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EXISTENCE OF SUPER-SOLUTIONS OF A DISCRETE EQUATION OF WOLFF TYPE

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Abstract. In this paper, we are concerned with the Wolff-type equation

$$u_i = \int_0^\infty \left(\frac{\sum_{j\in\mathbb{Z}^n, |j-i|<t} u_j^p}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}, \quad u_i > 0 \quad \text{for } i \in \mathbb{Z}^n,$$

where $n \geq 1$, $\min\{\beta, p\} > 0$, $\gamma > 1$ and $\beta\gamma < n$. Such an equation is related to the study in the theory of nonlinear PDEs and mathematical physics. Here we study the existence of positive super-solutions and obtain the critical exponent of the Serrin type.

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1. INTRODUCTION

Let $u = (u_i)_{i\in\mathbb{Z}^n}$ be a nonnegative sequence. The Wolff potential of u is (cf. [4])

$$W_{\beta,\gamma}(u)(i) = \int_0^\infty \left(\frac{\sum_{j\in\mathbb{Z}^n, |j-i|<t} u_j}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}, \tag{1.1}$$

where $n \geq 1$, $\beta > 0$, and $\gamma > 1$. In this paper, we study the existence of positive super-solutions of the equation

$$u_i = W_{\beta,\gamma}(u^p)(i), \quad i \in \mathbb{Z}^n, \tag{1.2}$$

where $n \geq 1$, $\min\{\beta, p\} > 0$, $\gamma > 1$ and $\beta\gamma < n$. A sequence $u = (u_i)_{i\in\mathbb{Z}^n}$ is called a positive super-solution of (1.2), if u satisfies

$$\begin{cases} u_i > 0 \ \ for \ \ all \ \ i \in \mathbb{Z}^n, \\ u_i < \infty \ \ when \ \ |i| < R \ \ for \ \ any \ \ R > 0, \end{cases} \tag{1.3}$$

and (1.2) with '$=$' replaced by '$\geq$' holds.

Recently, the authors in [11] obtained an optimal summability of positive solutions of the equation (1.2) by means of regularity lifting lemma and a Wolff-type inequality. Then, they also derived the decay rate of u_i when $|i| \to \infty$.

It is easy to see that (1.1) is one of the discrete forms of the Wolff potential of a locally integrable nonnegative function f

$$W_{\beta,\gamma}(f)(x) := \int_0^\infty \left[\frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}} \right]^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

This potential can be used to study nonlinear PDEs, such as $\mathcal{A}$-harmonic equations and k-Hessian equations. According to results in [6,7,15], if $\inf_{\mathbb{R}^n} u = 0$, there exists $C > 1$ such that some positive solution u of the Lane-Emden equation

$$Lu = u^p, \quad u > 0 \ \ in \ \ \mathbb{R}^n$$

satisfy

$$C^{-1} W_{\beta,\gamma}(u^p)(x) \le u(x) \le C W_{\beta,\gamma}(u^p)(x), \quad x \in \mathbb{R}^n. \tag{1.4}$$

Here $\beta = 1$ and $\gamma = q$ when $Lu = -div(A(x,u,\nabla u))$ or $Lu = -div(|\nabla u|^{q-2}\nabla u)$, and $\beta = 2k/(k+1)$ and $\gamma = k+1$ when $Lu = \sigma_k(D^2(-u))$. In view of (1.4), the following Wolff-type integral equation

$$u(x) = K(x) W_{\beta,\gamma}(u^p(y))(x), \quad u > 0 \ \ on \ \ \mathbb{R}^n \tag{1.5}$$

comes into play in those work. Here, a function $K(x)$ is called *double bounded*, if there exist positive constants c and C such that $c \le K(x) \le C$ for all $x \in \mathbb{R}^n$.

When $K(x) \equiv n - \alpha$, $\gamma = 2$ and $\beta = \alpha/2$, (1.5) is reduced to

$$u(x) = \int_{\mathbb{R}^n} \frac{u^q(y)dy}{|x-y|^{n-\alpha}}, \quad u > 0 \ \ on \ \ \mathbb{R}^n. \tag{1.6}$$

In addition, (1.6) is the Euler-Lagrange equation satisfied by the extremal functions of the Hardy-Littlewood-Sobolev inequality (cf. [3,12,13]).

