamma)L_{t_0}^\alpha\Big(b(s)\exp\big((\gamma-1)L_{t_0}^\alpha(a)(s)\big)\Big)(t)+y_0^{\beta-\beta\gamma}\Big|^{\gamma/(1-\gamma)} \\
&= a(t)z(t)+b(t)|z(t)|^\gamma.
\end{aligned}$$

□

If the function b or a is identically zero, Theorem 1 has the following consequences.

Corollary 1. *Let $y_0 \in \mathbb{R}$, I an interval with $t_0 \in I$, and let $a(t)$ be a continuous function on I. Then the function*

$$y(t) = y_0 \exp\Big(\frac{1}{\beta}L_{t_0}^\alpha(a)(t)\Big)$$

is a solution of the initial value problem

$$\frac{\mathrm{d}^\beta y}{\mathrm{d}t^\alpha}(t) = a(t)\big(y(t)\big)^\beta, \qquad y(t_0) = y_0.$$

Corollary 2. *Let $\gamma \in \mathbb{R}\setminus\{1\}$, $y_0 \in \mathbb{R}$, I an interval with $t_0 \in I$, and let $b(t)$ be a continuous function on I. The function*

$$y(t) = \begin{cases} \Big((1-\gamma)L_{t_0}^\alpha(b)(t)+y_0^{\beta-\beta\gamma}\Big)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma<1, \\ y_0\Big(y_0^{\beta\gamma-\beta}(1-\gamma)L_{t_0}^\alpha(b)(t)+1\Big)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma>1, \end{cases}$$

is a solution of the initial value problem

$$\frac{\mathrm{d}^\beta y}{\mathrm{d}t^\alpha}(t) = b(t)|y(t)|^{\beta\gamma}, \qquad y(t_0) = y_0,$$

with

$$(1-\gamma)L_{t_0}^{\alpha}(b)(t)+y_0^{\beta-\beta\gamma}\neq 0,$$

provided that $\gamma<0$*, whereas*

$$y_0^{\beta\gamma-\beta}(1-\gamma)L_{t_0}^{\alpha}(b)(t)+1\neq 0,$$

as long as $\gamma>1$.

If the functions $a(t)$ and $b(t)$ are constants, we have the following result:

Proposition 5. *Let* $\gamma\in\mathbb{R}\setminus\{1\}$, $y_0,a,b\in\mathbb{R}$ *with* $a\neq 0$*. The function*

$$y(t)=\begin{cases}\left(\left(y_0^{\beta-\beta\gamma}+\frac{b}{a}\right)\exp\left(a(1-\gamma)\left(t^{\alpha}-t_0^{\alpha}\right)\right)-\frac{b}{a}\right)^{1/(\beta-\beta\gamma)}, & \textit{if } \gamma<1,\\ y_0\left(\left(1+y_0^{\beta\gamma-\beta}\frac{b}{a}\right)\exp\left(a(1-\gamma)\left(t^{\alpha}-t_0^{\alpha}\right)\right)-y_0^{\beta\gamma-\beta}\frac{b}{a}\right)^{1/(\beta-\beta\gamma)}, & \textit{if } \gamma>1,\end{cases}$$

is a solution of the initial value problem

$$\frac{\mathrm{d}^{\beta}y}{\mathrm{d}t^{\alpha}}(t)=a\big(y(t)\big)^{\beta}+b\big|y(t)\big|^{\beta\gamma},\qquad y(t_0)=y_0,$$

when $t\geq t_0$*, and the following assumption are required:*

$$\left(y_0^{\beta-\beta\gamma}+\frac{b}{a}\right)\exp\left(a(1-\gamma)\left(t^{\alpha}-t_0^{\alpha}\right)\right)-\frac{b}{a}\neq 0,$$

if $\gamma<0$*, and*

$$\left(1+y_0^{\beta\gamma-\beta}\frac{b}{a}\right)\exp\left(a(1-\gamma)\left(t^{\alpha}-t_0^{\alpha}\right)\right)-y_0^{\beta\gamma-\beta}\frac{b}{a}\neq 0,$$

if $\gamma>1$.

