



VALLÉE-POUSSIN THEOREM FOR KATUGAMPOLA FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We propose Vallée-Poussin theorem in form of three equivalent assertions for Katugampola fractional functional differential equation. Choosing corresponding function, we obtain explicit test of negativity of Green’s function in form of algebraic inequality. We discuss particular cases of functional equation such as equations with deviation to illustrate application of our technique. Further, we demonstrate applications of Katugampola derivatives as it generalizes previous inequalities available in literature for Riemann–Liouville fractional boundary value problem and Hadamard fractional boundary value problem.

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1. INTRODUCTION

In this paper, we consider the following fractional functional differential equation

$$(D_{a+}^{\alpha, \rho} x)(t) + (Tx)(t) = f(t), \quad t \in [a, b], \quad (1.1)$$

with boundary condition

$$x(a) = x(b) = 0, \quad (1.2)$$

where $1 < \alpha \leq 2$ and $D_{a+}^{\alpha, \rho}$ is Katugampola derivative which depends on extra parameter ρ and generalizes the Riemann–Liouville and Hadamard fractional derivative. If $\rho = 1$, then it reduces to Riemann–Liouville fractional derivative, and if $\rho \rightarrow 0_+$, it becomes Hadamard fractional derivative [14, 15]. The operator $T : C \rightarrow L_\infty$ are linear continuous operators acting from the space of the continuous functions C to the space of essentially bounded functions L_∞ and $f \in L_\infty$. The operator T can be of the forms $(Tx)(t) = q(t)x(t - \tau(t))$, $x(\xi) = 0$, if $\xi \notin [a, b]$, $(Tx)(t) = \int_a^b Q(t, s)x(\theta(s))ds$, or $(Tx)(t) = \int_a^b x(s)d_s Q(t, s)$. The conditions describing assumptions about all their coefficients are explained in [5], which allows acting of these operators $T : C \rightarrow L_\infty$.

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For the application of fractional differential equations in various fields of science and engineering one can refer to the known monographs [16, 23]. In the last few decades, analysis of positive solutions and investigation of various inequalities for fractional differential equations has been an active area of research. Qualitative theory for fractional differential equations such as oscillation theory, zeros of solutions, disconjugacy and comparison theory for fractional differential equations can be studied and many results were obtained on the basis of various inequalities, for example, Lyapunov-type inequalities, De La Vallée-Poussin inequalities and Hartman–Wintner-type inequalities (see, for example [1, 20, 25]). There are various methods such as different fixed point theorems, topological methods, coincidence degree theory, upper and lower solution method and different numerical methods which are used to study fractional differential equations with different fractional derivatives, to mention some of them one can see articles for Riemann–Liouville [13, 24], Caputo [25], Hadamard [11], Caputo–Hadamard [3], Ψ -operators [18], along with relevant references therein.

Let us note some of the recent work using Katugampola fractional derivatives. In [20], Lupinska and Odziejewicz obtained Lyapunov inequality for Katugampola fractional differential equation. In [9], authors used Guo–Krasnoselskii and Banach fixed point theorems to study the existence and uniqueness of solutions for nonlinear Katugampola fractional differential equation

$$\begin{cases} -D_{a+}^{\alpha, \rho} x(t) = \beta f(t, x(t)), & 1 < \alpha \leq 2, \quad t \in [0, T], \\ x(0) = x(T) = 0, \end{cases}$$

where $\beta \in \mathbb{R}$ and $f : [0, T] \times [0, \infty) \rightarrow [h, \infty)$ is a continuous function with finite positive constants h, T . In [22], Łupinska and Schmeidel by proving the Lyapunov-type inequality deduced the conditions for the existence, and non-existence of the solutions for fractional differential equations under fractional boundary conditions with the Katugampola derivative.

$$\begin{cases} D_{a+}^{\alpha, \rho} x(t) + g(t)x(t) = 0, \\ x(a) = D_{a+}^{\alpha, \rho} x(b) = 0. \end{cases}$$

In [4], authors studied existence and uniqueness theorem for a fractional equation with Caputo–Katugampola derivative. Some other analysis work for equations with Katugampola derivatives can be found in [19, 21].

