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DYNAMICS OF TRAVELING WAVE SOLUTIONS AND CHAOS FOR GENERALIZED KDV-BURGERS EQUATION

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Abstract. This paper applies the qualitative theory of differential equations to explore the global structure of the generalized KdV-Burgers equation, including a complete description of the phase portrait at infinity and the amplitude estimation of the oscillating shock wave. And based on KCC theory, the deviation curvature tensor of the generalized KdV-Burgers equation is given. After determining the Jacobi stability at any point on the traveling wave solution, this paper applies the Melnikov method to further discuss the chaotic behavior when the dissipative term receiving periodic perturbations.

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1. Introduction

For nonlinear partial differential equations, it is very difficult to find its analytical solutions. A common way to analysis the partial differential equation is to transform the equation into two-dimensional ordinary differential equations through traveling wave transformation, and then analyze the traveling wave solutions of the equivalent system. Traveling wave solutions play a crucial role in nonlinear science, as they can effectively describe various natural phenomena such as vibrations [4] and propagating waves [20].

In this paper, we investigate the dynamics of the generalized KdV-Burgers equation in the following form [24].

$$\gamma\frac{\partial^3 u}{\partial x^3}+\beta\frac{\partial^2 u}{\partial x^2}+\alpha u^n\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=0, \tag{1.1}$$

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where $\gamma \neq 0$ is the dispersion coefficient, β is the dissipation coefficient and $\alpha \neq 0$ is the nonlinear term. This equation with $n \geq 1$ arises in modeling waves generated by a wavemaker in a channel and the waves incoming from deep water into nearshore zones [18]. In fact, if one takes different values for α, β, γ, n, the equation (1.1) includes many equations. For example, the KdV-Burgers equation with $n = 1$ [1], the Gardner equation with $n = 2$ [8] and the generalized KdV equation with a closed orbit when $\beta = 0$ [6].

The generalized KdV-Burgers equation also has two different traveling wave solutions. One of which is a monotonic shock wave [11]. The other one is an oscillating shock wave which is difficult to calculate the exact solution. Therefore, further research on the oscillation behavior of the oscillating shock wave is needed. This paper further explores the case of $n = 1$. The generalized KdV-Burgers equation is a more general case, leading to more general conclusions.

KCC theory (or Jacobi analysis) [2, 5, 13] is a geometric method for studying the stability of dynamic systems. As one of the KCC invariants, deviation curvature tensor is used to obtain Jacobi stability of the second-order ordinary differential equation. Nowadays, KCC theory has been applied to many three-dimensional systems, such as Rabinovich system [16], Yang-Chen system [17], Chen system [12, 19] and other unusual systems [3, 7, 14, 22, 23]. In these literatures, there is a close connection between the deviating curvature tensor and the chaotic behavior of trajectories. Besides, KCC theory has also been applied in traveling wave solutions [15]. This provides a new point for studying the dynamic behavior of traveling wave solutions. It helps to analyze the stability of the traveling wave solution at any point.

The rest of this paper is organized as follows. In Sec. 2, dynamics of the equivalent plane system near equilibria and at infinity is discussed. A more accurate amplitude estimation of the oscillating shock wave solution is given for (1.1) and the global structure diagram is presented. In Sec. 3, Jacobi stability and the deviation vector of the generalized KdV-Burgers equation is discussed. In Sec. 4, the chaotic behavior of traveling wave solutions when the dissipative term receiving periodic perturbations is analytically confirmed by Melnikov method. Conclusions are given in the last section.

2. GLOBAL ANALYSIS

This section will analyse the global dynamic behavior of the traveling wave solutions of the generalized KdV-Burgers equation.

2.1. *equilibrium points*

By traveling wave transformation $\zeta = x - ct$, $c \neq 0$, integrating both sides and making the constant of integration zero, the generalized KdV-Burgers equation becomes

$$\gamma u_{2\zeta} + \beta u_\zeta + \frac{\alpha}{n+1} u^{n+1} - cu = 0. \tag{2.1}$$

One can formalize (2.1) into the equivalent planar system.

$$\frac{du}{d\zeta} = v,$$
$$\frac{dv}{d\zeta} = -\frac{\beta}{\gamma}v - \frac{\alpha}{(n+1)\gamma}u^{n+1} + \frac{c}{\gamma}u. \tag{2.2}$$

When n is an odd number, the plane system (2.2) has two equilibrium points, $E_0 = (0,0)$ and $E_1 = \left(\frac{2c}{\alpha},0\right)$. If n is an even number, there has one equilibrium point $E_0 = (0,0)$ for $\frac{c}{\alpha} > 0$. For $\frac{c}{\alpha} < 0$, there has three equilibrium points $E_0 = (0,0)$, $E_1\left(\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}},0\right)$ and $E_2\left(-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}},0\right)$.

The Jacobi matrix corresponding to E_0 is

$$J(E_0) = \begin{pmatrix} 0 & 1 \\ \frac{c}{\gamma} & -\frac{\beta}{\gamma} \end{pmatrix},$$

and the eigenvalues are $\lambda_{1,2} = \frac{\beta}{2\gamma} \pm \frac{\sqrt{\beta^2+4c\gamma}}{2|\gamma|}$.

Similarly, the system (2.2) has the same Jacobi matrixes at E_1 and E_2.

$$J(E_1) = J(E_2) = \begin{pmatrix} 0 & 1 \\ -\frac{nc}{\gamma} & -\frac{\beta}{\gamma} \end{pmatrix},$$

and the eigenvalues are $\lambda_{3,4} = \frac{\beta}{2\gamma} \pm \frac{\sqrt{\beta^2-4c\gamma}}{2|\gamma|}$.

Local phase diagrams of the equilibrium points of (2.2) are depicted in Fig. 1.

If n is an even number, and $\frac{c}{\alpha} > 0$, system (2.2) has three equilibrium points E_0 and $E_{1,2}$, their stability is shown as the table 1.

If n is an even number, and $\frac{c}{\alpha} < 0$, system (2.2) has only one equilibrium point E_0, the stability is shown as the table 2.