Proof. Without loss of generality we can assume that $\gamma<1$. The case when $\gamma>1$ is analogous. We have

$$L_{t_0}^{\alpha}(a)(t)=a\int_{t_0}^{t}\alpha|s|^{\alpha-1}\,\mathrm{d}s=a\big(t^{\alpha}-t_0^{\alpha}\big).$$

Theorem 1 yields

$$\begin{aligned}y(t)&=\exp\left(\frac{1}{\beta}L_{t_0}^{\alpha}(a)(t)\right)\left((1-\gamma)L_{t_0}^{\alpha}\Big(b\exp\left((\gamma-1)L_{t_0}^{\alpha}(a)(s)\right)\Big)(t)+y_0^{\beta-\beta\gamma}\right)^{1/(\beta-\beta\gamma)}\\&=\exp\left(\frac{a}{\beta}\left(t^{\alpha}-t_0^{\alpha}\right)\right)\left(b(1-\gamma)L_{t_0}^{\alpha}\Big(\exp\left((\gamma-1)a\left(s^{\alpha}-t_0^{\alpha}\right)\right)\Big)(t)+y_0^{\beta-\beta\gamma}\right)^{1/(\beta-\beta\gamma)}.\end{aligned}$$

Since

$$\begin{aligned}(1-\gamma)L_{t_0}^{\alpha}\Big(\exp\big((\gamma-1)a\big(s^{\alpha}-t_0^{\alpha}\big)\big)\Big)(t) &= (1-\gamma)\int_{t_0}^{t}\alpha\,|s|^{\alpha-1}\exp\big((\gamma-1)a\big(s^{\alpha}-t_0^{\alpha}\big)\big)\,\mathrm{d}t\\ &= \frac{-1}{a}\int_{t_0}^{t}\alpha\,|s|^{\alpha-1}a(\gamma-1)\exp\big(a(\gamma-1)\big(s^{\alpha}-t_0^{\alpha}\big)\big)\,\mathrm{d}t\\ &= \frac{-1}{a}\Big[\exp\big(a(\gamma-1)\big(s^{\alpha}-t_0^{\alpha}\big)\big)\Big]_{s=t_0}^{s=t}\\ &= \frac{1}{a}\Big(1-\exp\big(a(\gamma-1)\big(t^{\alpha}-t_0^{\alpha}\big)\big)\Big),\end{aligned}$$

we conclude that

$$\begin{aligned}y(t) &= \exp\Big(\frac{a}{\beta}\big(t^{\alpha}-t_0^{\alpha}\big)\Big)\Big(b(1-\gamma)L_{t_0}^{\alpha}\Big(\exp\big((\gamma-1)a\big(s^{\alpha}-t_0^{\alpha}\big)\big)\Big)(t)+y_0^{\beta-\beta\gamma}\Big)^{1/(\beta-\beta\gamma)}\\ &= \exp\Big(\frac{a}{\beta}\big(t^{\alpha}-t_0^{\alpha}\big)\Big)\Big(\frac{b}{a}\Big(1-\exp\big(a(\gamma-1)\big(t^{\alpha}-t_0^{\alpha}\big)\big)\Big)+y_0^{\beta-\beta\gamma}\Big)^{1/(\beta-\beta\gamma)}\\ &= \Big(\Big(y_0^{\beta-\beta\gamma}+\frac{b}{a}\Big)\exp\big(a(1-\gamma)\big(t^{\alpha}-t_0^{\alpha}\big)\big)-\frac{b}{a}\Big)^{1/(\beta-\beta\gamma)}.\end{aligned}$$

□

We have the following consequence of Proposition 5 when $\gamma = 2$, $a = A$ and $b = -B$, which solves a kind of fractal differential equation.

