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EXTENDING A THEOREM OF DATKO FOR EVOLUTIONARY FAMILIES

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Abstract. In this note, we extend Datko's result in the paper [4, 1972]. In particular, the exponential stability of evolutionary families is characterized by its pointwise trajectories in which the norm mapping of each pointwise trajectory lies in a Banach function space.

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1. Introduction

The concept of an evolutionary family arises naturally from the well-posed theory of non-autonomous abstract differential equations on $\mathbb{R}$ or $\mathbb{R}_+$, it is also a quite natural generalization of strongly continuous semigroups. Readers can refer to Engel and Nagel [6], Chicone and Latushkin [2], Pazy [15], Daleckii and Krein [3] for this topic. In this note, X is a real or complex Banach space with a norm $\|\cdot\|$ and I is either $\mathbb{R}$ or $\mathbb{R}_+$.

Definition 1. A family of bounded linear operators $(U(t,s))_{t\geq s}$ on a Banach space X is an (strongly continuous and exponentially bounded) evolutionary family on I (means $t,s\in I$) if

(i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for all $t\geq r\geq s$ and $t,r,s\in I$.

(ii) The map $(t,s)\mapsto U(t,s)x$ is continuous for every $x\in X$.

(iii) There are constants $K,c>0$ such that $\|U(t,s)x\|\leq Ke^{c(t-s)}\|x\|$ for all $t\geq s$ and $x\in X$.

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Notice that a strongly continuous semigroup $(T(t))_{t\geq 0}$ naturally gives rise to the evolutionary family $U(t,s)=T(t-s)$ for $t\geq s$ and $t,s\in I$. Among the many interesting types of stability of evolutionary families, of interest in this note is the exponentially stable evolutionary families because such families are very useful when we study nonlinear problems associated with those families.

Definition 2. An evolutionary family $(U(t,s))_{t\geq s}$ on I will be called *exponentially stable* if there exist constants $K_1,\alpha>0$ such that $\|U(t,s)x\|\leq K_1e^{-\alpha(t-s)}\|x\|$ for all $x\in X$ and $t\geq s$.

By properties of (i) and (iii) from Definition 1, the exponential stability of evolutionary families is equivalent to the uniformly asymptotic stability of those families, see [4, Lemma 1]. To study the exponential stability of evolutionary families, there are two basic approaches. One is based on Perron's method, for instance [14]. The other is based on Datko's result, for example [18]. In [4], Datko pointed out that *an evolutionary family* $(U(t,s))_{t\geq s}$ *is exponentially stable on* $\mathbb{R}_+$ *if and only if for each* $x\in X$ *there exists a constant* $M(x)>0$ *such that*

$$\int_{t_0}^{\infty}\|U(t,t_0)x\|^2\mathrm{d}t\leq M(x)\quad\text{for all}\quad t_0\geq 0.$$

In case the evolutionary family $(U(t,s))_{t\geq s\geq 0}$ is a strongly continuous semigroup that means $U(t,s)=U(t-s,0)$ for $t\geq s\geq 0$ or $U(t,s)=T(t-s)$ with $(T(t))_{t\geq 0}$ is a strongly continuous semigroup, this result was extended by Pazy in [15] for the Lebesgue spaces $L^p(\mathbb{R}_+)$ with $p\in[1,\infty)$ and by Neerven in [18]. Neerven's result covers a wide class of function spaces. However, it only holds for strongly continuous semigroups. In addition, another limitation in Neerven's result is that it only offers a sufficient condition for exponential stability of strongly continuous semigroups, and his result is valid only for complex Banach spaces. In this case, we mention [19, Theorem 1.1] as an interesting application for Datko's result and Pazy's result.

Using the idea of replacing the squared function in the integral with a function of two variables that satisfies some certain conditions, Rolewicz [16, Theorem 2] had generalized Datko's result. With that same idea, Megan et al. extended Datko's result for linear skew-product semiflows and skew-evolution semiflows on Banach spaces in [11, Theorem 3.4] and [17, Theorem 1], respectively. In [13, Theorem 4.1 and Theorem 4.2], Megan et al. characterized the uniform exponential stability of periodic evolution families by the belonging of the associated trajectories to Banach sequence spaces and Banach function spaces having unbounded fundamental functions. In [12], Megan et al. characterized the uniform exponential stability of general evolution families on the half-line by the ownership of their associated trajectories to Banach function spaces that belong to a broad class of function spaces.

In [1, Theorem 0.1], Buse used normed function spaces, which were the same as in [18], to give a more general result for the evolutionary family on the half-line. More precisely, this result improves Theorem 4.2 in the paper [18]. Furthermore, Datko's result was also extended to nonuniform behavior of evolutionary family in papers [5] and [9].