If n is an odd number, system (2.2) has two equilibrium points E_0 and E_1, their stability conclusions are consistent with Table 1. But there is no equilibrium point E_2.

TABLE 1. Parameter Condition and Equilibrium Point Type for $\frac{c}{\alpha} > 0$

Condition			E_0	$E_{1,2}$
$\gamma > 0$	$c > 0$	$\beta \geq 2\sqrt{nc\gamma}$	saddle point	stable node
		$0 < \beta < 2\sqrt{nc\gamma}$		stable focus
		$\beta = 0$		center
		$-2\sqrt{nc\gamma} < \beta < 0$		unstable focus
		$\beta \leq -2\sqrt{nc\gamma}$		unstable node
	$c < 0$	$\beta \geq 2\sqrt{-c\gamma}$	stable focus	saddle point
		$0 < \beta < 2\sqrt{-c\gamma}$	stable node	
		$\beta = 0$	center	
		$-2\sqrt{-c\gamma} < \beta < 0$	unstable node	
		$\beta \leq -2\sqrt{-c\gamma}$	unstable focus	
$\gamma < 0$	$c > 0$	$\beta \geq 2\sqrt{-c\gamma}$	unstable node	saddle point
		$0 < \beta < 2\sqrt{-c\gamma}$	unstable focus	
		$\beta = 0$	center	
		$-2\sqrt{-c\gamma} < \beta < 0$	stable focus	
		$\beta \leq -2\sqrt{c\gamma}$	stable node	
	$c < 0$	$\beta \geq 2\sqrt{nc\gamma}$	saddle point	unstable focus
		$0 < \beta < 2\sqrt{nc\gamma}$		unstable node
		$\beta = 0$		center
		$-2\sqrt{nc\gamma} < \beta < 0$		stable node
		$\beta \leq -2\sqrt{nc\gamma}$		stable focus

2.2. *Dynamics at infinity*

In this subsection, one will discuss dynamics at infinity and give the global structure diagram of the generalized KdV-Burgers equation.

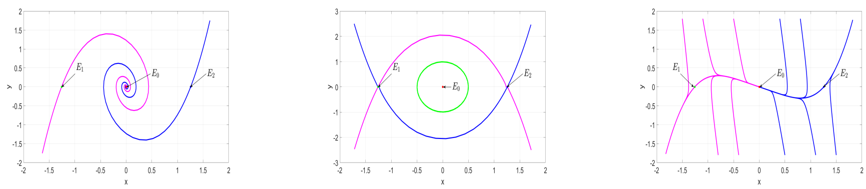
For the limit cycle, it can be proved that there is no limit cycle in the system (2.2). Due to the equation system (2.1), one has

$$v_u + \left(-\frac{\beta}{\gamma}v + \frac{c}{\gamma}u - \frac{\alpha}{(n+1)\gamma}u^{n+1}\right)_v = -\frac{\beta}{\gamma}.$$

According to the Bendixson criterion, when $\beta \neq 0$, the system does not have any closed orbits. Thus, there is no limit cycle.

TABLE 2. Equilibrium Point Type for $\frac{c}{\alpha} < 0$

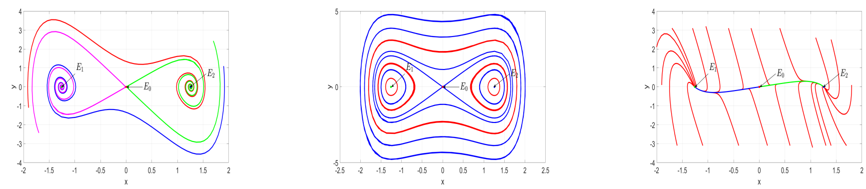
Condition			E_0
$\gamma > 0$	$c > 0$	For any β	saddle point
	$c < 0$	$\beta \geq 2\sqrt{-c\gamma}$	stable focus
		$0 < \beta < 2\sqrt{-c\gamma}$	stable node
		$\beta = 0$	center
		$-2\sqrt{-c\gamma} < \beta < 0$	unstable node
		$\beta \leq -2\sqrt{-c\gamma}$	unstable focus
$\gamma < 0$	$c > 0$	$\beta \geq 2\sqrt{-c\gamma}$	unstable node
		$0 < \beta < 2\sqrt{-c\gamma}$	unstable focus
		$\beta = 0$	center
		$-2\sqrt{-c\gamma} < \beta < 0$	stable focus
		$\beta \leq -2\sqrt{c\gamma}$	stable node
	$c < 0$	For any β	saddle point



(a) $\alpha = 8, \beta = -3.26, \gamma = 1, c = 4$

(b) $\alpha = 8, \beta = 0, \gamma = 1, c = 4$

(c) $\alpha = 8, \beta = -4.62, \gamma = 1, c = 4$



(d) $\alpha = 8, \beta = -3.26, \gamma = -1, c = 4$

(e) $\alpha = 8, \beta = 0, \gamma = -1, c = 4$

(f) $\alpha = 8, \beta = -4.62, \gamma = -1, c = 4$

FIGURE 1. (a) Saddle point - focus - Saddle point (b) Saddle point - center - Saddle point (c) Saddle point - node - Saddle point (d) focus - Saddle point - focus (e) center - Saddle point - center (f) node - Saddle point - node

For the infinite singularities, Poincaré Compaction Technology is a common tool to analyze it.

Making $u = \frac{1}{z}$, $v = \frac{w}{z}$, $dT = \frac{d\zeta}{z^n}$, system (2.2) becomes

$$\begin{aligned} \frac{dz}{dT} &= -z^{n+1}w, \\ \frac{dw}{dT} &= -w^2z^n - \frac{\beta}{\gamma}wz^n - \frac{\alpha}{(n+1)\gamma} + \frac{cz^n}{\gamma}. \end{aligned} \tag{2.3}$$

When $z = 0$, the system (2.3) has no equilibrium point.