Investigation of equations with Katugampola fractional derivatives is seldom in literature. It looks very natural in mathematical modeling to consider memory effects not only in the left-hand (i.e., in the “derivative part”), but also in another term. This leads us to equation (1.1). Another motivation presents a corresponding new step in the studying system: using representation of solution for one of the components of the solution vectors of the system to come to a scalar functional differential equation for another component of the solution-vector. For functional differential equation with

classical derivatives this idea was formulated in [2]. It should be stressed that even in the case of ordinary differential system, an equation for a corresponding component is an integro-differential (i.e., the functional differential equations). This motivates us to study problem (1.1)–(1.2). In this article, we apply Vallée-Poussin theorem about differential inequality to study problem (1.1)–(1.2). We obtain explicit tests of negativity of Green’s function in the form of algebraic inequalities. For n -th order functional differential equations, an analog of the Vallée-Poussin theorem and results on sign-constancy of Green’s functions on its basis were obtained in [6–8,10]. Recent work for fractional functional differential equations with operator T of the general form using Vallée-Poussin theorem can be found in [11–13,24].

Theorem 1 ([20]). *Let $0 < a < b < \infty$, $1 < \alpha \leq 2$, $\rho > 0$ and $q : [a, b] \rightarrow \mathbb{R}$ be a continuous function and x be a solution of the boundary value problem*

$$\begin{cases} (D_{a+}^{\alpha,\rho} x)(t) + q(t)x(t) = 0, \\ x(a) = x(b) = 0. \end{cases} \tag{1.3}$$

If $x(t) \neq 0$ for all $t \in (a, b)$, then we have the inequality

$$\int_a^b |q(s)| ds \geq \frac{\Gamma(\alpha)}{\max\{a^{\rho-1}, b^{\rho-1}\}} \left(\frac{4\rho}{b^\rho - a^\rho} \right). \tag{1.4}$$

Note that in [20], it was not assumed that $x(t) \neq 0$ for $t \in (a, b)$. We have (1.4) in the form

$$\min_{t \in [a,b]} q(t) \geq \frac{\Gamma(\alpha)}{\max\{a^{\rho-1}, b^{\rho-1}\}} \left(\frac{4\rho}{b^\rho - a^\rho} \right) \left(\frac{1}{b-a} \right). \tag{1.5}$$

From Corollary 3, obtained below in Section 4 (see Remark 1), we get that the inequality

$$q(t) < \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} \frac{\Gamma(\alpha+1)}{\rho^{2-\alpha}(b^\rho - a^\rho)}, \tag{1.6}$$

guarantees that the problem (1.3) has only the trivial solution and its Green’s function is negative for $(t, s) \in (a, b) \times (a, b)$. Note that inequality (1.6) we have constructed for a more general problem (4.1) in which we have $x(h(t))$ instead of $x(t)$ compared to problem (1.3). Inequality (1.6) means that in the case of zeros of solution $x(t)$ at the points a and b , we obtain that

$$\min_{t \in [a,b]} q(t) \geq \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} \frac{\Gamma(\alpha+1)}{\rho^{2-\alpha}(b^\rho - a^\rho)}, \tag{1.7}$$

since in the case of the coefficient q satisfying inequality (1.6) we exclude the existence of zero at the points a and b , i.e., one does not have the $x(a) = x(b) = 0$. Let us compare (1.5) and (1.7), computing the right-hand sides (RHS) in them, we obtain values estimating $\min_{t \in [a,b]} q(t)$ in Table 1 and graphical representation in Figure 1. We see that our estimate of q in right-hand side of inequality (1.5) gives sharper values of $\min_{t \in [a,b]} q(t)$ in compared to previous Lyapunov inequality (1.7).

