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STRONG ALMOST CONVERGENCE WITH RESPECT TO AN ORLICZ FUNCTION

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Abstract. In this paper, using an Orlicz function we extend the concept of strong almost convergence and show that strong almost convergence, uniform statistical convergence and strong almost convergence with respect to an Orlicz function are all equivalent on bounded sequences. The main tool in proving the result is to consider ideals in bounded sequences.

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1. Introduction

Convergence methods such as statistical and uniform statistical convergences, strong almost convergence are of some interest in mathematical analysis (see, e.g., [1, 5–7, 16–18, 20, 21]).

In the present paper, motivated by those of Demirci [7] and Şahin Bayram [6] we introduce the concept of strong almost convergence with respect to an Orlicz function and show that uniform statistical convergence, strong almost convergence and strong almost convergence with respect to an Orlicz function are all equivalent on bounded sequences.

We first collect some basic concept and notation.

Let $E \subseteq \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. By $E(m,n)$ we denote the cardinality of the set of natural numbers i so that $m \le i \le n$, where $m,n \in \mathbb{N}$. We now consider the numbers

$$\underline{\mathrm{d}}(E) = \liminf_n \frac{E(1,n)}{n}, \quad \bar{\mathrm{d}}(E) = \limsup_n \frac{E(1,n)}{n}.$$

$\underline{\mathrm{d}}(E)$ and $\bar{\mathrm{d}}(E)$ are respectively called the lower and upper densities of the set E. If $\underline{\mathrm{d}}(E) = \bar{\mathrm{d}}(E)$ then the common value $\mathrm{d}(E)$ is called the asymptotic density of E.

The lower and upper uniform densities of $E \subseteq \mathbb{N}$ have been respectively introduced in ([2],[3]) as follows;

$$\underline{\mathrm{u}}(E) = \lim_n \min_{i \geqslant 0} \frac{E(i+1, i+n)}{n}, \quad \bar{\mathrm{u}}(E) = \lim_n \max_{i \geqslant 0} \frac{E(i+1, i+n)}{n}.$$

If $\mathrm{u}(E) := \underline{\mathrm{u}}(E) = \bar{\mathrm{u}}(E)$ then $\mathrm{u}(E)$ is called the uniform density of E. It is known [1] that

$$\underline{\mathrm{u}}(E) \leq \underline{\mathrm{d}}(E) \leq \bar{\mathrm{d}}(E) \leq \bar{\mathrm{u}}(E).$$

Let $x := (x_n)$ be a sequence of real numbers and let

$$K_\varepsilon = \{n \in \mathbb{N} : \ |x_n - L| \geq \varepsilon\}, \quad (\varepsilon > 0).$$

If, for every $\varepsilon > 0$, $\mathrm{d}(K_\varepsilon) = 0$ then x is said to be statistically convergent to L (see, e.g., [8],[11],[22]); and $\mathrm{u}(K_\varepsilon) = 0$ then x is said to be uniformly statistically convergent to L (see, e.g., [1],[21]).

By S and S_u we respectively denote the spaces of all statistically convergent and uniformly statistically convergent sequences. A connection between strong convergence, statistical convergence may be found in [4] (see also [13],[14]) and strong convergence with respect to a modulus function is also made in [5].

There is also a close connection between statistical convergence and almost convergence [19],[23]. Recall that a sequence $x = (x_k)$ is almost convergent to L if and only if

$$\lim_n \frac{1}{n} \sum_{k=i+1}^{i+n} x_k = L, \quad (uniformly\ in\ i)$$

(see [16]). If

$$\lim_n \frac{1}{n} \sum_{k=i+1}^{i+n} |x_k - L| = 0$$

then x is said to be strongly almost convergent to L (see, e.g., [1],[10],[17]).

By c, f, $[f]$, $[f]_0$, l^∞ we respectively denote the spaces of convergent, almost convergent, strongly almost convergent, strongly almost convergent to 0 and bounded sequences then we have the inclusions [17] that

$$c \subset [f] \subset f \subset l^\infty.$$

Combining this inclusion result with Theorem 2 of [1] we have the following

$$[f] = S_u \cap l^\infty. \tag{1.1}$$

Some inclusion relations between strong almost convergence and lacunary statistical convergence is considered in [12]. Later on Pehlivan [21] gave inclusion results between uniform statistical convergence and strong almost convergence with respect to a modulus function.