Let $u = \frac{w}{z}$, $v = \frac{1}{z}$, $dT = \frac{d\zeta}{z}$, and $z' = z^n$, it is obtained that

$$\begin{aligned} \frac{dz'}{dT} &= \frac{z'^2 n\beta}{\gamma} + \frac{w^{n+1}z'n\alpha}{(n+1)\gamma} - \frac{cnwz'^2}{\gamma}, \\ \frac{dw}{dT} &= \frac{wz'\beta}{\gamma} + \frac{w^{n+2}\alpha}{(n+1)\gamma} - \frac{cw^2z'}{\gamma} + z'. \end{aligned} \tag{2.4}$$

The origin is an equilibrium point of the system (2.4). Suppose

$$p_2 = \frac{wz'\beta}{\gamma} + \frac{w^{n+2}\alpha}{(n+1)\gamma} - \frac{cw^2z'}{\gamma}, q_2 = \frac{z'^2 n\beta}{\gamma} + \frac{w^{n+1}z'n\alpha}{(n+1)\gamma} - \frac{cnwz'^2}{\gamma}.$$

Assuming that the equation

$$z + p_2 = 0 \text{ has the solution } z = \varphi(w) = c_1 w + c_2 w^2 + \cdots,$$

one has

$$c_1 w + \left(c_2 + \frac{\beta}{\gamma}c_1\right)w^2 + \left(c_3 + \frac{\beta}{\gamma}c_2 - \frac{c}{\gamma}c_1\right)w^3 + \cdots = 0.$$

One can get

$$c_1 = c_2 = c_3 = \cdots = c_{n+1} = 0, c_{n+2} = -\frac{\alpha}{(n+1)\gamma}.$$

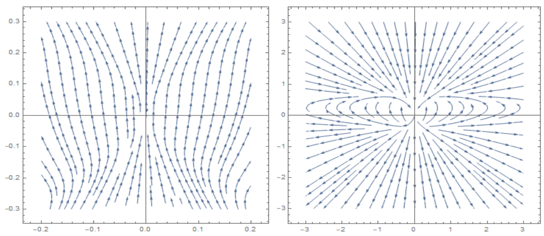
Thus, there is $z' = -\frac{\alpha}{(n+1)\gamma}w^{n+2} + \cdots$.

The equation (2.4) becomes

$$\frac{dz'}{dT} = \frac{z'^2 n\beta}{\gamma} + \frac{w^{n+1}z'n\alpha}{(n+1)\gamma} - \frac{cnwz'^2}{\gamma} = -\frac{n\alpha^2}{(n+1)^2\gamma^2}w^{2n+3} + \cdots.$$

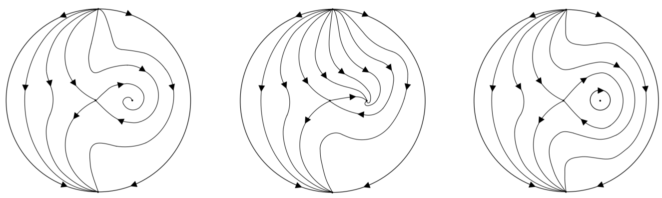
One obtains that

$$\begin{aligned} [(p_2)_w + (q_2)_{z'}] &= \frac{\beta}{\gamma}z' + \frac{(n+2)\alpha}{(n+1)\gamma}w^{n+1} - \frac{2c}{\gamma}wz' + \frac{2n\beta}{\gamma}z' + \frac{n\alpha}{(n+1)\gamma}w^{n+1} - \frac{2nc}{\gamma}wz' \\ &= \left(\frac{2\alpha}{\gamma}\right)w^{n+1} + \cdots, \end{aligned}$$



(a) (b)

FIGURE 2. For $\beta = 1, \gamma = 1, n = 4$, the phase diagrams near the singularity at infinity: (a)$c = 4$, $\alpha = 8$; (b)$c = -4$, $\alpha = -8$.



(a) $c > \frac{\beta^2}{4\gamma}$ (b) $c \leq \frac{\beta^2}{4\gamma}$ (c) $\beta = 0$

FIGURE 3. The global structure diagram of system (2.2) when n is an odd number.

and

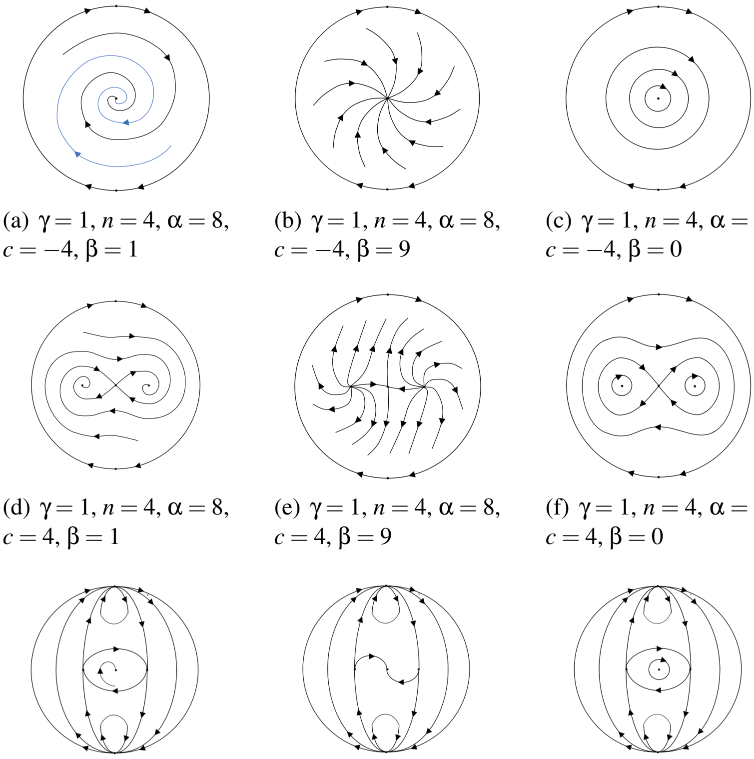
$$\lambda = b_{n+1}^2 + 4(m+1)a_{2n+3} = \frac{4\alpha^2}{\gamma^2}\left[1 - \frac{n(n+2)}{(n+1)^2}\right] > 0.$$

Therefore, the singularity is an unstable node when n is an odd number, as shown in Fig. 2. For n is an even number, the sufficient small domain of the singularity consists of a hyperbolic sector and an elliptical sector. If α and γ have the same sign, the hyperbolic sector is above the elliptical sector. Otherwise, the elliptical sector is above the hyperbolic sector. Due to n being an even number, the image below the V-axis is symmetrical from the image above the V-axis.