For the coupling system

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x) \\ v(x) = W_{\beta,\gamma}(u^p)(x), \end{cases} \tag{1.7}$$

Chen and Li [2] proved the radial symmetry for the integrable solutions. Afterward, Ma, Chen and Li [14] used the regularity lifting lemmas to obtain the optimal integrability and the Lipschitz continuity. Based on these results, [16] obtained the decay rates of the integrable solutions when $|x| \to \infty$.

The main result in this paper is the following theorem.

Theorem 1. *The Wolff-type equation* (1.2) *has a positive super-solution if and only if* $p > \frac{n(\gamma-1)}{n-\beta\gamma}$.

Remark 1. When $\beta = \alpha/2$ and $\gamma = 2$, the discrete form of (1.7) is reduced to

$$\begin{cases} u_i = \sum_{j \in \mathbb{Z}^n, i \neq j} \frac{v_j^q}{|i-j|^{n-\alpha}}, \\ v_i = \sum_{j \in \mathbb{Z}^n, i \neq j} \frac{u_j^p}{|i-j|^{n-\alpha}}, \end{cases}$$

This system is associated with the best constant of the discrete Hardy-Littlewood-Sobolev inequality (cf. [5]). Theorem 1 is consistent with Theorem 1.3 in [10]. Although Corollary 2.1 in [14] provides a Wolff-type inequality, the Euler-Lagrange equation corresponding to this extremum function is not (1.2). In fact, we have not found the Euler-Lagrange equation that satisfies the extremum function of the inequality corresponding to (1.2) at present.

Remark 2. The existence results and the critical exponents of integral equations (1.5), (1.6) and (1.7) can be seen in [1, 8, 9].

Remark 3. Another discrete Wolff potential of a nonnegative Borel measure ω is (cf. [4])

$$\tilde{W}_{\beta,\gamma}(\omega)(x) = \sum_{Q \in \mathcal{D}} \left[\frac{\omega(Q)}{|Q|^{1-\beta\gamma/n}} \right]^{\frac{1}{\gamma-1}} \chi_Q(x),$$

where $\mathcal{D} = \{Q\}$ and $Q = 2^i(k + [0,1)^n)$, $i \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. The corresponding discrete equation of the Wolff type is

$$u(x) = \tilde{W}_{\beta,\gamma}(u^p)(x) + f(x),$$

where $f \in L^p_{loc}(\mathbb{R}^n)$ is a nonnegative function. The existence results and the critical exponents can be seen in [15].

Clearly, the discrete equation of (1.5) is

$$u_i = c_i \int_0^\infty \left(\sum_{j \in \mathbb{Z}^n, |j-i|<t} u_j^p \right)^{\frac{1}{\gamma-1}} t^{\frac{\beta\gamma-n}{\gamma-1}} \frac{dt}{t}, \quad i \in \mathbb{Z}^n, \tag{1.8}$$

where c_i is a double bounded sequence. Namely, there is a constant $C > 1$ such that $C^{-1} \leq c_i \leq C$ for all $i \in \mathbb{Z}^n$.

Theorem 1 is a corollary of the following two lemmas.

Lemma 1. *If $p > \frac{n(\gamma-1)}{n-\beta\gamma}$, (1.8) has positive solutions for some double bounded c_i.*

Lemma 2. *If $0 < p \leq \frac{n(\gamma-1)}{n-\beta\gamma}$, (1.8) has no positive solution satisfying (1.3) for any double bounded c_i.*

2. Proof of Theorem 1

Proof of Lemma 1.