α	RHS in inequality (1.5)	RHS in inequality (1.7)
2	4	8
1.9888	3.91993	7.85644
1.95	3.65704	7.37800
1.9	3.34906	6.80198
1.88	3.23840	6.58362
1.85	3.07229	6.62682
1.8	2.82342	5.773238

TABLE 1. Comparing our results with the known ones.

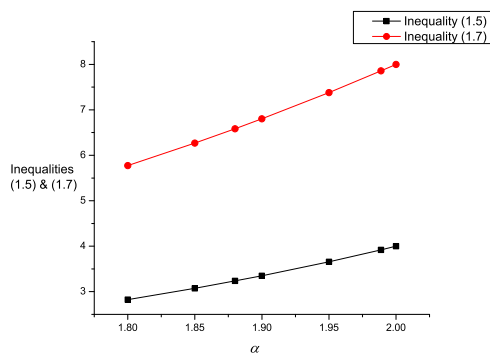


FIGURE 1. Describing that our inequality (1.7) is more exact than (1.5)

2. PRELIMINARIES

In [14], to define the generalized fractional derivative Katugampola considered the space $X_c^p(a, b)$ (where $c \in \mathbb{R}$, $1 \leq p \leq \infty$) of those Lebesgue measurable functions f on $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty, \quad (c \in \mathbb{R}, 1 \leq p \leq \infty)$$

and for the case $p = \infty$

$$\|f\|_{X_c^\infty} = \text{ess sup}_{a \leq t \leq b} [t^c |f(t)|], \quad c \in \mathbb{R}.$$

In particular, when $c = \frac{1}{p}$, ($1 \leq p \leq \infty$) the space $X_c^p(a, b)$ coincides with the classical $L_p(a, b)$ -space. In this article, we consider the case when $p \rightarrow \infty$, which brings us to the classical L_∞ space.

Definition 1 (see [9, 20]). Let $\alpha > 0$, $\rho > 0$, $0 < a < b \leq \infty$, $f(t) \in L_\infty$. The operator

$$I_{a+}^{\alpha,\rho} = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau,$$

for $t \in (a, b)$ is called left-side Katugampola integral of fractional order α .

Definition 2 (see [20]). Let $\alpha > 0$, $\rho > 0$, $n = [\alpha] + 1$, $0 < a < t < b \leq \infty$, $f \in L_\infty$. The operator

$$\begin{aligned} D_{a+}^{\alpha,\rho} f(t) &= \left(t^{1-\rho} \frac{d}{dt} \right)^n I_{a+}^{n-\alpha,\rho} f(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt} \right)^n \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{\alpha-n+1}} f(\tau) d\tau \end{aligned}$$

for $t \in (a, b)$ is called left-side Katugampola derivatives of fractional order α .

Lemma 1 (see [20]). Let $\alpha, \rho > 0$, $n = [\alpha] + 1$, where $[\alpha]$ is the integer part of α , $f \in L_\infty$. The fractional differential equation

$$(D_{a+}^{\alpha,\rho} x)(t) = f(t)$$

has the general solution of the form

$$x(t) = \sum_{i=1}^n c_i \left(\frac{t^\rho - a^\rho}{\rho} \right)^{i-n+\alpha} - I_{a+}^{\alpha,\rho} f(t),$$

where c_i are real constants.

Lemma 2 (see [20]). Assume $1 < \alpha \leq 2$ and $f \in L_\infty$. Then the unique solution to the problem

$$(D_{a+}^{\alpha,\rho} x)(t) = f(t) \tag{2.1}$$

with boundary condition (1.2) is given by

$$x(t) = \int_a^b G(t,s) f(s) ds \tag{2.2}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} -\frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-1} + \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}, & a \leq s \leq t \leq b \\ -\frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-1}, & a \leq t \leq s \leq b. \end{cases} \tag{2.3}$$

Now, in the following lemma we use the technique of [20], where authors proved the sign-constancy of Green's function.

