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UNIVERSAL DISTANCE DOMINATION IN RANDOM GRAPHS

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Abstract. Given a graph $G = (V,E)$, a *dominating set* is a subset $D \subseteq V$ such that every vertex in $V \setminus D$ is adjacent with at least one vertex in D. The *domination number* of G, traditionally denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.

For any natural number k, a set D is *k-distance dominating*—called kdd set for short—if every vertex not in D is at distance at most d from some vertex in D. We define the *universal k-distance domination number* as

$$u_k(G) := \min\{d : \forall D \subseteq V \text{ with } |D| \geq d, D \text{ is a kdd set in } G\}.$$

In sharp contrast to most of the standard domination parameters, determining the universal k-distance domination number $u_k(G)$ turns out to be computable in polynomial time. We also characterize the graphs satisfying $\gamma(G) = u_1(G)$, and investigate the behavior of the function u_k on random graphs.

It remains a challenging open problem to characterize the equality of k-distance domination and universal k-distance domination numbers for a general k. A further problem is to sharpen the estimates on the edge probability in the auxiliary graph $H = G^k$.

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1. INTRODUCTION

The key notion studied here is k-distance domination, that involves vertex subsets in graphs, from which every vertex is at distance at most k apart. The history of distance domination began half a century ago, as introduced by Meir and Moon [13] (using the term "k-covering" at that time). Since then, more than 100 papers appeared on the subject. Beside theoretical interest, applications concerning various kinds of communication networks are also discussed in the literature; see, e.g., [8] and [9]. The interested reader can obtain a very detailed picture of the area by putting together the survey [15] and the (non-numbered) chapter [11] of a recent monograph.

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Those two works are complementary; the authors of the survey intentionally concentrated on the part of the literature not discussed in the book chapter. Beyond them, distance domination has been studied in vertex-partitioned graphs, as well; for results of that kind please see [10] and the references therein. Further issues concerning the relations between domination and distances are discussed in the papers [4–6], where the role of induced subgraphs is analyzed. [2] and [3] are also related to this paper.

A closer analysis of a proof in a paper by Wieland and Godbole [16] reveals that under some circumstances not only a k-distance dominating set of a certain size can be found, but in fact any vertex set of that size is k-distance dominating. This observation leads us to the new concept of universal distance domination, that we introduce in the present study. In this work both a deterministic and a random model of this concept will be discussed.

The decision version of the Set Cover problem is included in Karp's 21 NP-complete problems [12], and there is an easy direct reduction from it to (distance) domination. Hence, intractability of the latter is an immediate consequence from the early literature on the subject.

In sharp contrast to this, here we prove that determining the universal k-distance domination number $u_k(G)$, that we define in the next section, can be done in polynomial time, very efficiently. This follows from a result in Section 2 where we prove a minimax formula involving $u_k(G)$.

It is also a very interesting problem to characterize the graphs whose universal k-distance domination number is equal to their k-distance domination number. In Section 3 we solve the case $k = 1$, leaving the general question open. Let us note that a problem of similar flavor was recently studied by Boruzanlı Ekinci and Bujtás [14] who investigated the equality of domination and 2-domination numbers (the latter being the smallest size of a vertex set S such that every vertex not in S has at least two neighbors in S).

Concerning random graphs, in Section 4 we introduce a function defined recursively, and prove that it is an upper bound on $u_k(G)$ almost surely. Then the distribution of $u_k(G)$ for any given constant k is analyzed in Section 5.

2. Definitions and a minimax formula

Let us begin with quoting the definition of the fundamental notion of graph domination theory.

Definition 1. A vertex set D is *dominating* in the graph $G = (V,E)$ if for any $v \in V \setminus D$ there exists some $w \in D$ adjacent to v. The minimum cardinality of a dominating set is termed the *domination number*, and is denoted by $\gamma(G)$.

A natural generalization of this concept is the following.

Definition 2. A vertex set D is *k-distance dominating* in the graph $G = (V,E)$ if for any $v \in V \setminus D$ there exists some path of length at most k between some $w \in D$

and v. We use the shorthand kdd set to abbreviate "k-distance dominating set". We denote

$$\gamma_k(G) := \min\{|D| : D \text{ is a kdd set in } G\}.$$

In particular, we have

$$\gamma(G) := \gamma_1(G).$$

Motivated by Wieland and Godbole's work [16], here we introduce the following variant.