Combining the conclusions of Section 2.2 and Theorem 1, the global phase diagram corresponding to each situation is shown in Fig. 3 and Fig. 4.

2.3. *Amplitude estimation*

In order to understand the dynamic behavior of the generalized KdV-Burgers equation, one further analyzes the amplitude of the oscillating shock wave.



(a) $\gamma = 1, n = 4, \alpha = 8, c = -4, \beta = 1$

(b) $\gamma = 1, n = 4, \alpha = 8, c = -4, \beta = 9$

(c) $\gamma = 1, n = 4, \alpha = 8, c = -4, \beta = 0$

(d) $\gamma = 1, n = 4, \alpha = 8, c = 4, \beta = 1$

(e) $\gamma = 1, n = 4, \alpha = 8, c = 4, \beta = 9$

(f) $\gamma = 1, n = 4, \alpha = 8, c = 4, \beta = 0$

(g) $\gamma = 1, n = 4, \alpha = -8, c = -4, \beta = 1$

(h) $\gamma = 1, n = 4, \alpha = -8, c = -4, \beta = 9$

(i) $\gamma = 1, n = 4, \alpha = -8, c = -4, \beta = 0$

FIGURE 4. The global structure diagram of system (2.2) when n is an even number.

For convenience, one fixes $\gamma > 0, \beta > 0, c > \frac{\beta^2}{4n\gamma}$ and transfers the focus to the origin. Let $v(s) = u(\zeta) - \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}$, we have

$$\gamma v_{2s} + \beta v_s + \frac{\alpha}{n+1}\left(v + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)^{n+1} - c\left(v + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right) = 0. \tag{2.5}$$

Let expand $\left(v + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)^{n+1}$,

$$\left(v + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)^{n+1} = v^{n+1} + \cdots + C_{n+1}^{n} v \frac{(n+1)c}{\alpha} + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{n+1}{n}}. \tag{2.6}$$

Substitute equation (2.6) into equation (2.5),

$$\gamma v_{2s}+\beta v_s+\frac{\alpha}{n+1}f(v)+ncv=0, \tag{2.7}$$

where

$$f(v)=v^{n+1}+\cdots+C_{n+1}^{n-1}v^2\frac{(n+1)c}{\alpha}^{\frac{n+1}{n-1}}+\left(\frac{(n+1)c}{\alpha}\right)^{\frac{n+1}{n}}<\left(v+\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)^{n+1}.$$

Then $v(s)$ has a series of peaks and valleys. So one can set the first and largest wave peak u_0 and obtain the initial value condition

$$\begin{cases} v(0) & =u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}, \\ v_s(0) & =0. \end{cases} \tag{2.8}$$

Therefore, equation (1.1) becomes an initial value problem

$$\begin{cases} \gamma v_{2s}+\beta v_s+ncv & =-\frac{\alpha}{n+1}f(v), \\ v(0) & =u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}, v_s(0)=0. \end{cases} \tag{2.9}$$

On the one hand, the problem

$$\begin{cases} \gamma w^{oo}(s)_{2s}+\beta w^{oo}(s)_s+ncw & =0, \\ w^{oo}(s)(0) & =u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}, w_s^{oo}(0)=0, \end{cases} \tag{2.10}$$

has a solution $w^o(s)$. Because of the parameter condition $c>\frac{\beta^2}{4n\gamma}$, the solution is

$$w^{oo}(s)=e^{-\frac{\beta}{2\gamma}s}\left[(u_0-(\frac{(n+1)c}{\alpha})^{\frac{1}{n}})\cos\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s\right.$$
$$\left.+\left(\frac{\beta(u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}})}{\sqrt{4nc\gamma-\beta^2}}\right)\sin\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s\right].$$

On the other hand, the nonlinear initial value problem

$$\begin{cases} \gamma w_{2s}^{**}+\beta w_s^{**}+cw^{**} & =0, \\ w^{**}(0)=0, w_s^{**}(0) & =-\frac{\alpha}{n+1}f(v), \end{cases} \tag{2.11}$$

has the solution

$$w^{**}(s,\iota)=-e^{-\frac{\beta}{2\gamma}(s-\iota)}\frac{2\alpha\gamma}{(n+1)\sqrt{4nc\gamma-\beta^2}}f(v(\iota))\sin\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}(s-\iota).$$

The solution of the initial value problem (2.9) is

$$w(s)=w^{oo}(s)+\int_0^s w^{**}(s,\iota)d\iota,$$

To obtain the upper bound of $w(s)$, one discusses the upper bound of $w^{oo}(s)$ and $\int_0^s w^{**}(s,\iota)d\iota$.

For $w^{oo}(s)$, one takes the derivative of the part within the parentheses,

$$-\left(u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}\sin\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s$$
$$+\beta\frac{u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}}{2\gamma}\cos\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s=0.$$

After simplification, it can be obtained that $\tan\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s=\frac{\beta}{\sqrt{4nc\gamma-\beta^2}}$.