Lemma 3. Green's function represented by (2.3) is negative for $s, t \in (a, b)$.

Proof. For $a < t < s < b$, it is clear that $G(t, s) < 0$. For $a < s < t < b$,

$$\begin{aligned} G(t, s) &= -\frac{s^\rho \rho^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-1} \left[(b^\rho - s^\rho)^{\alpha-1} + \left((t^\rho - s^\rho) \left(\frac{b^\rho - a^\rho}{t^\rho - a^\rho} \right) \right)^{\alpha-1} \right] \\ &= -\frac{s^\rho \rho^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-1} \left[(b^\rho - s^\rho)^{\alpha-1} \right. \\ &\quad \left. + \left(b^\rho - \left(a^\rho + \frac{(s^\rho - a^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \right) \right)^{\alpha-1} \right]. \end{aligned}$$

Observing that

$$s^\rho \leq a^\rho + \frac{(s^\rho - a^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \leq b^\rho,$$

because of the fact that

$$\frac{(s^\rho - a^\rho)(b^\rho) - a^\rho}{t^\rho - a^\rho} \geq 0 \quad \text{and} \quad \frac{(s^\rho - a^\rho)(b^\rho) - a^\rho}{t^\rho - a^\rho} \leq 0,$$

we get $G(t, s) < 0$ also for $s < t$. Hence, $G(t, s) < 0$ for $t, s \in (1, e)$. \square

3. MAIN RESULTS

Let us define the operator $K : L_\infty \rightarrow L_\infty$ by the equality

$$(Kz)(t) = -T \left[\int_a^b G(\cdot, s) z(s) ds \right] (t). \quad (3.1)$$

We use here and below the notation $T[\gamma(t)]$ meaning that the operator T acts on the continuous function γ , i.e., $T[\gamma(t)] = (T\gamma)(t)$. We assume in this paper the positivity of operators in the standard sense, i.e., the operator K is positive if $(Kz)(t) \geq 0$ for $t \in [a, b]$ for every nonnegative $z \in L_\infty$.

The following assertion can be considered as an analog of the Vallée-Poussin theorem on differential inequality [6].

Theorem 2. *Let $T : C \rightarrow L_\infty$, be positive operator, $1 < \alpha \leq 2$. Then the following assertions are equivalent:*

- 1) *there exist a positive number ε and a function $v \in X_c^\rho(a, b) \cap C$ such that $v(t) > 0$ for $t \in (a, b)$, $v(a) = 0$, $v(b) = 0$ and*

$$(D_{a+}^{\alpha, \rho} v)(t) + (Tv)(t) \equiv \psi(t) \leq -\varepsilon < 0 \quad \text{for } t \in (a, b); \quad (3.2)$$

- 2) *the spectral radius $r(K)$ of the operator $K : L_\infty \rightarrow L_\infty$ is less than 1;*
- 3) *problem (1.1)–(1.2), is uniquely solvable for any $f \in L_\infty$ and its Green's function $G(t, s)$ is negative for $(t, s) \in (a, b) \times (a, b)$.*

Proof. 1) \Rightarrow 2). The function v in condition 1) satisfies the boundary value problem

$$\begin{cases} (D_{a+}^{\alpha,p}x)(t) = z(t), \\ x(a) = 0, x(b) = v(b), \end{cases} \quad (3.3)$$

where z from L_∞ is $z(t) = \Psi(t) - (Tv(t))$. Thus, there exists $\delta > 0$ such that $z(t) \leq -\delta$ for $t \in (a, b)$. It is clear that

$$x(t) = \int_a^b G(t,s)z(s)ds + u(t), \quad (3.4)$$

where u is a solution of the homogeneous equation

$$\begin{cases} (D_{a+}^{\alpha,p}u)(t) = 0, \quad t \in [a, b], \\ u(a) = 0, \quad u(b) = 0. \end{cases} \quad (3.5)$$