Definition 3. The *universal k-distance domination number* is

$$u_k(G) := \min\{d : \forall D \subseteq V \text{ with } |D| \geq d, D \text{ is a kdd set in } G\}$$

i.e. the minimum of d such that any vertex set of cardinality at least d vertices k-distance-dominates G.

Claim 1. Clearly, for any integer $k \geq 1$, every graph G satisfies the inequality

$$u_k(G) \geq \gamma_k(G).$$

As an illustration for $k = 1$, we mention the following fact.

Observation 1. *A graph G on n vertices has $u_1(G) \leq n-1$ if and only if there are no isolated vertices in G.*

Notation 1. For any two vertices u and v of a graph G, the distance between them will be denoted by $d_G(u,v)$, or by $d(u,v)$ for short when G is understood.

Let x be an arbitrary vertex of a graph G. The set $N_k[x]$ means $\{y \in V(G) : d(x,y) \leq k\}$, the *closed k-neighborhood* of x. It contains x, by definition. Let $C_k(x)$ be the vertex set $V(G) \setminus N_k[x]$ and $c_k(x) := 1 + |C_k(x)|$.

We state here a simple but useful minimax relation.

Theorem 1. *For the universal k-distance domination number $u_k(G)$ we have*

$$\begin{aligned} u_k(G) &= \min\{d : \forall D \subseteq V \text{ with } |D| \geq d, D \text{ is a kdd set in } G\} \\ &= \max\{c_k(x) : x \in V(G)\} \end{aligned} \tag{2.1}$$

for every graph G.

Proof. First, we prove that in (2.1), min $\geq$ max is valid.

Let us take a vertex x_0, yielding the maximum at the right-hand side. The set $C_k(x_0)$ does not k-distance dominate x_0. So it is not a kdd set, and the minimum $u_k(G)$ is strictly larger than $|C_k(x_0)|$. This means min $\geq$ max.

Second, we prove that the converse inequality min $\leq$ max also holds in (2.1).

By the definition of $u_k(G)$, there exists some vertex set W not k-distance dominating with $|W| = u_k(G) - 1$. This implies the existence of some vertex x_0 such that $W \cap N_k[x_0] = \varnothing$. Consequently, $|N_k[x_0]| \leq n - |W| \leq n - u_k(G) + 1$ and therefore

$$u_k(G) \leq 1 + n - |N_k[x_0]| = 1 + |C_k(x_0)| = c_k(x_0) = \max\{c_k(x) : x \in V(G)\}.$$

This completes the proof of the theorem. □

Theorem 1 makes it possible to design an efficient algorithm for the determination of *all* universal distance domination numbers of a graph.

Theorem 2. *Given a generic input graph $G = (V,E)$, the universal k-distance domination numbers $u_k(G)$ can be determined for all $k \geq 1$ in polynomial time (in at most $C * |V| * |E|$ steps, with an absolute constant C).*

Proof. First run any linear-time graph traversal algorithm that can identify the connected components of G. If G is disconnected, say it has components $G_1, \ldots, G_c$, then $u_k(G) = u_k(G_1) + \ldots + u_k(G_c)$, so the computation can be done separately on the components of G.

Assume now that the input graph $G = (V,E)$ is connected, and has n vertices. Then we run the standard Breadth-First Search algorithm n times, starting with each vertex $x \in V$ as root. During BFS we register the sizes $n_i(x) := |N_i[x]|$ of closed i-neighborhoods of x for $i = 1,2,\ldots$. Then, for any k, we have

$$u_k(G) = \max_{x \in V} c_k(x) = 1 + \left(n - \min_{x \in V} n_k(x) \right).$$

Since each run of BFS takes $O(|V| + |E|)$ time, and we need to make at most $O(n^2)$ comparisons at the end, the claimed total running time follows. □

3. On the equation $u_k(G) = \gamma_k(G)$

As noted in Claim 1, $u_k(G) \geq \gamma_k(G)$ holds for every graph G and every positive integer k. In this section we make the first step in characterizing the cases of equality.

Theorem 3. *A graph G satisfies the equality $u_1(G) = \gamma(G)$ if and only if G is complete or empty or it is the complement of a matching.*

Proof. We shall refer to the three types of graphs listed in the assertion as "sample graphs". It can easily be seen that every sample graph satisfies the eqality condition $u_1 = \gamma$.