Corollary 3. *Let $y_0, A, B \in \mathbb{R}$ with $A > 0$ and $y_0 B \geq 0$. The function given by the expression*

$$y(t) = y_0\Big(\Big(1-y_0^{\beta}\frac{B}{A}\Big)\exp\big(-A\big(t^{\alpha}-t_0^{\alpha}\big)\big)+y_0^{\beta}\frac{B}{A}\Big)^{-1/\beta}$$

is a solution for $t \geq t_0$ of the initial value problem

$$\frac{\mathrm{d}^{\beta}y}{\mathrm{d}t^{\alpha}}(t) = \big(y(t)\big)^{\beta}\big(A-B\big(y(t)\big)^{\beta}\big), \qquad y(t_0) = y_0.$$

Proof. By Proposition 5, we need to prove that the function

$$u(t) = \Big(1-y_0^{\beta}\frac{B}{A}\Big)\exp\big(-A\big(t^{\alpha}-t_0^{\alpha}\big)\big)+y_0^{\beta}\frac{B}{A}$$

is not 0 for every $t \geq t_0$. Since $A > 0$ and $y_0 B \geq 0$, we have that $y_0^{\beta}B/A \geq 0$ and

$$u(t) = \exp\big(-A\big(t^{\alpha}-t_0^{\alpha}\big)\big)+y_0^{\beta}\frac{B}{A}\Big(1-\exp\big(-A\big(t^{\alpha}-t_0^{\alpha}\big)\big)\Big) > 0, \quad t \geq t_0.$$

□

4. A real data fitting with Matlab

First we will graph the solutions to the following initial value problem for the fractal Bernoulli differential equation by varying α and β:

$$\frac{d^{\beta}y}{dt^{\alpha}}(t) = t(y(t))^{\beta}, \qquad y(0) = 1. \tag{4.1}$$

In Figure 1, black curve is related to the usual derivative ($\beta = 1, \alpha = 1$). The yellow, green, blue, red and brown curves correspond to the following (α, β) values: $(1.1, 1.5)$, $(1.8, 3.0)$, $(1.5, 0.8)$, $(0.2, 0.7)$ and $(0.4, 2.1)$, respectively.

We also plot the solutions to the following equation

$$\frac{d^{\beta}y}{dt^{\alpha}}(t) = 2(y(t))^{\beta} - |y(t)|^{1.8\beta} \tag{4.2}$$

with the same initial condition as the problem (4.1). In Figure 2, black curve is also related to the ordinary derivative ($\beta = 1, \alpha = 1$). However, in this case the yellow, green, blue, red and brown curves correspond to the following (α, β) values: $(0.7, 1.0)$, $(1.5, 1.0)$, $(1.2, 1.4)$, $(0.7, 0.8)$ and $(0.8, 1.5)$, respectively.

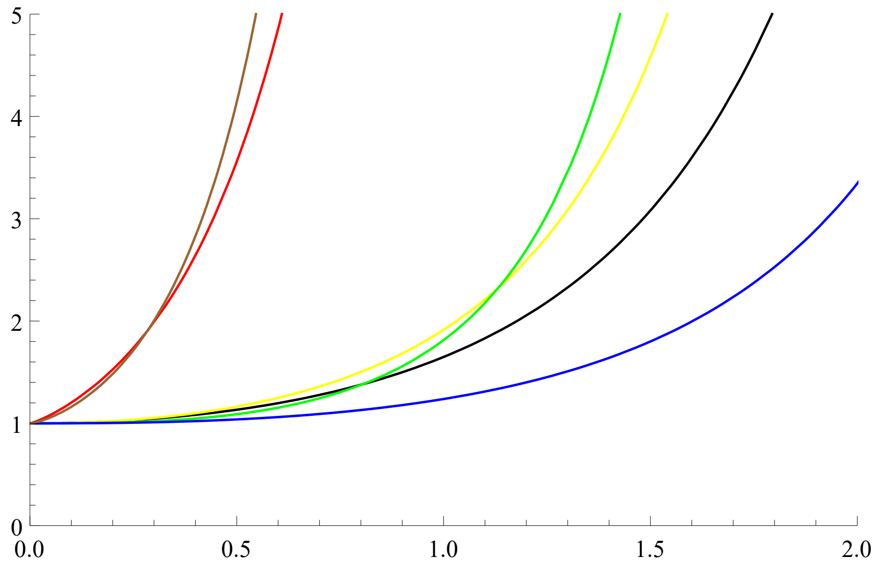

FIGURE 1. Fractal Bernoulli differential model (4.1) for arbitrary values of α and β