Substituting this representation in the place of v into (3.2) yields

$$z(t) + T \left[\int_a^b G(t,s)z(s)ds \right] + (Tu)(t) = \Psi(t), \quad (3.6)$$

and

$$z(t) - (Kz)(t) = \Psi(t) - (Tu)(t), \quad t \in (a, b). \quad (3.7)$$

Let us prove that $u(t) \equiv 0$ for $t \in (a, b)$. From condition 1) of Theorem 2 we have $v(a) = 0$ and $v(b) = 0$. It is clear that $u(t) \equiv 0$ for $t \in (a, b]$ according to Lemma 2. Thus, $\Psi(t) = \Psi(t) - (Tu)(t) \leq -\varepsilon < 0$. The function $w = -z$ satisfies the inequalities

$$w(t) - (Kw)(t) = -\Psi(t) > 0,$$

and

$$w(t) > (Kw)(t).$$

According to [17, Theorem 5.8 on p. 84], we obtain $r(K) < 1$.

This completes the proof of the implication 1) \Rightarrow 2).

Let us prove now the implication 2) \Rightarrow 3). Consider the boundary value problem (1.1). Let us use the substitution

$$x(t) = \int_a^b G_0(t,s)z(s)ds, \quad (3.8)$$

where $G_0(t, s)$ is Green's function of the problem consisting of the equation

$$(D_{a+}^{\alpha,p}x)(t) = z(t) \quad (3.9)$$

with the boundary conditions (1.2). Substituting representation (3.8) into (1.1), we get (3.7), where $\Psi(t) = f(t)$ and $u(t) \equiv 0$ for $t \in [a, b]$. If $r(K) < 1$, then (3.7) is uniquely solvable and its solution is

$$z(t) = (I - K)^{-1}\Psi(t) = ((I + K + K^2 + K^3 + \dots)\Psi)(t). \quad (3.10)$$

We obtain that the solution x defined by (3.8) exists and is unique, and this proves that the problem (1.1) is uniquely solvable. We see also that $(I - K)^{-1}$ is a positive operator if K is positive. The assumption about positivity of the operator $T : C \rightarrow L_\infty$ and Lemma 2 imply that $K : L_\infty \rightarrow L_\infty$ is positive. Then from $\psi(t) \leq 0$, it follows that $z(t) \leq 0$ for $t \in [a, b]$. Thus, if $f(t) \leq 0$, then $z(t) \leq 0$ for $t \in [a, b]$. If $z(t) \leq 0$, then from the fact of nonpositivity of Green's function $G(t, s)$ in the formula (3.8), we get $x(t) \geq 0$ for $t \in [a, b]$. This is possible only in the case when Green's function $G(t, s)$ of problem (1.1)–(1.2) satisfies the inequality $G(t, s) \leq 0$. From the inequalities

$$\begin{aligned} x(t) &= \int_a^b G_0(t, s)z(s)ds = \int_a^b G_0(t, s)(I - K)^{-1}\psi(s)ds \\ &= \int_a^b G_0(t, s)[I + K + K^2 + K^3 + \dots]\psi(s)ds \end{aligned}$$

and the fact that $G_0(t, s) < 0$, it follows that $G(t, s) \leq G_0(t, s) < 0$ for $t, s \in (a, b)$. This completes the proof of the implication 2) \Rightarrow 3).

In order to prove the implication 3) \Rightarrow 1), we set

$$v(t) = - \int_a^b G(t, s)ds.$$

Since $G(t, s) < 0$ for $(t, s) \in (a, b)$, we get $v(t) > 0$ for $t \in (a, b)$. It is clear that $v(a) = v(b) = 0$, $\psi(t) = -1$. This completes the proof of the implication 3) \Rightarrow 1).