Follow the second approach in studying the exponential stability of an evolutionary family, our purpose is the generalization of Datko's result for an evolutionary family on the line or the half-line and a broader class of function spaces. To accomplish this task, we introduce the concept of Banach function space in Subsection 2.1. Using the concept of Banach function space, we get the expected results which are stated in Subsection 2.2. Our results hold for both Banach spaces over complex and real fields. Because we provide both necessary and sufficient conditions for the exponential stability of the evolutionary family, the class of Banach function spaces in this note is smaller than the class of function spaces in [18]. However, it is still very wide and most of the known function spaces belong to the class of Banach function spaces that we define in this note.

2. Banach function spaces and main results

2.1. *Banach function spaces*

In the monograph book [10, Chapter 2], Massera and Schäffer introduced several classes of function spaces that play a fundamental role in studying differential equations. Based on the concepts of function spaces given by Massera and Schäffer, we have collected some basic properties to give the notion of Banach function space. This definition method is also similar to that in [7, Section 2].

Definition 3. Let $\mathcal{B}$ be the Borel σ-algebra and let μ be the Lebesgue measure on $\mathbb{R}$. A vector space E of real-valued measurable functions on $\mathbb{R}$ is called a Banach function space if

i. $(E, \|\cdot\|_E)$ is a Banach space and $\|\cdot\|_E$ guarantees the property: if $\varphi_2 \in E$ and φ_1 is real-valued measurable function such that $|\varphi_1(t)| \le |\varphi_2(t)|$ almost everywhere on $\mathbb{R}$ then $\varphi_1 \in E$ and $\|\varphi_1\|_E \le \|\varphi_2\|_E$;

ii. the characteristic function $\chi_{[a,b]} \in E$ for all $[a,b] \subset \mathbb{R}$ and $\inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0$;

iii. there is a constant $M \ge 1$ such that
$$\frac{1}{b-a}\int_a^b |\varphi(t)|\mathrm{d}t \le \frac{M\|\varphi\|_E}{\|\chi_{[a,b]}\|_E} \text{ for all } \varphi \in E \text{ and } [a,b] \subset \mathbb{R}; \tag{2.1}$$

iv. the function $\Lambda_1\varphi$ defined by $(\Lambda_1\varphi)(t) = \int_t^{t+1} \varphi(\tau)\mathrm{d}\tau$ belongs to E for each $\varphi \in E$;

v. E is T_τ-invariant and there exists a constant $N > 0$ such that $\|T_\tau\| \le N$ for all $\tau \in \mathbb{R}$, where T_τ is shift operator on E defined by $(T_\tau\varphi)(t) = \varphi(t+\tau)$ for $t \in \mathbb{R}$.

Denote by $L_{1,\mathrm{loc}}(\mathbb{R})$ space of real-valued locally integrable functions on $\mathbb{R}$. A family of seminorms defining the topology of $L_{1,\mathrm{loc}}(\mathbb{R})$ is given by

$$\left\{p_n\colon n\in\mathbb{Z} \text{ and } p_n(\varphi)=\int_n^{n+1}|\varphi(t)|\mathrm{d}t\right\}.$$

Then, $L_{1,\mathrm{loc}}(\mathbb{R})$ is a Fréchet space. Therefore, by (2.1) then $E\hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R})$.

By direct inspection, it can be easily seen that class of Banach function spaces includes the Lebesgue spaces $L^p(\mathbb{R})$ with $p\in[1,\infty]$, the Lorentz spaces $L^{p,q}(\mathbb{R})$ with $p,q\in[1,\infty]$, the Orlicz spaces, etc, and the space

$$\mathbf{M}(\mathbb{R})=\left\{\varphi\in L_{1,\mathrm{loc}}(\mathbb{R})\colon \sup_{t\in\mathbb{R}}\int_t^{t+1}|\varphi(\tau)|\mathrm{d}\tau<\infty\right\}$$

with the norm $\|\varphi\|_{\mathbf{M}}:=\sup_{t\in\mathbb{R}}\int_t^{t+1}|\varphi(\tau)|\mathrm{d}\tau$. By (2.1) and the item ii. in Definition 3, we have the estimate

$$\|\varphi\|_{\mathbf{M}}\le\frac{M}{\inf_{t\in\mathbb{R}}\|\chi_{[t,t+1]}\|_E}\|\varphi\|_E, \tag{2.2}$$

for all $\varphi\in E$. Thus, $E\hookrightarrow\mathbf{M}(\mathbb{R})$.

Throughout this note, function $\chi_{D(\varphi)}\varphi$ has the domain $\mathbb{R}$ and defines as follows:

$$(\chi_{D(\varphi)}\varphi)(t)=\begin{cases}\varphi(t), & \text{if } t\in D(\varphi),\\ 0, & \text{otherwise,}\end{cases}$$

where $D(\varphi)\subset\mathbb{R}$ is the domain of the function φ.

2.2. *Main results*

Let $(U(t,s))_{t\ge s}$ be an evolutionary family on I. For $t_0\in I$ and $x\in X$, put

$$g_{t_0,x}(t)=\|U(t,t_0)x\| \quad\text{for}\quad t\ge t_0.$$

Because the evolutionary family is strongly continuous, $\chi_{[t_0,\infty)}g_{t_0,x}$ is measurable function. By the properties of the Banach function space, the exponential stability of the evolutionary family will now be characterized by its pointwise trajectories. First, we give the necessary condition.