From this, it can be obtained that

$$\sin\frac{\sqrt{4c\gamma-\beta^2}}{2\gamma}s=\frac{\beta}{\sqrt{4nc\gamma}}\text{ and }\cos\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s=\frac{\sqrt{4nc\gamma-\beta^2}}{\sqrt{4nc\gamma}}.$$

So the maximum value of $w^{oo}(s)$ is

$$|w^{oo}(s)|\le e^{-\frac{\beta}{2\gamma}s}\left[\left(u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)\frac{\sqrt{4nc\gamma-\beta^2}}{\sqrt{4nc\gamma}}+\frac{u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}}{\sqrt{4nc\gamma-\beta^2}}\frac{\beta^2}{\sqrt{4nc\gamma}}\right].\tag{2.12}$$

Let $f(v)<(v+(\frac{(n+1)c}{\alpha})^{\frac{1}{n}})^{n+1}<u_0^{n+1}$, the upper bound of $\int_0^s w^{**}(s,\iota)d\iota$ is

$$\int_0^s w^{**}(s,\iota)d\iota\le u_0^n\frac{2\alpha\gamma}{(n+1)\sqrt{4nc\gamma-\beta^2}}e^{-\frac{\beta}{2\gamma}s}\int_0^s|v(\iota)|e^{\frac{\beta}{2\gamma}\iota}d\iota.\tag{2.13}$$

According to the existence and uniqueness of the solution, one has

$$|w(s)|\equiv|v(s)|\le|w^{oo}(s)|+\int_0^s w^{**}(s,\iota)d\iota.\tag{2.14}$$

According to Gronwall inequality and returning to the original situation, one can get

$$|u(s)|\le\exp\left\{u_0^n\frac{4\alpha\gamma^2}{(n+1)\beta\sqrt{4nc\gamma-\beta^2}}\left[1-\exp\left(-\frac{\beta}{2\gamma}s\right)\right]-\frac{\beta}{2\gamma}s\right\}$$
$$\times\left[\frac{\left(u_0-(\frac{(n+1)c}{\alpha})^{\frac{1}{n}}\right)\sqrt{4nc\gamma}}{\sqrt{4nc\gamma-\beta^2}}\right]+\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}.$$

The amplitude of the generalized KdV-Burgers equation is

$$|f(s)| \leq \exp\left\{u_0^n \frac{4\alpha\gamma^2}{(n+1)\beta\sqrt{4nc\gamma-\beta^2}}\left[1-\exp\left(-\frac{\beta}{2\gamma}s\right)\right]-\frac{\beta}{2\gamma}s\right\}$$
$$\times\left[\frac{(u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}})\sqrt{4nc\gamma}}{\sqrt{4nc\gamma-\beta^2}}\right].$$

It can be seen that the oscillating shock wave is exponentially stable, and the spiral will converge to the equilibrium point at this speed.

Remark 1. When n becomes a variable, the calculation process is different from the case when $n=1$. This paper makes changes in the selection of the maximum wave amplitude and calculates the amplitude estimation of the KdV-Burgers equation with n as a variable.

3. JACOBI ANALYSIS

In this section, Jacobi stability of the traveling wave solutions at any point of the trajectory is analysed on the basis of KCC theory. In addition, the deviation vector is calculated.

3.1. *KCC theory and Jacobi stability*

In the subsection, one introduces the basic concepts of the KCC theory, including nonlinear connection, Berwald connection, deviation curvature tensor and the definition of Jacobi stability, et.al. [9][21]. The Einstein summation convention is used throughout.

Let $(x^i)=(x^1,x^2,\ldots,x^n)\in\mathbf{R}^n$ and $(y^i)=(y^1,y^2,\ldots,y^n)\in\mathbf{R}^n$. The coordinates y^i are defined by $y^i=dx^i/d\zeta$. One considers second-order differential equations of the form

$$\frac{d^2x^i}{d\zeta^2}+2G^i(x,y,\zeta)=0, i=1,2,\cdots,n, \tag{3.1}$$

where $(x^i,y^i,\zeta)\in\Omega\subset\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}$, Ω is an open connected set, and each function $G^i(x,y,\zeta)$ is C^∞ in a neighborhood at any point.

The coefficient of the nonlinear connection N^i_j is defined by $N^i_j=\partial G^i/\partial y^j$. The value of N are closely related to the linear stability of the equilibrium point [10].

One considers the disturbed trajectories $x^i(\zeta)$ of the system (3.1) as

$$\tilde{x}^i(\zeta)=x^i(\zeta)+\eta\xi^i(\zeta), \tag{3.2}$$

where $|\eta|$ is a small parameter, and $\xi^i(\zeta)$ are the components of a contravariant vector field defined along the path $x^i(\zeta)$. Substituting Eq. (3.2) into Eq. (3.1) and taking the limit $\eta \to 0$, one obtains the equation in the form

$$\frac{d^2\xi^i}{d\zeta^2} + 2N^i_j \frac{d\xi^j}{d\zeta} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0. \tag{3.3}$$

Converting to the KCC covariant differential, Eq. (3.3) becomes

$$\frac{D^2\xi^i}{d\zeta^2} = P^i_j \xi^j, \tag{3.4}$$

where

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j + \frac{\partial N^i_j}{\partial \zeta}, \tag{3.5}$$

and $G^i_{jl} = \partial N^i_j / y^l$ is Berwald connection coefficients. The tensor P^i_j is called the second KCC invariant or the deviation curvature tensor. The trajectories of (3.1) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation curvature tensor P^i_j are strictly negative everywhere. Otherwise, the trajectories are Jacobi unstable.

According to the formula (3.5), one can get KCC differential invariants:

$$N = \frac{\beta}{2\gamma},\ G = 0,\ P = -\frac{\alpha}{\gamma}u^n + \frac{\beta^2 + 4c\gamma}{4\gamma^2}.$$

Therefore, the following propositions can be proved.

Proposition 1.

(1) *If β and γ have the same sign, the equilibrium points are linear stable.*
(2) *If β and γ have different signs, the equilibrium points are linear unstable.*

From equation (3.5), the Jacobi stability of any point on the system (2.2) orbit can be obtained. When n is an odd, we can obtain Proposition 2.

Proposition 2.

(1) *If $\frac{\alpha}{\gamma} > 0$ and $u < \left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}}$, the orbit of the system (2.2) is* Jacobi *unstable. Otherwise, the orbit is* Jacobi *stable.*
(2) *If $\frac{\alpha}{\gamma} < 0$ and $u < \left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}}$, the orbit of the system (2.2) is* Jacobi *unstable. Otherwise, the orbit is* Jacobi *stable.*

When n is an even, we can obtain Proposition 3.