The proof is now complete. \square

Corollary 1. *If $1 < \alpha \leq 2$ and the following inequality is fulfilled*

$$T [\rho^{2-\alpha}(t^\rho - a^\rho)^{\alpha-1} [(b^\rho - a^\rho) - (t^\rho - a^\rho)]] < \Gamma(\alpha + 1) \quad (3.11)$$

then problem (1.1)–(1.2) is uniquely solvable for any $f \in L_\infty$ and its Green's function $G(t, s)$ is negative for $(t, s) \in (a, b)$.

Proof. For $\delta > 0$, consider the auxiliary equation

$$\begin{cases} (D_{a+}^{\alpha, \rho} v)(t) = -\delta, \\ v(a) = v(b) = 0, \end{cases}$$

Using Lemma 1, we write

$$v(t) = c_1 \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + c_2 \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-2} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (-\delta) d\tau$$

for some real constants c_1 and c_2 . Applying boundary condition, we obtain $c_2 = 0$ and so,

$$v(t) = c_1 \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} (-\delta) d\tau$$

We have to “connect” c_1 with δ to guarantee the inequality

$$v(t) > 0 \quad (3.12)$$

In order to continue the proof, let us describe in more detail an idea of this “connection”. It is clear that condition (3.12) is fulfilled for sufficiently large c_1 , but to achieve inequality (3.2), we need sufficiently small c_1 . Thus, we have to choose a minimal possible c_1 such that inequality (3.12) is fulfilled.

$$\begin{aligned} v(t) &= c_1 \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} - \frac{\delta \rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} d\tau \\ &= c_1 \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} - \frac{\delta \rho^{2-\alpha}}{\Gamma(\alpha+1)} (t^\rho - a^\rho)^\alpha \end{aligned}$$

Using the boundary condition $v(b) = 0$, we obtain

$$c_1 = \frac{\delta \rho}{\Gamma(\alpha+1)} (b^\rho - a^\rho)$$

putting this c_1 back in $v(t)$, we get

$$\begin{aligned} v(t) &= \frac{\delta \rho}{\Gamma(\alpha+1)} (b^\rho - a^\rho) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} - \frac{\delta \rho^{2-\alpha}}{\Gamma(\alpha+1)} (t^\rho - a^\rho)^\alpha \\ &= \frac{\delta \rho^{2-\alpha}}{\Gamma(\alpha+1)} (t^\rho - a^\rho)^{\alpha-1} [(b^\rho - a^\rho) - (t^\rho - a^\rho)] \end{aligned}$$

Now, considering $(D_{a+}^{\alpha,\rho} v)(t) = -\delta$ and obtained $v(t)$, we get (3.11). Using now the equivalence of the assertions 1) and 3) of Theorem 2, we obtain negativity of Green’s function $G(t, s)$ of problem (1.1), (1.2). \square

Let us consider the equation with deviations

$$(D_{a+}^{\alpha,\rho} x)(t) + \sum_{j=1}^m q_j(t)x(h_j(t)) = f(t), \quad t \in (a, b), \tag{3.13}$$

where

$$x(\xi) = 0, \quad \xi \notin (a, b), \tag{3.14}$$

$q_j, f \in L_\infty$, h is a measurable function. We obtain the following assertion.

Corollary 2. *Let $q_j(t) \geq 0$, $1 < \alpha < 2$ and assume*

$$\begin{aligned} \sum_{j=1}^m \chi(a < h_j(t) < b) q_j(t) \rho^{2-\alpha} (h_j(t)^\rho - a^\rho)^{\alpha-1} [(b^\rho - a^\rho) - (h_j(t)^\rho - a^\rho)] \\ < \Gamma(\alpha+1), \quad t \in (a, b) \end{aligned}$$

where

$$\chi(a < h_j(t) < b) = \begin{cases} 1, & h_j(t) \in (a, b), \\ 0, & h_j(t) \notin (a, b). \end{cases}$$

Then the problem consisting of equation (3.13) and the boundary conditions (1.2) is uniquely solvable for any $f \in L_\infty$ and its Green's function $G(t, s)$ is negative for $t, s \in (a, b)$.