To prove the "only if" part, let $G = (V,E)$ be any graph with $u_1(G) = \gamma(G)$, say on n vertices. As a useful notation, let us write

$$X(x) := \{y \in V : y \notin N_1[x]\},$$

where x is any vertex of G.

We first prove that G is regular. On the one hand, the vertex set $X(x) \cup \{x\}$ is dominating in G, which implies

$$\gamma(G) \leq n - d_G(x)$$

for every vertex x. On the other hand, the vertex set $X(x)$ is not dominating since it does not dominate x. Consequently, $u_1(G) > |X(x)| = n - d_G(x) - 1$. This fact and

the equality condition in the theorem imply

$$\gamma(G) \geq n - d_G(x)$$

for every x. Summarizing, the degree of each vertex is equal to $n-\gamma(G)$, thus G is regular.

Let us denote $w := u_1(G) - 1$. Note that the degree of regularity in G is equal to $n-1-w$, and w is in the range $0 \leq w \leq n-1$. Moreover, each set W of w vertices is non-dominating since $\gamma(G)$ is the minimum size of dominating vertex sets in the whole graph and, by assumption, we have $w < u_1(G) = \gamma(G)$.

Definition 4. For an arbitrary non-dominating vertex set Y, and for any vertex η not dominated by Y, we call η the *witness* of Y. (Certainly, Y may have several witnesses.)

Let us consider now a set W that has cardinality exactly w. This W is non-dominating. For any witness η of W, we have $W \subseteq X(\eta)$.

Suppose now that W is a *proper* subset of $X(\eta)$. In this case, there exists some $\xi \in X(\eta) \setminus W$ and adding it to W, we obtain a set Z of cardinality $u_1(G)$. By the definition of $u_1(G)$, Z is dominating. On the other hand, W does not dominate η, and ξ is also not adjacent to it. Thus, Z is not dominating, and we got a contradiction with the assumption "proper subset".

We have shown that, taking any vertex set W of cardinality w in the graph, it coincides with $X(\eta)$ for some vertex η. Consequently,

$$\binom{n}{w} \leq n\,.$$

This inequality is valid for $w = 0, 1, n-1$ only, that yields a clique, a co-matching and an empty graph, respectively. These are exactly the sample graphs. □

Here we prove an extension of Theorem 3 for general k.

We emphasize that Theorem 4 below is not an explicit structural characterization of those graphs G which satisfy $u_k(G) = \gamma_k(G)$.

Theorem 4. *A graph $G = (V,E)$ satisfies the equality $u_k(G) = \gamma_k(G)$ if and only if one of the following cases holds:*

- (*i*) *G has no edges, hence $u_k(G) = \gamma_k(G) = |V|$.*
- (*ii*) *G has diameter at most k, hence $u_k(G) = \gamma_k(G) = 1$.*
- (*iii*) *G has diameter $k+1$, and for every $x \in V$ there exists precisely one $\bar{x} \in V$ at distance $k+1$ apart from x. In this case, $u_k(G) = \gamma_k(G) = 2$.*

Proof. Here we introduce an auxiliary graph H.

Definition 5. In $H := G^k$, let $V(H) = V(G)$ and let $xy \in E(H)$ if and only if $d_G(x,y) \leq k$.

The definitions imply that the closed neighborhood of any vertex x in H coincides with $N_k[x]$ in G. Consequently, we have $u_1(H) = u_k(G)$ and $\gamma(H) = \gamma_k(G)$. Thus, $u_k(G) = \gamma_k(G)$ holds if and only if $u_1(H) = \gamma(H)$, and we may apply Theorem 3 for H.

For the cases when H is an empty graph or complete, respectively, we obtain the cases (i) and (ii).

For the third possibility, let (a_i, b_i), $i = 1, 2, \ldots, n/2$, be the edges of the perfect matching whose complementary graph is H. Then, in G, b_i is the unique vertex farther than k from a_i, and vice versa. Picking any neighbor u of a_i in G, we also see that a_i and b_i have distance

$$d_G(a_i, b_i) \le d_G(a_i, u) + d_G(u, b_i) \le 1 + k,$$

hence exactly $k+1$. This yields (iii). □

4. A PROBABILISTIC UPPER BOUND ON u_k

In this section we begin the study of the universal k-distance domination number u_k of the random graph $\mathbf{G}_{n,p}$ with n vertices and edge-probability p. We write $q := 1/(1-p)$. Let us consider now the function $f(n) := \lceil \log_q n + a_n \rceil$ where a_n is an arbitrary positive sequence tending to infinity, such that $f(n) < n$.