Proposition 3.

(1) *If $\frac{\alpha}{\gamma} > 0$, $\beta^2 + 4c\gamma > 0$ and $-\left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}} < u < \left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}}$, the orbits of the system (2.2) are* Jacobi *unstable. Otherwise, the orbits are* Jacobi *stable.*
(2) *If $\frac{\alpha}{\gamma} > 0$, $\beta^2 + 4c\gamma < 0$, the orbits of the system (2.2) are* Jacobi *stable.*
(3) *If $\frac{\alpha}{\gamma} < 0$, $\beta^2 + 4c\gamma > 0$, the orbits of the system (2.2) are* Jacobi *unstable.*

(4) *If* $\frac{\alpha}{\gamma} < 0$, $\beta^2 + 4c\gamma < 0$, *and* $-\left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}} < u < \left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}}$, *the orbits of the system* (2.2) *are* Jacobi *stable. Otherwise, the orbits are* Jacobi *unstable.*

Remark 2. When α and c have different signs, system (2.2) does not have bounded traveling wave solutions.

One discusses the deviation curvature tensor at the equilibrium points.

Proposition 4. *At the point* E_0*, it is obtained that* $P = \frac{\beta^2+4c\gamma}{4\gamma^2}$.

(1) *If* $\gamma > 0$ *and* $c \geq -\frac{\beta^2}{4\gamma}$*, the equilibrium point* E_0 *is Jacobi unstable.*
(2) *If* $\gamma > 0$ *and* $c < -\frac{\beta^2}{4\gamma}$*, the equilibrium point* E_0 *is Jacobi stable.*
(3) *If* $\gamma < 0$ *and* $c \geq -\frac{\beta^2}{4\gamma}$*, the equilibrium point* E_0 *is Jacobi stable.*
(4) *If* $\gamma < 0$ *and* $c < -\frac{\beta^2}{4\gamma}$*, the equilibrium point* E_0 *is Jacobi unstable.*

Similarly, one can analyze the Jacobi stability of equilibrium point $E_{1,2}$.

Proposition 5. *At the point* $E_{1,2}$*, it is obtained that* $P = \frac{\beta^2-4nc\gamma}{4\gamma^2}$.

(1) *If* $\gamma > 0$ *and* $c > \frac{\beta^2}{4n\gamma}$*, the equilibrium point* $E_{1,2}$ *is Jacobi stable.*
(2) *If* $\gamma > 0$ *and* $c \leq \frac{\beta^2}{4n\gamma}$*, the equilibrium point* $E_{1,2}$ *is Jacobi unstable.*
(3) *If* $\gamma < 0$ *and* $c > \frac{\beta^2}{4n\gamma}$*, the equilibrium point* $E_{1,2}$ *is Jacobi unstable.*
(4) *If* $\gamma < 0$ *and* $c > \frac{\beta^2}{4n\gamma}$*, the equilibrium point* $E_{1,2}$ *is Jacobi stable.*

When n is an odd, the boundary line $P = 0$ will go through the following stages. In the first stage $\beta \leq -2\sqrt{cn\gamma}$, the boundary line appears to the right side of the two equilibrium points. As β gradually increases to $2\sqrt{cn\gamma} > \beta > -2\sqrt{cn\gamma}$, the boundary moves between two equilibrium points. Specially, when β increases to 0, the linear stability of equilibrium points changes. As β continues to increase to $\beta \geq 2\sqrt{cn\gamma}$, the boundary line will move back to the right of the equilibrium points.

When n is even, in the first stage $\beta \leq -2\sqrt{cn\gamma}$, the boundary line appears outside the two equilibrium points. As β gradually increases to $2\sqrt{cn\gamma} > \beta > -2\sqrt{cn\gamma}$, the boundary moves between the equilibrium points $E_{1,2}$ and E_0. Specially, when β increases to 0, the linear stability of equilibrium points changes. As β continues to increase to $\beta \geq 2\sqrt{cn\gamma}$, the boundary line will move back to the outside of the equilibrium points. This conclusion can correspond the type of equilibrium points mentioned earlier.

3.2. *Dynamic behavior of deviation vectors*

In this subsection, the dynamic behavior of deviation vectors is analyzed. According to the equation (3.3), the dynamical equation of the deviation vector is

$$\frac{d^2\xi}{d\zeta^2}+\frac{\beta}{\gamma}\frac{d\xi}{d\zeta}+\left(\frac{\alpha}{\gamma}u^n-\frac{c}{\gamma}\right)\xi=0. \tag{3.6}$$

For convenience, one makes $\gamma>0, \beta>0$. About other situations, conclusions can be obtained similarly.

3.2.1. *Dynamic behavior of deviation vectors at E_0*

For equilibrium point $E_0(0,0)$, according to expression (3.6), it can be obtained that

$$\frac{d^2\xi_0}{d\zeta^2}+\frac{\beta}{\gamma}\frac{d\xi_0}{d\zeta}-\frac{c}{\gamma}\xi_0=0, \tag{3.7}$$

According to the initial conditions $\xi_0(0)=0, \dot{\xi}_0(0)=\xi_{01}$, The solution of the equation (3.2) is

$$\xi_0=\begin{cases}\exp\left(\frac{-\beta+\sqrt{\beta^2+4c\gamma}}{2\gamma}\zeta\right)-\exp\left(\frac{-\beta-\sqrt{\beta^2+4c\gamma}}{2\gamma}\zeta\right), & c>-\frac{\beta^2}{4\gamma},\\ \zeta\exp\left(-\frac{\beta}{2\gamma}\zeta\right), & c=-\frac{\beta^2}{4\gamma},\\ \exp\left(-\frac{\beta}{2\gamma}\zeta\right)\sin(-\beta^2-4c\gamma)\zeta, & c<-\frac{\beta^2}{4\gamma}.\end{cases} \tag{3.8}$$