Motivated by those of Demirci [7] and Şahin Bayram [6] we establish relationships between uniform statistical convergence and strong almost convergence with respect to an Orlicz function.

We collect some basic concepts and notation.

Let $F:[0,\infty)\to[0,\infty)$ be a continuous, nondecreasing and convex function with $F(0)=0$, $F(x)>0$ for $x>0$ and $F(x)\to\infty$ as $x\to\infty$.

Such a function is called Orlicz function [15]. If the convexity condition of F is replaced by $F(x+y)\le F(x)+F(y)$ then F is called modulus function (see, e.g., [18]).

Throughout the paper let e denote the sequence which is identically 1 and let

$$w:=\{all\ real\ valued\ sequences\}.$$

Given an Orlicz function F we introduce the following sequence spaces;

$$[f,F]_0=\left\{x\in w:\lim_n\frac{1}{n}\sum_{k=i+1}^{i+n}F(|x_k|)=0,\ uniformly\ in\ i\right\},$$
$$[f,F]=\{x\in w:x-Le\in[f,F]_0\ for\ some\ L\}.$$

Note that when $F(x)=x$ then we get that

$$[f,F]_0=[f]_0\ and\ [f,F]=[f].$$

If $x\in[f,F]$ we say that x is strongly almost convergent to L with respect to an Orlicz function F.

Recall that $x=(x_k)$ is uniformly statistically convergent to L if $\chi_{K(x-Le;\varepsilon)}$ is contained in $[f]_0$ for every $\varepsilon>0$ where $\chi_{K(x;\varepsilon)}$ is the characteristic function of the set

$$K(x;\varepsilon)=\{k\in\mathbb{N}:|x_k|\geqslant\varepsilon\}.$$

Let F be an Orlicz function. If there is a constant $H>0$ such that

$$F(2u)\le HF(u),\ (for\ all\ u>0)$$

we say that the Orlicz function F satisfies Δ_2-condition. This condition is equivalent to the fact that

$$F(tu)\le HtF(u), \tag{1.2}$$

for all $u\geqslant0$ and for $t\geqslant1$ (see, e.g., [15]).

2. Main Results

In this section we are concerned with some inclusion results among the spaces $[f]$, $[f,F]$ and S_u; and show that strong almost convergence, uniform statistical convergence and strong almost convergence with respect to a modulus function F are equivalent on bounded sequences provided that F satisfies Δ_2-condition.

In order to prove our first result we use the idea given by Parashar and Choudhary [20].

Proposition 1. *Let F be an Orlicz function satisfying Δ_2-condition. Then we have the inclusions*

$$[f]_0 \subset [f,F]_0 \text{ and } [f] \subset [f,F].$$

Proof. It is enough to prove that $[f]_0 \subset [f,F]_0$. Let $x=(x_k) \in [f]_0$ and F be an Orlicz function satisfying Δ_2-condition. Since F is right continuous at zero, given $\varepsilon > 0$ there exists δ with $0<\delta<1$ such that $F(t)<\varepsilon$ whenever $0 \le t < \delta$.

Hence we get

$$\frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k|) = \frac{1}{n}\sum_{\substack{k=i+1\\ |x_k|<\delta}}^{i+n} F(|x_k|) + \frac{1}{n}\sum_{\substack{k=i+1\\ |x_k|\geqslant\delta}}^{i+n} F(|x_k|) < \frac{1}{n}\varepsilon n + \frac{1}{n}\sum_{\substack{k=i+1\\ |x_k|\geqslant\delta}}^{i+n} F(|x_k|). \quad (2.1)$$

Since $0<\delta<1$, we get for every $k \in \mathbb{N}$, that $|x_k| < \frac{1}{\delta}|x_k| < 1 + \frac{|x_k|}{\delta}$.

On the other hand F is an Orlicz function satisfying Δ_2-condition, so we have by (1.2) that

$$F(|x_k|) < F\left(1+\frac{|x_k|}{\delta}\right) = F\left(\frac{1}{2}2+\frac{1}{2}\frac{2|x_k|}{\delta}\right) \le \frac{1}{2}F(2) + \frac{H|x_k|}{2\delta}F(2)$$
$$< (1+H)F(2)\frac{|x_k|}{\delta}. \quad (2.2)$$

Combining (2.1) and (2.2) we get

$$\frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k|) < \varepsilon + \frac{(1+H)}{\delta}F(2)\frac{1}{n}\sum_{k=i+1}^{i+n}|x_k|.$$

Since $x=(x_k) \in [f]_0$ we conclude that $x \in [f,F]_0$. This proves the result. □

The next result is an analog of Lemma 1 of Demirci [7].