Let θ be a positive constant which will be determined later. Inserting

$$v_i = (1+|i|^2)^{-\theta} \tag{2.1}$$

into $W_{\beta,\gamma}(v^p)(i)$, we obtain

$$\begin{aligned} W_{\beta,\gamma}(v^p)(i) = & \int_0^{|i|/2} [\sum_{|j-i|<t} (1+|j|^2)^{-p\theta} t^{\beta\gamma-n}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ & + \int_{|i|/2}^{\infty} [\sum_{|j-i|<t} (1+|j|^2)^{-p\theta} t^{\beta\gamma-n}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ := & I_1(i) + I_2(i). \end{aligned}$$

When $|i| \le R$ for some $R>0$, then u_i is proportional to $W_{\beta,\gamma}(u^p)(i)$. So we also only consider suitably large $|i|$.

Clearly, when $t \in (0,|i|/2)$, $|j-i|<t$ implies $|i|/2<|j|<3|i|/2$. Therefore, we can find positive constants c and C such that

$$\frac{c}{(1+|i|^2)^{\frac{p\theta}{\gamma-1}}} \int_0^{|i|/2} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} \le I_1(i) \le \frac{C}{(1+|i|^2)^{\frac{p\theta}{\gamma-1}}} \int_0^{|i|/2} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t}.$$

Namely,

$$c(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}} \le I_1(i) \le C(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}}. \tag{2.2}$$

Take the slow rate

$$2\theta = \frac{\beta\gamma}{p-\gamma+1}. \tag{2.3}$$

There holds $\beta\gamma < 2p\theta < n$ by $p > \frac{n(\gamma-1)}{n-\beta\gamma}$. In addition, when $t \ge |i|/2$,

$$\{j \in \mathbb{Z}^n; |j-i|<t\} \subset \{j \in \mathbb{Z}^n; |j|<3t\}.$$

Therefore,

$$I_2(i) \le C \int_{|i|/2}^{\infty} \left(\frac{t^{n-2p\theta}}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \le C(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}}.$$

Thus,

$$c(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}} \le I_1(i)+I_2(i) \le C(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}}. \tag{2.4}$$

Set

$$c_i := (1+|i|^2)^{\frac{2p\theta-\beta\gamma}{2(\gamma-1)}} [I_1(i)+I_2(i)]$$

Then $I_1(i)+I_2(i) = c_i v_i$ in view of (2.1) with (2.3). In addition, (2.4) implies that c_i is double bounded. Thus, v_i is a solution of (1.8).

Similarly, we also find a fast decaying solution. In fact, taking

$$2\theta = \frac{n-\beta\gamma}{\gamma-1}, \tag{2.5}$$

from $p > \frac{n(\gamma-1)}{n-\beta\gamma}$, we also have

$$2p\theta > n, \tag{2.6}$$

and

$$\frac{\beta\gamma-n}{\gamma-1} > \frac{\beta\gamma-2p\theta}{\gamma-1}. \tag{2.7}$$

When $t > 2|i|$, there holds

$$\{j \in Z^n; |j-i| < t\} \supset \{j \in Z^n; |j| \le 1\}.$$

Therefore,

$$I_2(i) \ge \int_{2|i|}^{\infty} [\sum_{|j|\le 1} (1+|j|^2)^{-p\theta}]^{\frac{1}{\gamma-1}} t^{\frac{\beta\gamma-n}{\gamma-1}} \frac{dt}{t} \ge c\int_{2|i|}^{\infty} t^{\frac{\beta\gamma-n}{\gamma-1}} \frac{dt}{t} \ge c(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}. \tag{2.8}$$

On the other hand,

$$\sum_{|j|\le 1, |j-i|<t} (1+|j|^2)^{-p\theta} \le \sum_{|j|\le 1} 1,$$

and by (2.6), there holds

$$\sum_{|j|>1, |j-i|<t} (1+|j|^2)^{-p\theta} \le \sum_{|j|>1} (1+|j|^2)^{-p\theta} \le C.$$

Therefore,

$$I_2(i) = \int_{|i|/2}^{\infty} \left(\frac{\Sigma_{|j|\le 1, |j-i|<t} (1+|j|^2)^{-p\theta} + \Sigma_{|j|>1, |j-i|<t} (1+|j|^2)^{-p\theta}}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}$$

$$\le C\int_{|i|/2}^{\infty} t^{-\frac{n-\beta\gamma}{\gamma-1}} \frac{dt}{t} \le C(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}.$$