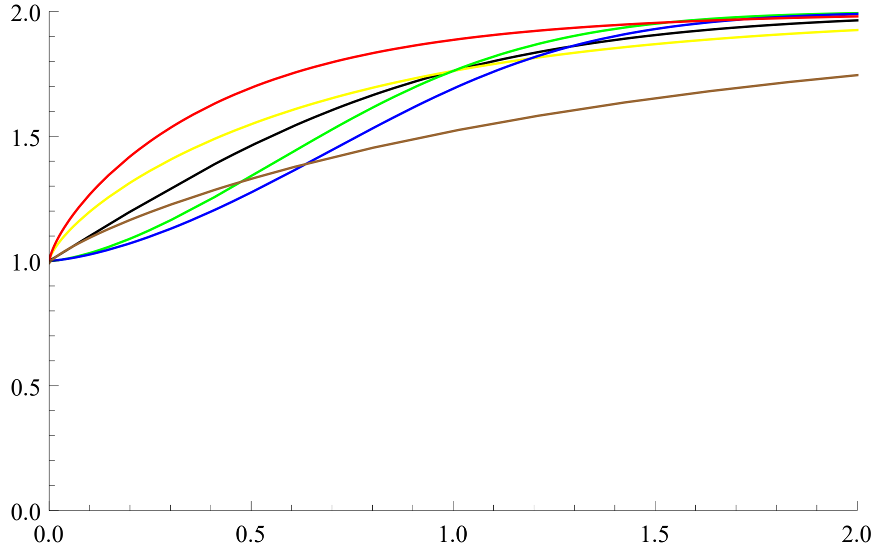

FIGURE 2. Fractal Bernoulli differential model (4.2) for arbitrary values of α and β

In [20], the authors proposed a generalized conformal fractional derivative G_T^α given by the formula

$$(G_T^\alpha f)(t) = \lim_{h\to 0} \frac{f(t) - f(t - he^{(\alpha-1)t})}{h},$$

and applied it to study a Gompertz model. In addition, a real data set on tuberculosis in Mexico was studied and used to solve the inverse problem to estimate the order of the proposed fractional derivative and compare it with the usual derivative and other fractional derivatives (e.g. Khalil and Caputo derivatives). The results they yielded were surprising because the proposed fractional conformable approximation minimizes the error in the adjustments of the parameters associated to the Gompertz model. In what follows we will see that a particular fractal Bernoulli differential equation best fits the data set studied in [20]. For this study we used the same data of the percentage of people with tuberculosis in Mexico between the years 1990 and 2015 [1].

Figure 3 shows the data associated with the percentage of people with tuberculosis in Mexico (black asterisk) and the corresponding fits to the Gompertz model

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = 0.2008 \cdot y(t) \ln\left(\frac{1}{y(t)}\right), \tag{4.3}$$