At first glance, the property that every vertex set of size at least d is k-dominating in the whole graph, seems to be too strong. However, Proposition 1 shows that its probability of having this property is relatively large.

Let us define the integer functions ϕ_k, $k \ge 1$, recursively: $\phi_1(n) := f(n)$, and $\phi_k(n) := f[\phi_{k-1}(n)]$ for all $k > 1$.

Proposition 1. *Let $k \ge 1$ be a fixed integer, and let $0 < p < 1$ be a fixed real number. Then*

$$\mathbb{P}(u_k(\mathbf{G}_{n,p}) \le \phi_k(n)) \to 1$$

as $n \to \infty$.

Proof. The argument works by induction on k. Essentially, we apply the proof of [16], page 2, where the assertion was proved for $k = 1$. This case serves as anchor for the induction.

Suppose that the proposition has been verified for $k-1$. Pick any set of vertices D with size $\phi_k(n)$ in $G := \mathbf{G}_{n,p}$. We will show that D is a kdd set in G, with high probability.

Let us extend D to a vertex set D' with size $\phi_{k-1}(n)$ arbitrarily. Consider

$$\mathbb{P}(D \text{ is a kdd set in } G) \ge \mathbb{P}(A_n B_n)$$

for the events

$$A_n := D' \text{ is } (k-1)\text{-distance dominating in } G$$

and

$$B_n := D \text{ is dominating in } D'.$$

We state the following two facts.
(i)

$$\mathbb{P}(A_n) \to 1,$$

it follows from the induction hypothesis.
(ii)

$$\mathbb{P}(B_n) \to 1.$$

Instead of n, in the proof of (ii) we use the variable ν, meaning the order of an arbitrary graph (possibly different from the original one), tending to ∞. Applying the abbreviations $M := \phi_{k-1}(\nu)$, $r := \phi_k(\nu)$, and recalling $q = 1/(1-p)$, from the Binomial Theorem we obtain

$$\rho := \mathbb{P}(D \text{ is dominating in } D') = [1-(1-p)^r]^{M-r}$$

$$\geq [1-(1-p)^r]^{\nu-r} \geq 1-(\nu-r)(1-p)^r,$$

$$\rho \geq 1-\nu(1-p)^r \geq 1-\nu(1-p)^{\log_q \nu + a_\nu},$$

because $(1-p)^{\log_q \nu} = 1/\nu$ and $\rho \geq 1-(1-p)^{a_\nu} \to 1$ as ν tends to ∞.

Clearly, (i) and(ii) together imply that $P(A_nB_n) \to 1$ holds, and from this

$$\mathbb{P}(D \text{ is } k\text{-distance dominating in } G) \to 1.$$

□

Since $u_k(G)$ is an upper bound on $\gamma_k(G)$ for every graph G and every positive integer k, Proposition 1 has the following immediate consequence.

Corollary 1. *For every integer $k \geq 1$ and every fixed edge probability p,*

$$\mathbb{P}(\gamma_k(\mathbf{G}_{n,p}) \leq \phi_k(n)) \to 1,$$

as $n \to \infty$.

5. On the distribution of u_k

In this section we analyze the asymptotic behavior of u_k in the random graph $\mathbf{G}_{n,p}$.

5.1. *The closed k-neighborhood*

For arbitrary fixed $k \geq 0$, let us consider the size $|N_k[x_0]|$ of the closed k-neighborhood for a vertex x_0 as a random variable. We compute here its distribution.

Recall that $d_G(u,v)$ or simply $d(u,v)$ denotes the distance between two vertices u, v of G.

We define now the *levels* with respect to x_0.

$$L_\mu(x_0) := \{y \in V(G) \mid d(y,x_0) = \mu\}$$

for $\mu = 0, 1, \ldots, k$.

The following is clearly true.