Combining with (2.8), we get

$$c(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}} \le I_2(i) \le C(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}.$$

This result, together with (2.2) and (2.7), implies

$$c(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}} \le I_1(i) + I_2(i) \le C(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}. \tag{2.9}$$

Set

$$c_i = (1+|i|^2)^{\frac{n-\beta\gamma}{2(\gamma-1)}} [I_1(i) + I_2(i)].$$

Then, there also holds

$$I_1(i) + I_2(i) = c_i(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}} = c_i v_i,$$

in view of (2.1) with (2.5), and c_i is bounded in view of (2.9). Thus, v_i is also a solution of (1.8).

Proof of Lemma 2.

Suppose that u solves (1.8). We will deduce a contradiction.
Step 1. Let

$$0 < p < \frac{n(\gamma-1)}{n-\beta\gamma}. \tag{2.10}$$

From (1.8) it follows

$$u_i \geq c\int_{2|i|}^{\infty} t^{\frac{\beta\gamma-n}{\gamma-1}} \frac{dt}{t} = \frac{c}{|i|^{a_0}},$$

since $\Sigma_{|j|<1} u_j^p \geq c$, where $a_0 = \frac{n-\beta\gamma}{\gamma-1}$. By this estimate, we have

$$u_i \geq c\int_{2|i|}^{\infty} \left(\frac{\sum_{|j|<t-|i|} |j|^{-pa_0}}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \geq c\int_{2|i|}^{\infty} t^{\frac{\beta\gamma-pa_0}{\gamma-1}} \frac{dt}{t}. \tag{2.11}$$

When $\frac{p}{\gamma-1} \in (0, \frac{\beta\gamma}{n-\beta\gamma}]$, we have $\beta\gamma - pa_0 \geq 0$. Eq. (2.11) implies $u_i = \infty$. This contradicts with (1.3).

Next, we consider the case $\frac{p}{\gamma-1} \in (\frac{\beta\gamma}{n-\beta\gamma}, \frac{n}{n-\beta\gamma})$. Now (2.11) leads to

$$u_i \geq \frac{c}{|i|^{a_1}},$$

where $a_1 = \frac{p}{\gamma-1}a_0 - \frac{\beta\gamma}{\gamma-1}$. Write

$$a_k = \frac{p}{\gamma-1}a_{k-1} - \frac{\beta\gamma}{\gamma-1},\ k = 1, 2, \cdots. \tag{2.12}$$

We claim that there must be $k_0 > 0$ such that $a_{k_0} \leq 0$. This leads to $u_i = \infty$, which is impossible.

In fact, by (2.12) we get

$$a_k = \left(\frac{p}{\gamma-1}\right)^k a_0 - \left[1 + \frac{p}{\gamma-1} + \cdots + \left(\frac{p}{\gamma-1}\right)^{k-1}\right]\frac{\beta\gamma}{\gamma-1}.$$

If $\frac{p}{\gamma-1} = 1$, then we can find a large k_0 such that

$$a_{k_0} = a_0 - k_0\frac{\beta\gamma}{\gamma-1} \leq 0.$$

If $\frac{p}{\gamma-1} \in (1, \frac{n}{n-\beta\gamma})$, then using $a_0 - \frac{\beta\gamma}{p-\gamma+1} < 0$ which is implied by (2.10), we can find a large k_0 such that

$$a_{k_0} = \left(\frac{p}{\gamma-1}\right)^{k_0} a_0 - \frac{(\frac{p}{\gamma-1})^{k_0} - 1}{\frac{p}{\gamma-1} - 1}\frac{\beta\gamma}{\gamma-1}$$

$$= \left(\frac{p}{\gamma-1}\right)^{k_0}\left(a_0 - \frac{\beta\gamma}{p-\gamma+1}\right) + \frac{\beta\gamma}{p-\gamma+1} \le 0.$$

If $\frac{p}{\gamma-1} \in (0,1)$, letting $k \to \infty$, we get

$$a_k = \left(\frac{p}{\gamma-1}\right)^k a_0 - \frac{1-(\frac{p}{\gamma-1})^k}{1-\frac{p}{\gamma-1}}\frac{\beta\gamma}{\gamma-1} \to \frac{\beta\gamma}{p-\gamma+1} < 0.$$

Thus, there must be k_0 such that $a_{k_0} \le 0$.