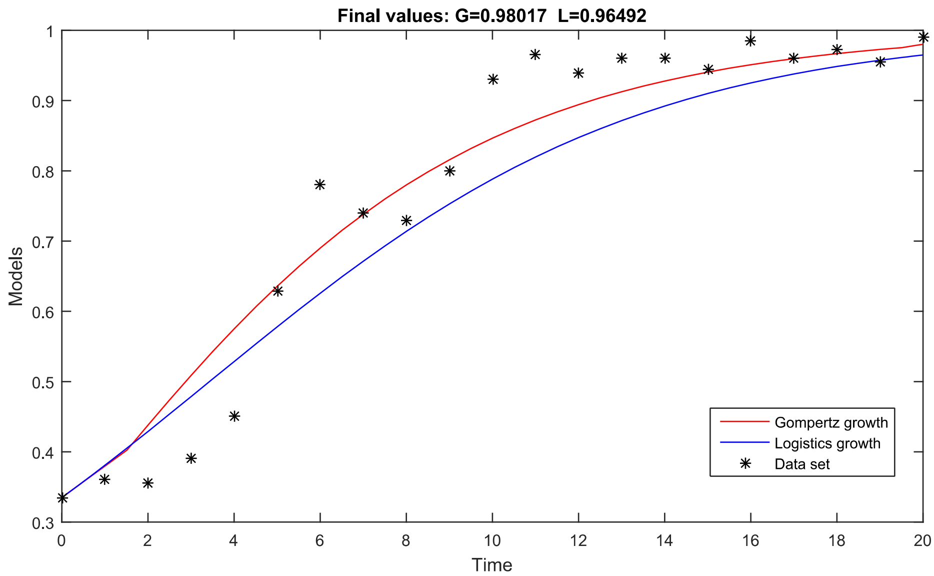

FIGURE 3. Data and estimates of tuberculosis infectious using Gompertz and logistics models

and logistics model

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = 0.2008 \cdot y(t)\,(1 - y(t))\,, \tag{4.4}$$

both with initial condition $y(0) = 0.335$. Using Matlab we have written a script to obtain numerical calculation of the solution of fractal Bernoulli differential equations. For its implementation we used the efficient *ode45* function based on an explicit Runge-Kutta formula, the Dormand-Prince pair. We arrive that the following fractal Bernoulli differential model:

$$\frac{\mathrm{d}^{0.9}y(t)}{\mathrm{d}t^{1.4}} = \left(1 - \frac{1}{0.99}\right) \cdot \left[0.087(y(t))^{0.9} + 0.008(y(t))^{0.63}\right], \tag{4.5}$$
$$y(0) = 0.335,$$

best fits the data of the percentage of people with tuberculosis in Mexico with respect to the previous models (see Figure 4).

Table 1 shows the fitting errors of the studied models: the mean absolute error (MAE) and the mean squared error (MSE). It should also be noted that the fitting

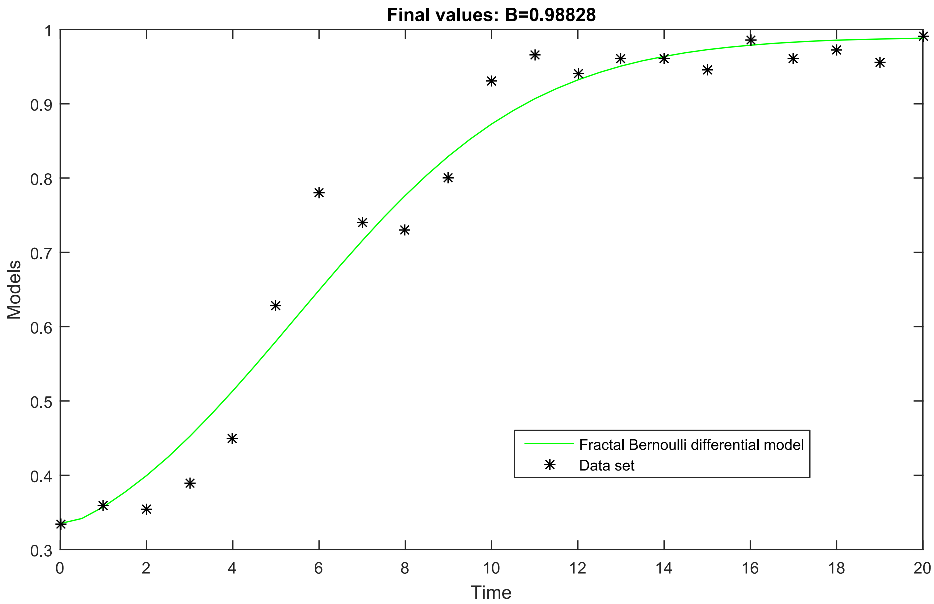

FIGURE 4. Data and estimates of tuberculosis infectious using the fractal Bernoulli differential model (4.5)

errors of the generalized Gompertz model proposed in [20] were relatively larger than those obtained by our model (4.5). The MAE and MSE had values of 0.0416 and 0.0027, respectively.