3.2.2. *Dynamic behavior of deviation vectors at $E_{1,2}$*

For equilibrium point $E_{1,2}(\frac{2c}{\alpha},0)$, it can be obtained that

$$\frac{d^2\xi_1}{d\zeta^2}+\frac{\beta}{\gamma}\frac{d\xi_1}{d\zeta}+\frac{nc}{\gamma}\xi_1=0. \tag{3.9}$$

Similarly, one utilizes initial conditions $\xi_1(0)=0, \dot{\xi}_1(0)=\xi_{11}$. The solution of the equation (3.4) is

$$\xi_1=\begin{cases}\exp\left(\frac{-\beta+\sqrt{\beta^2-4nc\gamma}}{2\gamma}\zeta\right)-\exp\left(\frac{-\beta-\sqrt{\beta^2-4nc\gamma}}{2\gamma}\zeta\right), & c>\frac{\beta^2}{4\gamma},\\ \zeta\exp\left(-\frac{\beta}{2\gamma}\zeta\right), & c=\frac{\beta^2}{4\gamma},\\ \exp\left(-\frac{\beta}{2\gamma}\zeta\right)\sin(-\beta^2+4nc\gamma)\zeta & c<\frac{\beta^2}{4\gamma}.\end{cases} \tag{3.10}$$

4. Melnikov Analysis of Chaotic Behaviors

In this section, one begins by the equation (2.1) when the dissipation term is periodically disturbed.

$$\gamma u_{2\zeta}+\beta(1+\delta\cos\omega\zeta)u_\zeta+\frac{\alpha}{n+1}u^{n+1}-cu=0. \tag{4.1}$$

one transforms the equation (4.1) into an equivalent planar system.

$$\begin{aligned}\frac{du}{d\zeta} &= v\\ \frac{dv}{d\zeta} &= -\frac{\beta}{\gamma}(1+\delta\cos\omega\zeta)v - \frac{\alpha}{(n+1)\gamma}u^{n+1} + \frac{c}{\gamma}u\end{aligned} \tag{4.2}$$

When β is small enough, one can define $\varepsilon = -\frac{\beta}{\gamma}$ as disturbance amplitude. So the system (3.6) can be transformed into the following form.

$$\begin{aligned}\frac{du}{d\zeta} &= v,\\ \frac{dv}{d\zeta} &= -\frac{\alpha}{(n+1)\gamma}u^{n+1} + \frac{c}{\gamma}u + \varepsilon(\delta\cos\omega\zeta v - v).\end{aligned} \tag{4.3}$$

For equation (4.3) without disturbance, one obtains Hamiltonian function

$$H = \frac{v^2}{2} - \frac{c}{2\gamma}u^2 + \frac{\alpha}{(n+1)(n+2)\gamma}u^{n+2}.$$

Let $H=0$, the explicit expression for the homoclinic orbit can be obtained as

$$q_0(\zeta) = \left(\left(\frac{(n+1)(n+2)c}{2\alpha}\right)^{\frac{1}{n}} \operatorname{sech}^{\frac{2}{n}}\zeta, -\frac{2}{n}\left(\frac{(n+1)(n+2)c}{2\alpha}\right)^{\frac{1}{n}} \operatorname{sech}^{\frac{2}{n}}\zeta\tanh\zeta\right).$$

Therefore, Melnikov function for the homoclinic orbit $q_0(\zeta)$ becomes

$$M = \int_{-\infty}^{\infty} v(q_0)[\delta\cos\omega\zeta v(q_0) - v(q_0)]d\zeta$$

where

$$\begin{aligned}v(q_0) &= -\frac{2}{n}\left(\frac{(n+1)(n+2)c}{2\alpha}\right)^{\frac{1}{n}}\operatorname{sech}^{\frac{2}{n}}\zeta\tanh\zeta.\\ M &= \frac{4}{n^2}\left(\frac{(n+1)(n+2)c}{2\alpha}\right)^{\frac{2}{n}}\left[\int_{-\infty}^{\infty}\delta\operatorname{sech}^{\frac{4}{n}}\zeta\tanh^2\zeta\cos\omega(\zeta+\zeta_0)d\zeta\right.\\ &\qquad \left.-\int_{-\infty}^{\infty}\operatorname{sech}^{\frac{4}{n}}\zeta\tanh^2\zeta d\zeta\right].\end{aligned}$$

It is clear that the homoclinic Melnikov function M has simple zeros for $\cos\omega(\zeta+\zeta_0) = \frac{1}{\delta}$. This means the equation (4.1) has chaos in the sense of Smale horseshoe.

5. Conclusion

This paper analyzes the cases where n is an odd and an even respectively and devotes to the dynamic behavior of traveling wave solutions of the generalized KdV-Burgers equation. The global structure diagrams of the system (2.2) under different parameters are sketched. An amplitude estimation of the generalized KdV-Burgers equation (1.1) is obtained. The KdV-Burgers equation is a special case when $n = 1$. Therefore, the obtained results are more accurate and universal. The conditions of Jacobi stability of the system (2.1) at any point on the trajectory are obtained. When $\gamma > 0$ and $c < -\frac{\beta^2}{4\gamma}$, or $\gamma < 0$ and $c \geq -\frac{\beta^2}{4\gamma}$, the equilibrium point E_0 is Jacobi stable, and for $\gamma > 0$ and $c > \frac{\beta^2}{4n\gamma}$, or $\gamma < 0$ and $c \leq \frac{\beta^2}{4n\gamma}$, the equilibrium points $E_{1,2}$ are Jacobi stable. For the equation (2.1) with the dissipative term receiving periodic perturbations, it is discovered that the system is chaotic in the sense of Smale horseshoe when $|\delta| > 1$.