Lemma 1. *Let F be an Orlicz function satisfying Δ_2-condition. Then $[f,F]_0$ is an ideal in l^∞.*

Proof. Let $x \in [f,F]_0$ and $y \in l^\infty$. We prove that $xy \in [f,F]_0 \cap l^\infty$. Since $y \in l^\infty$, there is $H_1 > 1$ so that $\|y\| \le H_1$. Since F is nondecreasing and satisfies Δ_2-condition we have

$$F(|x_k y_k|) \le F(H_1|x_k|) \le H(1+H_1)F(|x_k|),\ (H>0).$$

Hence, one can get

$$\frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k y_k|) \le H(1+H_1)\frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k|)$$

from which we conclude that

$$\lim_n \frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k y_k|) = 0, \text{ uniformly in } i.$$

□

We will require the following lemmas:

Lemma 2. *Let M be an ideal in l^∞ and let $x \in l^\infty$.Then x is in the closure of M in l^∞ if and only if $\chi_{K(x;\varepsilon)} \in M$ for all $\varepsilon > 0$ (see, [5]).*

As in Lemma 1 one can see that $[f]_0$ is an ideal in l^∞. On the other hand we can get the following result by using the idea that Freedman and Sember [9] used.

Lemma 3. *$[f]_0$ is closed ideal in l^∞.*

Our next result concerns the equality $[f] = [f,F] \cap l^\infty$.

Theorem 1. *Let F be an Orlicz function which satisfies Δ_2-condition. Then we have*

$$[f] = [f,F] \cap l^\infty.$$

Proof. We just prove that $[f]_0 = [f,F]_0 \cap l^\infty$. By Proposition 1 we have $[f]_0 \subset [f,F]_0$. In order to prove the opposite inclusion we first note that

$$\frac{1}{n}\sum_{k=i+1}^{i+n} F\left(\chi_{K(x;\varepsilon)}(k)\right) = F(1)\frac{1}{n}\sum_{k=i+1}^{i+n} \chi_{K(x;\varepsilon)}(k). \tag{2.3}$$

Let $x \in [f,F]_0 \cap l^\infty$ and $\varepsilon > 0$. Now define a sequence $y = (y_k)$ by $y_k = \dfrac{1}{x_k}$ if $|x_k| \geqslant \varepsilon$ and $y_k = 0$ otherwise. Since $xy = \chi_{K(x;\varepsilon)}$ and $[f,F]_0 \cap l^\infty$ is an ideal in l^∞ we get that $\chi_{K(x;\varepsilon)} \in [f,F]_0 \cap l^\infty$ and therefore

$$\lim_n \frac{1}{n}\sum_{k=i+1}^{i+n} F\left(\chi_{K(x;\varepsilon)}(k)\right) = 0, \text{ uniformly in } i.$$

Now (2.3) implies that

$$\lim_n \frac{1}{n}\sum_{k=i+1}^{i+n} \chi_{K(x;\varepsilon)}(k) = 0.$$

Combining this result with Lemma 2 and Lemma 3 we conclude that $x \in [f]_0$. This proves the theorem. □

By (1.1) and Theorem 1 we get the following

Corollary 1. *If F is an Orlicz function satisfying Δ_2-condition then we have*

$$[f] = [f,F] \cap l^\infty = S_u \cap l^\infty.$$

Our final result shows that $[f,F]$ lies between $[f]$ and S_u.

Theorem 2. *Let F be an Orlicz function satisfying Δ_2-condition. Then we have*

$$[f] \subset [f,F] \subset S_u.$$

Proof. The first inclusion is given in Proposition 1. We now prove that $[f,F] \subset S_u$. Let $x \in [f,F]_0$ and $y \in l^\infty$. By Lemma 1 we have that $xy \in [f,F]_0$. Let $\varepsilon > 0$ and $x \in [f,F]_0$. We define a bounded sequence $y = (y_k)$ by $y_k = \frac{1}{x_k}$, if $|x_k| \geqslant \varepsilon$ and $y_k = 0$ otherwise. Hence $xy = \chi_{K(x;\varepsilon)} \in [f,F]_0$. Now Theorem 1 implies that $\chi_{K(x;\varepsilon)} \in [f]_0$. So inclusion (1.1) implies that x is uniformly statistically convergent. □

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