	Fitting errors	
Models	MAE	MSE
Gompertz growth (4.3)	0.0495	0.0047
Logistics growth (4.4)	0.0614	0.0057
Fractal Bernoulli differential model (4.5)	0.0329	0.0020

TABLE 1. Fitting errors for the studied models

The script is shown below:

```
function fbernoulli
global alpha beta
%%% Set parameters
```

```
alpha=1.4 ; % FILL IN A VALUE FOR ALPHA
beta=0.9; % FILL IN A VALUE FOR BETA
%%% Solve equations
pt = linspace(0,20,100); % Generate t for p
p = (alpha/beta)*pt.^(alpha-1); % Generate p(t)
Tspan = [0 20]; % Solve from t=0 to t=20
IC = 0.335; % y(t=0)=1
[T L] = ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC)
%%%Errors
err2=immse([0.335,0.36,0.355,0.39,0.45,0.628,0.78,0.74,0.73,0.8,0.93,0.965,
0.94,0.96,0.96,0.945,0.985,0.96,0.973,0.955,0.99],[0.335,deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),1),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),2),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),3),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),4),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),5),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),6),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),7),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),8),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),9),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),10),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),11),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),12),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),13),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),14),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),15),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),16),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),17),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),18),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),19),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),20)])
mae2=mae([0.335,0.36,0.355,0.39,0.45,0.628,0.78,0.74,0.73,0.8,0.93,0.965,
0.94,0.96,0.96,0.945,0.985,0.96,0.973,0.955,0.99],[0.335,deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),1),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),2),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),3),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),4),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),5),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),6),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),7),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),8),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),9),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),10),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),11),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),12),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),13),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),14),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),15),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),16),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),17),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),18),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),19),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),20)])
%%% Plot results
figure;
plot(T,L,'green',0,0.335,'k*',1,0.36,'k*',2,0.355,'k*',3,0.39,'k*',4,0.45,
'k*',5,0.628,'k*',6,0.78,'k*',7,0.74,'k*',8,0.73,'k*',9,0.8,'k*',10,0.93,
'k*',11,0.965,'k*',12,0.94,'k*',13,0.96,'k*',14,0.96,'k*',15,0.945,
'k*',16,0.985,'k*',17,0.96,'k*',18,0.973,'k*',19,0.955,'k*',20,0.99,'k*');
```

```
title('Plot of y as a function of time');
xlabel('Time');
ylabel('Models');
legend('Fractal Bernoulli differential model','Data set')
function dl\mathrm{d}t = myode2(t,l,pt,p)
global beta
f = interp1(pt,p,t); % Interpolate the data set (pt,p) at time t
dl\mathrm{d}t =
f.*(abs(l))^(1-beta)*0.1*(0.87*l^beta+0.08*l^(0.7*beta))*(1-(l/0.99));
```

5. Conclusions

The advantage of using fractal differential equations in general is that we can incorporate the variable order of this operator, unlike ordinary differential equations where the simulation and modeling is referred only to the time variable. Thus we have a theoretical-practical panorama that is much broader, more general and richer than that of ordinary differential equations, and which has demonstrated its effectiveness and efficiency in the solution of multiple problems. In this work we studied Bernoulli-type differential equations using fractal derivatives. We were able to show the goodness-of-fit of a particular fractal Bernoulli differential model for the study of real data related to tuberculosis in Mexico. This last fact is remarkable because with such fractal derivatives the curve fit is better than with classic models using usual derivatives. The question of finding a fractal model that further minimizes the fitting error of the data remains open to us. As future work, we plan to use Bayesian statistics to estimate the alpha and beta orders with the aim of finding an optimal model, following the ideas of the paper [16].

References

[1] *Secretaría de Salud, Acciones y Programas*, 2023-08-02.