Step 2. Let $p = \frac{n(\gamma-1)}{n-\beta\gamma}$. We deduce the contradiction if u is a positive solution of (1.8).

Let $R > 0$. Clearly, when $t \in (R/2, R)$ and $|i| < R/4$, there holds

$$\{j \in \mathbb{Z}^n; |j-i| < t\} \supset \{j \in \mathbb{Z}^n; |j| < R/4\}.$$

From (1.8), it follows that

$$u_i \ge c\int_{R/2}^R (\sum_{|j-i|<t} u_j^p)^{\frac{1}{\gamma-1}} t^{\frac{\beta\gamma-n}{\gamma-1}}\frac{dt}{t}$$

$$\ge c\int_{R/2}^R (\sum_{|j|<R/4} u_j^p)^{\frac{1}{\gamma-1}} t^{\frac{\beta\gamma-n}{\gamma-1}}\frac{dt}{t} \ge cR^{-\frac{n-\beta\gamma}{\gamma-1}}\left(\sum_{|j|<R/4} u_j^p\right)^{\frac{1}{\gamma-1}}.$$

Therefore, we get

$$u_i^p \ge cR^{p\frac{\beta\gamma-n}{\gamma-1}}(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}}. \tag{2.13}$$

Summing for $|i| < R/4$ and noting $p = \frac{n(\gamma-1)}{n-\beta\gamma}$, we get

$$\sum_{|j|<R/4} u_i^p \ge cR^{p\frac{\beta\gamma-n}{\gamma-1}}(\sum_{|j|<R/4} 1)(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}} \ge c(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}}.$$

Here $c > 0$ is independent of R. Letting $R \to \infty$ and noting $p > \gamma - 1$, we have

$$\sum_{j\in\mathbb{Z}^n} u_j^p < \infty. \tag{2.14}$$

Summing (2.13) for $R/8 < |i| < R/4$ yields

$$\sum_{R/8<|i|<R/4} u_i^p \ge cR^{p\frac{\beta\gamma-n}{\gamma-1}}(\sum_{R/8<|i|<R/4} 1)(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}}.$$

By $p = \frac{n(\gamma-1)}{n-\beta\gamma}$, it follows

$$\sum_{R/8<|i|<R/4} u_i^p \ge c(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}},$$

where $c > 0$ is independent of R. Letting $R \to \infty$, and noting (2.14), we obtain

$$\sum_{j \in \mathbb{Z}^n} u_j^p = 0,$$

which contradicts with (1.3).

Proof of Theorem 1.

Necessity. Replacing (1.8) by (1.2) in the proof of Lemma 2, we easily see that (1.2) has no super-solution when $p \in (0, \frac{n(\gamma-1)}{n-\beta\gamma}]$.

Sufficiency. By Lemma 1, we assume that v_i solves (1.8) for some c_i. Since c_i is double bounded, we can find a constant $b_0 > 0$ such that $c_i \geq b_0$ for all $i \in \mathbb{Z}^n$.

Set $u_i = \lambda v_i$, where $\lambda > 0$ will be determined later. Thus, by (1.8),

$$u_i \geq \lambda b_0 W_{\beta,\gamma}(\lambda^{-p} u^p)(i) = b_0 \lambda^{1-\frac{p}{\gamma-1}} W_{\beta,\gamma}(u^p)(i).$$

Take $\lambda = b_0^{1/[p/(\gamma-1)-1]}$. Then $b_0 \lambda^{1-\frac{p}{\gamma-1}} = 1$ and hence u_i is a super-solution of (1.2).

The proof is complete.

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