Observation 2. *For $z \in V(G)$ and $m \geq 1$, we have $d(x_0,z) = m+1$ (i.e., $z \in L_{m+1}$) if and only if*

(1) *z has at least one neighbor in L_m*
and
(2) *z has no neighbors in $N_{m-1}[x_0]$.*

For any sequence $j_1, \ldots, j_k$ of k nonnegative integers and any $1 \leq \mu \leq k$, let $\prod_\mu$ be the product of the following events: $|L_1(x_0)| = j_1, \ldots, |L_\mu(x_0)| = j_\mu$. For convenience we also put $j_0 := |L_0(x_0)| = 1$.

Let us consider the following events: $|L_1(x_0)| = j_1, \ldots, |L_\mu(x_0)| = j_\mu$. Let be Π_μ the product of these events.

We introduce a sequence $f_1, f_2, \ldots, f_k$ of auxuliary functions, that help in calculating the distribution of u_k. For $\mu = 1, \ldots, k$ the function f_μ has μ positive integer variables, and is defined as the probability $\mathbb{P}(\prod_\mu)$.

$$J_\mu := 1 + \sum_{\iota=1}^{\mu} j_\iota = \sum_{\iota=0}^{\mu} j_\iota \qquad (\mu = 1,2,\ldots k).$$

$$J_\mu := 1 + \sum_{\iota=1}^{\mu} j_\iota = \sum_{\iota=0}^{\mu} j_\iota \qquad (\mu = 1,2,\ldots k).$$

The meaning of J_μ is the number of vertices having distance at most μ from x_0.

We calculate these functions recursively.

Proposition 2. *For every sequence $j_1, \ldots, j_k$ the functions $f_1, f_2, \ldots, f_k$ can be computed as*

$$f_1(j_1) = \binom{n-1}{j_1} p^{j_1} (1-p)^{n-J_1},$$

$$f_2(j_1,j_2) = f_1(j_1) \binom{n-J_1}{j_2} (1-(1-p)^{j_1})^{j_2} (1-p)^{(n-J_2)j_1},$$

$$\vdots$$

$$f_\mu(j_1,\dots,j_\mu) = f_{\mu-1}(j_1,\dots,j_{\mu-1})\binom{n-J_{\mu-1}}{j_\mu}(1-(1-p)^{j_{\mu-1}})^{j_\mu}(1-p)^{(n-J_\mu)j_{\mu-1}},$$

$$\vdots$$

$$f_k(j_1,\dots,j_k) = f_{k-1}(j_1,\dots,j_{k-1})\binom{n-J_{k-1}}{j_k}(1-(1-p)^{j_{k-1}})^{j_k}(1-p)^{(n-J_k)j_{k-1}}.$$

Proof. We argue concerning the general step, applying Observation 2 with $\mu = m+1$. Having computed the probabilities and the positions of the levels up to $L_{\mu-1}$, there are $\binom{n-J_{\mu-1}}{j_\mu}$ ways to select the j_μ vertices of L_μ. Then each selected $x \in L_\mu$ has to be adjacent to at least one vertex in $L_{\mu-1}$; the probability that x satisfies this requirement is $1-(1-p)^{j_{\mu-1}}$. This is not only necessary but also sufficient for $x \in L_\mu$ because in the preceding steps we ensured that x does not belong to levels of smaller indices. The exponent j_μ of this expression in the recursive formula is due to the fact that for distinct vertices of L_μ these events are totally independent.

In addition, it has to be ensured that the unselected $n-J_\mu$ vertices are nonadjacent to $L_{\mu-1}$. This means that all edges of a complete bipartite graph with $n-J_\mu$ and $j_{\mu-1}$ vertices in its partite sets are missing from G, hence yielding the last factor in the formula, namely the power of $(1-p)$ with the claimed exponent. (Non-adjacency to levels of smaller indices were decided in earlier steps.) □

Now we are in the position to calculate the distibution we wanted.

Let us introduce the notation

$$W_{k,i} := \mathbb{P}(|N_k[x_0]| = i).$$

Theorem 5.

$$W_{k,i} = \sum_{\substack{j_1,j_2,\dots,j_k \geq 1 \\ j_1+j_2+\dots+j_k = i-1}} f_k(j_1,j_2,\dots,j_k).$$

Clearly, the following statement is valid.

Observation 3.

$$\max\{c_k(x) : x \in V(G)\} = n+1-\min\{|N_k[x]| : x \in V(G)\}.$$

For the distribution of the minimum in Observation 3, first it seems that the method in [1] can be applied. Unfortunately, the complete independence is supposed there. Here we show a simple case where even pairwise independence is not valid.