[2] R. Abreu Blaya, A. Fleitas, J. E. Nápoles, R. Reyes, J. M. Rodríguez, and J. Sigarreta, "On the conformable fractional logistic models," *Math. Methods Appl. Sci.*, vol. 43, pp. 4156–4167, 2020, doi: 10.1002/mma.6180.

[3] F. B. Adda and J. Cresson, "About non-differentiable functions," *J. Math. Anal. Appl.*, vol. 263, pp. 721–737, 2001, doi: 10.1006/jmaa.2001.7656.

[4] F. B. Adda and J. Cresson, "Fractional differential equations and the Schrödinger equation," *Applied Mathematics and Computation*, vol. 161, pp. 323–345, 2005, doi: 10.1016/j.amc.2003.12.031.

[5] D. Alfonso Santiesteban, A. Portilla, J. Rodríguez, and J. Sigarreta, "On Fractal Derivatives and Applications," *Math. Methods Appl. Sci.*, vol. 48, pp. 10 726–10 739, 2025, doi: 10.1002/mma.10914.

[6] X.-J. Y. and F. Gao, "The fundamentals of local fractional derivative of the one-variable non-differentiable functions," *World Sci-Tech R&D*, vol. 31(5), pp. 920–921, 2009.

[7] A. Atangana and Q. Sania, "Modeling attractors of chaotic dynamical systems with fractal-fractional operators," *Chaos, Solitons and Fractals*, vol. 123, pp. 320–337, 2019, doi: 10.1016/j.chaos.2019.04.020.

[8] A. Babakhani and V. Daftardar-Gejji, “On calculus of local fractional derivatives,” *J. Math. Anal. Appl.*, vol. 270(1), pp. 66–79, 2002, doi: 10.1016/S0022-247X(02)00048-3.

[9] J. Bernoulli, “Explicationes, Annotationes & Additiones ad ea, quae in Actis sup. de Curva Elastica, Isochrona Paracentrica, & Velaria, hinc inde memorata, & paratim controversa legundur; ubi de Linea mediarum directionum, alliisque novis,” *Acta Erudit. Lipsiae*, vol. Anno MDCXCV, pp. 537–553, 1695.

[10] M. Braun, *Differential Equations and Their Applications: An Introduction to Applied Mathematics*, Springer, Ed. Springer, 1993. doi: 10.1007/978-1-4612-4360-1.

[11] A. Carpinteri and P. Cornetti, “A fractional calculus approach to the description of stress and strain localization in fractal media,” *Chaos Soliton. Fract.*, vol. 13(1), pp. 85–94, 2002, doi: 10.1016/S0960-0779(00)00238-1.

[12] W. Chen, “Time-space fabric underlying anomalous diffusion,” *Chaos Soliton. Fract.*, vol. 28(4), pp. 923–929, 2006, doi: 10.1016/j.chaos.2005.08.199.

[13] W. Chen, H. Sun, X. Zhang, and D. Korovsak, “Anomalous diffusion modeling by fractal and fractional derivatives,” *Comput. Math. Appl.*, vol. 59(5), pp. 1754–1758, 2010, doi: 10.1016/j.camwa.2009.08.020.

[14] Y. Chen, Y. Yan, and K. Zhang, “On the local fractional derivative,” *J. Math. Anal. Appl.*, vol. 362, pp. 17–33, 2010, doi: 10.1016/j.jmaa.2009.08.014.

[15] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, Ed. McGraw-Hill, 1955.

[16] M. Cruz de la Cruz, D. A. Santiesteban, L. M. Álvarez, R. A. Blaya, and J. H. Gómez, “On a generalized Klausmeier model,” *Mathematical Biosciences and Engineering*, vol. 20(9), pp. 16 447–16 470, 2023, doi: 10.3934/mbe.2023734.

[17] P. Georgiou, S. Yaliraki, E. Drakakis, and M. Barahona, “Quantitative measure of hysteresis for memristors through explicit dynamics,” *Proc. R. Soc. A.*, vol. 468, pp. 2210–2229, 2012, doi: 10.1098/rspa.2011.0585.