References

[1] D. Benney, "Long waves on liquid films." *Journal of Mathematics and Physics.*, vol. 45, no. 1-4, pp. 150–155, 1966, doi: 10.1002/sapm1966451150.
[2] É. Cartan, "Observations sur le mémoire précédent." *Mathematische Zeitschrift.*, vol. 37, no. 1, pp. 619–622, 1933.
[3] B. Chen, Y. Liu, Z. Wei, and C. Feng, "New insights into a chaotic system with only a Lyapunov stable equilibrium." *Mathematical Methods in the Applied Sciences.*, vol. 43, no. 15, pp. 9262–9279, 2020, doi: 10.1002/mma.6619.
[4] Z.-S. Chen and S. H. Rhee, "Effect of traveling wave on the vortex-induced vibration of a long flexible pipe." *Applied Ocean Research.*, vol. 84, pp. 122–132, 2019, doi: 10.1016/j.apor.2018.12.011.
[5] S. S. Chern, "Sur la géométrie d'un système d'équations différentielles du second ordre." *Bulletin des Sciences Mathematiques.*, vol. 63, pp. 206–212, 1939.
[6] F. Cooper, H. Shepard, and P. Sodano, "Solitary waves in a class of generalized Korteweg–de Vries equations." *Physical Review E.*, vol. 48, no. 5, p. 4027, 1993, doi: 10.1103/PhysRevE.48.4027.
[7] C. Feng, Q. Huang, and Y. Liu, "Jacobi analysis for an unusual 3D autonomous system." *International Journal of Geometric Methods in Modern Physics.*, vol. 17, no. 04, p. 2050062, 2020, doi: 10.1142/S0219887820500620.
[8] Z. Fu, S. Liu, and S. Liu, "New kinds of solutions to gardner equation." *Chaos, Solitons & Fractals.*, vol. 20, no. 2, pp. 301–309, 2004, doi: 10.1016/S0960-0779(03)00383-7.
[9] T. Harko, C. Y. Ho, C. S. Leung, and S. Yip, "Jacobi stability analysis of the Lorenz system." *International Journal of Geometric Methods in Modern Physics.*, vol. 12, no. 07, p. 1550081, 2015, doi: 10.1142/S0219887815500814.
[10] T. Harko, P. Pantaragphong, and S. V. Sabau, "Kosambi–Cartan–Chern (KCC) theory for higher-order dynamical systems." *International Journal of Geometric Methods in Modern Physics.*, vol. 13, no. 02, p. 1650014, 2016, doi: 10.1142/S0219887816500146.
[11] M. Hassan, "Exact solitary wave solutions for a generalized Kdv–Burgers equation." *Chaos, Solitons & Fractals.*, vol. 19, no. 5, pp. 1201–1206, 2004, doi: 10.1016/S0960-0779(03)00309-6.
[12] Q. Huang, A. Liu, and Y. Liu, "Jacobi stability analysis of the Chen system." *International Journal of Bifurcation and Chaos.*, vol. 29, no. 10, p. 1950139, 2019, doi: 10.1142/S0218127419501396.
[13] D. D. Kosambi, "Parallelism and path-spaces." *Mathematische Zeitschrift.*, vol. 37, no. 1, pp. 608–618, 1933.

[14] A. Liu, B. Chen, and Y. Wei, "Jacobi analysis of a segmented disc dynamo system." *International Journal of Geometric Methods in Modern Physics.*, vol. 17, no. 14, p. 2050205, 2020, doi: 10.1142/S0219887820502059.

[15] Y. Liu, B. Chen, and X. Huang, "Qualitative geometric analysis of traveling wave solutions of the modified equal width Burgers equation." *Mathematical Methods in the Applied Sciences.*, vol. 45, no. 16, pp. 9560–9577, 2022, doi: 10.1002/mma.8323.

[16] Y. Liu and Q. Huang, "Jacobi analysis of the Rabinovich system." *Acta Mathematica Scientia.*, vol. 41, no. 3, pp. 783–796, 2021, doi: 10.3969/j.issn.1003-3998.2021.03.016.

[17] Y. Liu, Q. Huang, and Z. Wei, "Dynamics at infinity and Jacobi stability of trajectories for the Yang-Chen system." *Discrete and Continuous Dynamical Systems-B.*, vol. 26, no. 6, pp. 3357–3380, 2020, doi: 10.3934/dcdsb.2020235.

[18] D. Lu, B. Hong, and L. Tian, "New solitary wave and periodic wave solutions for general types of KdV and KdV–Burgers equations." *Communications in Nonlinear Science & Numerical Simulation.*, vol. 14, no. 1, pp. 77–84, 2009, doi: 10.1016/j.cnsns.2007.08.007.

[19] X. Lu, Y. Liu, A. Liu, and C. Feng, "New geometric viewpoints to Chen chaotic system." *Miskolc Mathematical Notes.*, vol. 23, no. 1, pp. 339–362, 2022, doi: 10.18514/MMN.2022.3787.

[20] J. Rinzel and J. B. Keller, "Traveling wave solutions of a nerve conduction equation." *Biophysical Journal.*, vol. 13, no. 12, pp. 1313–1337, 1973, doi: 10.1016/s0006-3495(73)86065-5.

[21] C. Udrişte and I. R. Nicola, "Jacobi stability of linearized geometric dynamics." *Journal of Dynamical Systems and Geometric Theories.*, vol. 7, no. 2, pp. 161–173, 2009, doi: 10.1080/1726037X.2009.10698571.

[22] F. Wang, T. Liu, K. N. V, and Z. Wei, "Jacobi stability analysis and the onset of chaos in a two-degree-of-freedom mechanical system." *International Journal of Bifurcation and Chaos.*, vol. 31, no. 05, p. 2150075, 2021, doi: 10.1142/S0218127421500759.

[23] Z. Wei, F. Wang, H. Li, and W. Zhang, "Jacobi stability analysis and impulsive control of a 5D self-exciting homopolar disc dynamo." *Discrete and Continuous Dynamical Systems-B.*, vol. 27, no. 9, pp. 5029–5045, 2022, doi: 10.3934/dcdsb.2021263.

[24] Z. Zhu, "Stability of solitary wave solution of Kdv type equation and travelling wave solution of Kdv–Burgers type equation." *Acta Physica Sinica.*, vol. 45, no. 7, p. 4, 1996, doi: 10.1088/0256-307X/13/3/018.

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