Example 1. Let $1 \leq k \leq n-1$ and $\xi_i := |N_k[x_i]|$. Then $P(\xi_0 = n) > 0$, $P(\xi_1 = 1) > 0$, but $P(\xi_0 = n \text{ and } \xi_1 = 1) = 0$.

Due to the lack of independence among the random variables in question, an alternative method is developed in the sequel.

5.2. *The new edge probability*

In Theorem 1 we reduced the problem concerning u_k to the minimum of $|N_k(x)|$. Now we go further, to the distribution of minimum degree in an auxiliary graph and finally reducing it to a significant result of Bollobás.

Here we present the key notion of the process, the auxiliary graph $H := G^k$. The aim of introducing it, is to take a quantity much simpler than $|N_k(x)|$.

By definition, let $xy \in E(H)$ if and only if $d_G(x,y) \le k$. We obtained H from a random graph with edge probability p and thus H is a random graph again, with some edge probability, say, $s = s(n,p,k)$.

We give recursive estimates on s. Having n and p fixed, let us introduce the notation $\pi_\kappa(u,v) := P(d_G(u,v) \le \kappa)$.

Proposition 3. *For $k \ge 2$, the following recursive lower and upper bounds are valid for $s(n,p,k) = \pi_k(x,y)$.*

(i) $\pi_k(x,y) \ge p * \max\{\pi_{k-1}(z,y) \mid z \in V(H) \setminus \{x,y\}\}$,

(ii) $\pi_k(x,y) \le p * \sum_{z \in V(H)\setminus\{x,y\}} \pi_{k-1}(z,y)$.

Proof. In order to verify the lower bound we observe that $\pi_k(x,y) = s(n,p,k) \ge \max\{P(xz \in E(G)|) * \pi_{k-1}(z,y) \mid z \in V(H) \setminus \{x,y\}\}$. Moreover, we have here disjoint edge sets and thus, independent events. The reason for the validity of the upper bound is similar. □

Remark 1. In fact, the proof of Proposition 3 is based on the following principle.

Observation 4. *The event $xy \in E(H)$ is the sum of the events $d_G(y,z) \le k-1$ for $z \in V(H) \setminus \{x,y\}$ and the event $xy \in E(G)$.*

5.3. *The minimum degree*

Supposing that the edge probability s in the auxuiliary random graph H is in our hand, it is enough to determine the distribution of minimum degree $\delta(H)$ in $H = G^k$.

Below, we abbreviate to G_s the notation for the random graph of order n and edge probability s, where $s = s(n,p,k)$ as defined above. From the theorem on page 65 of [7], the following formulas can be obtained.

Theorem 6 ([7], Theorem 3.5.)**.** *Let $0 < s < 1$ be given. Let us express it in the form*

$$s = (\log n + h \log\log n + y)/n,$$

where n is large enough, $h \ge 2$ is an integer and $y \in \mathbb{R}$ is a real number.

Then

$$P(\delta(G_s) = h) \to 1 - e^{-e^{-y/h!}}$$

and

$$P(\delta(G_s) = h+1) \to e^{-e^{-y/h!}},$$

if $n \to \infty$.

Remark 2. We can express y in terms of n, h and s:

$$y = ns - \log n - h \log\log n$$

5.4. *Summarizing*

Our original target was to determine the distribution of $u_k(G)$. First we calculated its exact value by a recursive formula in Subsection 5.1. Then we reduced distance domination to an auxiliary degree condition in Subsection 5.2. Finally in Subsection 5.3, we gave its limit distribution, using a classical result.

6. Concluding remarks

In this paper we introduced and studied universal k-distance dominating sets of graphs. Many new problems arise, here we mention those most related to the present study.

Problem 1. *Give a structural characterization of those graphs G for which $u_k(G) = \gamma_k(G)$ holds.*

Remark 3. Theorem 3 yields the answer for $k = 1$, and necessary and sufficient conditions are given in Theorem 4. In fact the latter is closely related to the long-standing open problem to characterize the so-called antipodal graphs. There are several variations of those graphs in the literature, and to the best of our knowledge, none of them has been characterized so far.

Problem 2. *Sharpen the estimates given in Proposition 3 concerning the asymptotics of the edge probability of the graph $H = (\mathbf{G}_{n,p})^k$.*

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