[18] A. K. Golmankhaneh, *Fractal Calculus and its Applications*. Singapor: World Scientific Pub Co Inc., 2022. doi: 10.1142/12988.

[19] A. K. Golmankhaneh and D. Bongiorno, “Exact solutions of some fractal differential equations,” *Applied Mathematics and Computation*, vol. 472, p. 128633, 2024, doi: 10.1016/j.amc.2024.128633.

[20] J. H. Gómez, R. Reyes, J. Rodríguez, and J. Sigarreta, “Fractional model for the study of the tuberculosis in Mexico,” *Math. Methods Appl. Sci.*, vol. 45, pp. 10 645–10 688, 2022, doi: 10.1002/mma.8392.

[21] J.-H. He, “A new fractal derivation,” *Therm. Sci.*, vol. 15, pp. 145–147, 2011, doi: 10.2298/TSCI11S1145H.

[22] R. Khalil, M. A. Horani, A. Yousef, and M. Sababhed, “A new definition of fractional derivative,” *Comput. Appl. Math.*, vol. 264, pp. 65–70, 2014, doi: 10.1016/j.cam.2014.01.002.

[23] K. Kolwankar and A. Gangal, “Fractional differentiability of nowhere differentiable functions and dimensions,” *Chaos*, vol. 6(4), pp. 505–513, 1996, doi: 10.1063/1.166197.

[24] K. Kolwankar and A. Gangal, “Hölder exponents of irregular signals and local fractional derivatives,” *Pramana*, vol. 48(1), pp. 49–68, 1997, doi: 10.1007/BF02845622.

[25] K. Kolwankar and A. Gangal, “Local fractional Fokker-Planck equation,” *Phys. Rev. Lett.*, vol. 80(2), p. 214, 1998, doi: 10.1103/PhysRevLett.80.214.

[26] R. Merton, “Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case,” *The Review of Economics and Statistics*, vol. 51(3), pp. 247–257, 1969, doi: 10.2307/1926560.

[27] D. Mirko, A. Lai, and P. Loreti, “Solutions of Bernoulli Equations in the Fractional Setting,” *Fractal Fract.*, vol. 5(2), p. 57, 2021, doi: 10.3390/fractalfract5020057.

[28] S. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*, W. Press, Ed. Westview Press, 2014.
[29] Y. Xiao-Jun, D. Baleanu, and H. Srivastava, *Local Fractional Integral Transforms and Their Applications*, A. P. is an imprint of Elsevier, Ed. Academic Press is an imprint of Elsevier, 2016. doi: 10.1016/C2014-0-04768-5.
[30] X.-J. Yang, "Local fractional integral transforms," *Prog. Nonlinear Sci.*, vol. 4(1), pp. 1–225, 2011.
[31] X.-J. Yang, *Advanced Local Fractional Calculus and Its Applications*, N. Y. World Science, Ed. World Science, New York, 2012.
[32] X.-J. Yang, D. Baleanu, and J. Machado, "Systems of Navier-Stokes equations on Cantor sets," *Math. Probl. Eng.*, vol. 769724, pp. 1–8, 2013, doi: 10.1155/2013/769724.

Authors' addresses

Daniel Alfonso Santiesteban
Universidad Autónoma de Guerrero, Facultad de Matemáticas, Lázaro Cárdenas 88, Col.21, 39086 Chilpancingo, Mexico
E-mail address: danielalfonso950105@gmail.com

Ricardo Abreu Blaya
Universidad Autónoma de Guerrero, Facultad de Matemáticas, Lázaro Cárdenas 88, Col.21, 39086 Chilpancingo, Mexico
E-mail address: rabreublaya@yahoo.es

Yamilet Quintana
Universidad Carlos III de Madrid, Departamento de Matemáticas, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
E-mail address: yaquinta@math.uc3m.es

José María Sigarreta Almira
(**Corresponding author**) Universidad Autónoma de Guerrero, Facultad de Matemáticas, Carlos E. Adame No.54, Col. Garita, 39650 Acalpulco, Mexico
E-mail address: jsmathguerrero@gmail.com