4. REMARKS AND EXAMPLES

Consider the particular case of (1.1) given as

$$\begin{cases} (D^{\alpha, \rho} x)(t) + q(t)x(h(t)) = f(t), & t \in (a, b), \\ x(a) = x(b) = 0. \end{cases} \quad (4.1)$$

Corollary 3. If $1 < \alpha < 2$, $q(t) \geq 0$ for $t \in (a, b)$ and

$$\left\{ \operatorname{ess\,sup}_{t \in [a, b]} q(t) \right\} \rho^{2-\alpha} ((h(t))^\rho - a^\rho)^{\alpha-1} [(b^\rho - a^\rho) - ((h(t))^\rho - a^\rho)] < \Gamma(\alpha + 1), \quad t \in (a, b) \quad (4.2)$$

is fulfilled, then problem (4.1) is uniquely solvable for any $f \in L_\infty$ and its Green's function is negative for $(t, s) \in (a, b) \times (a, b)$.

Remark 1. Taking into account

$$\max_{a \leq t \leq b} \rho^{2-\alpha} (t^\rho - a^\rho)^{\alpha-1} [(b^\rho - a^\rho) - (t^\rho - a^\rho)] = \rho^{2-\alpha} (b^\rho - a^\rho)^\alpha \frac{(\alpha - 1)^{\alpha-1}}{\alpha^\alpha}, \quad (4.3)$$

which is achieved at the point $t^\rho = a^\rho + \frac{(\alpha-1)}{\alpha}(b^\rho - a^\rho)$, we get

$$q(t) < \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{\Gamma(\alpha + 1)}{\rho^{2-\alpha} (b^\rho - a^\rho)}. \quad (4.4)$$

Remark 2. Inequality (4.4) cannot be improved. Actually, assume that

$$(h(t))^\rho = a^\rho + \frac{(\alpha - 1)}{\alpha} (b^\rho - a^\rho)$$

in (4.1) and consider the equation

$$\begin{aligned} (D_{a+}^{\alpha, \rho} x)(t) + \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{\Gamma(\alpha + 1)}{\rho^{2-\alpha} (b^\rho - a^\rho)} x \left(\left(a^\rho + \frac{(\alpha - 1)}{\alpha} (b^\rho - a^\rho) \right)^{\frac{1}{\rho}} \right) \\ = 0, \quad t \in (a, b), \quad 1 < \alpha < 2. \end{aligned} \quad (4.5)$$

The function $x(t) = \rho^{2-\alpha} (t^\rho - a^\rho)^{\alpha-1} [(b^\rho - a^\rho) - (t^\rho - a^\rho)]$ satisfies (4.5) and indeed (4.5) has an infinite number of solutions of the form $c(\rho^{2-\alpha} (t^\rho - a^\rho)^{\alpha-1} [(b^\rho - a^\rho) - (t^\rho - a^\rho)])$ for every real number c .

Corollary 4. If $1 < \alpha \leq 2$, $h(t) < \varepsilon$ for $a = 0$, $b = 1$, for $t \in (0, 1)$, then the inequality (4.4) becomes

$$q(t) < \frac{\Gamma(\alpha + 1)}{(1 - \varepsilon)\varepsilon^{\alpha-1}} \quad (4.6)$$

and implies the unique solvability for any $f \in L_\infty$ of the problem (4.1).

Example 1. If we take $\epsilon = 0.1, 0.001, 0.0001, \alpha = 1.5$ in inequality (4.6) of Corollary 4, then particular bounds of inequalities calculated in Table 1 and represented in Figure 2.

ϵ	$\frac{\Gamma(\alpha+1)}{(1-\epsilon)\epsilon^{\alpha-1}}$
0.09	4.869379
0.05	6.257885
0.009	14.139736
0.005	18.8941829
0.0009	44.351262
0.0005	59.47964

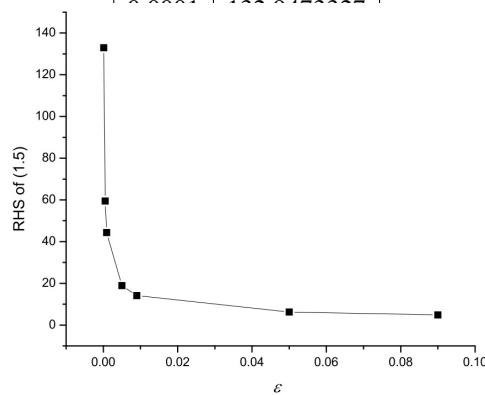


FIGURE 2.

Corollary 5. *In the case of superposition of integral and deviation operators*

$$(Tx)(t) = \int_a^b k(t,s)x(h(s))ds,$$

we get to

$$\int_a^b k(t,s)[\rho^{2-\alpha}(s-a^\rho)][(b^\rho - a^\rho) - (t^\rho - a^\rho)]ds < \Gamma(\alpha + 1).$$

Using estimate of (4.3), we get to

$$\int_a^b k(t,s)ds < \frac{1}{\max \left\{ \rho^{2-\alpha}(b^\rho - a^\rho) \alpha \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} \right\}},$$

which implies that problem

$$\begin{cases} (D^{\alpha,\rho}x)(t) + \int_a^b k(t,s)x(h(s))ds = f(t), & t \in (a,b), \\ x(a) = x(b) = 0. \end{cases}$$

is uniquely solvable for any $f \in L_\infty$ and its Green's function is negative for $(t,s) \in (a,b) \times (a,b)$.

5. APPLICATIONS

If we take $\rho = 1$ in Definition 2, then we obtain the Riemann–Liouville fractional derivative [16, 23]

$$D_{a+}^{\alpha,1} f(t) = {}^{RL} D_{a+}^{\alpha} f(t) = \left(\frac{d}{dt} \right)^n \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau.$$

Now, for $\rho \rightarrow 1$, $a = 0$ and $b = 1$, problem (1.1)–(1.2) becomes a Riemann–Liouville fractional boundary value problem which coincides with the problem studied in [13] for $k = 0$,

$$\begin{cases} ({}^{RL} D_{0+}^{\alpha} x)(t) + (Tx)(t) = f(t), \\ x(0) = x(1) = 0, \end{cases} \quad (5.1)$$

where $({}^{RL} D_{0+}^{\alpha})$ is Riemann–Liouville fractional derivative.

Corollary 6. For $\rho \rightarrow 1$ and $a = 0$, $b = 1$, we get inequality (3.11) as [13, Corollary 6]

$$T [t^{\alpha-1}(1-t)] < \Gamma(\alpha+1)$$

Similarly, if we take $\rho \rightarrow 0_+$ in Definition 2, then we get the Hadamard fractional derivative [16, 23]

$$\lim_{\rho \rightarrow 0_+} D_{1+}^{\alpha,\rho} f(t) = {}^H D_{1+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt} \right)^n \int_a^t \left(\log \frac{t}{\tau} \right)^{n-\alpha-1} f(\tau) \frac{d\tau}{\tau}.$$

Next, for $\rho \rightarrow 0_+$, $a = 1$ and $b = e$, problem (1.1)–(1.2) becomes a Hadamard fractional boundary value problem which coincides with the problem studied in [11]

$$\begin{cases} ({}^H D_{1+}^{\alpha} x)(t) + (Tx)(t) = f(t), \\ x(1) = x(e) = 0, \end{cases} \quad (5.2)$$

where $({}^H D_{1+}^{\alpha})$ is Hadamard fractional derivative.

Corollary 7. For $\rho \rightarrow 0_+$ and $a = 1$, $b = e$, we get inequality (3.11) as [11, Corollary 3.2]

$$T [(1nt)^{\alpha-1}(1-\ln t)] < \Gamma(\alpha+1)$$

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