Theorem 1. *Let E be any Banach function space. If $(U(t,s))_{t\ge s}$ is exponentially stable on I, then for each $x\in X$ the function $\chi_{[t_0,\infty)}g_{t_0,x}$ belongs to E and there is a constant $M(x)>0$ such that*

$$\|\chi_{[t_0,\infty)}g_{t_0,x}\|_E\le M(x),$$

for all $t_0\in I$.

Proof. First we show $e^{-\alpha|t|} \in E$. Put

$$v(t) = \int_{-\infty}^{t} e^{-\alpha(t-\tau)}\chi_{[0,1]}(\tau)\mathrm{d}\tau + \int_{t}^{\infty} e^{-\alpha(\tau-t)}\chi_{[0,1]}(\tau)\mathrm{d}\tau.$$

Then,

$$v(t) = \begin{cases} \frac{e^{-\alpha t}(e^{\alpha}-1)}{\alpha}, & \text{if } t \geq 1, \\ \frac{e^{\alpha t}(1-e^{-\alpha})}{\alpha}, & \text{if } t \leq 0, \\ \frac{1-e^{-\alpha t}}{\alpha} + \frac{1-e^{-\alpha(1-t)}}{\alpha}, & \text{if } t \in (0,1). \end{cases}$$

Therefore, $e^{\alpha|t|}v(t) \geq \frac{1-e^{-\alpha}}{\alpha}$ for all $t \in \mathbb{R}$. On the other hand,

$$\begin{aligned} v(t) &= \sum_{k=0}^{\infty} \int_{t-(k+1)}^{t-k} e^{-\alpha(t-\tau)}\chi_{[0,1]}(\tau)\mathrm{d}\tau + \sum_{k=0}^{\infty} \int_{t+k}^{t+k+1} e^{-\alpha(\tau-t)}\chi_{[0,1]}(\tau)\mathrm{d}\tau \\ &\leq \sum_{k=0}^{\infty} e^{-\alpha k} \int_{t-(k+1)}^{t-k} \chi_{[0,1]}(\tau)\mathrm{d}\tau + \sum_{k=0}^{\infty} e^{-\alpha k} \int_{t+k}^{t+k+1} \chi_{[0,1]}(\tau)\mathrm{d}\tau \\ &= \sum_{k=0}^{\infty} e^{-\alpha k}(T_{-k-1}(\Lambda_1\chi_{[0,1]}))(t) + \sum_{k=0}^{\infty} e^{-\alpha k}(T_k(\Lambda_1\chi_{[0,1]}))(t) =: \varphi(t). \end{aligned}$$

This implies that

$$e^{-\alpha|t|} \leq \frac{\alpha}{1-e^{-\alpha}}v(t) \leq \frac{\alpha}{1-e^{-\alpha}}\varphi(t) \quad \text{for all } t \in \mathbb{R}.$$

We also have the following estimates.

$$\begin{aligned} \sum_{k=0}^{\infty} e^{-\alpha k}\|T_{-k-1}(\Lambda_1\chi_{[0,1]})\|_E + \sum_{k=0}^{\infty} e^{-\alpha k}\|T_k(\Lambda_1\chi_{[0,1]})\|_E &\leq \sum_{k=0}^{\infty} e^{-\alpha k}\, 2N\|\Lambda_1\chi_{[0,1]}\|_E \\ &= \frac{2N}{1-e^{-\alpha}}\|\Lambda_1\chi_{[0,1]}\|_E. \end{aligned}$$

So, function series φ is absolutely convergent in the Banach function space E. Therefore, $\varphi \in E$ and

$$\|\varphi\|_E \leq \frac{2N\|\Lambda_1\chi_{[0,1]}\|_E}{1-e^{-\alpha}}.$$

By the item i. in Definition 3, we get $e^{-\alpha|t|} \in E$ and

$$\|e^{-\alpha|\cdot|}\|_E \leq \frac{2N\alpha\|\Lambda_1\chi_{[0,1]}\|_E}{(1-e^{-\alpha})^2}.$$

Because $(U(t,s))_{t\geq s}$ is exponentially stable on I,

$$\begin{aligned} (\chi_{[t_0,\infty)}g_{t_0,x})(t) &\leq K_1\chi_{[t_0,\infty)}(t)e^{-\alpha(t-t_0)}\|x\| \\ &\leq K_1\|x\|(T_{-t_0}e^{-\alpha|\cdot|})(t), \end{aligned} \tag{2.3}$$

for all $t \in \mathbb{R}$. By the items i., v. in Definition 3 and (2.3), we obtain $\chi_{[t_0,\infty)} g_{t_0,x} \in E$ and

$$\|\chi_{[t_0,\infty)} g_{t_0,x}\|_E \le \frac{2N^2 K_1 \alpha \|\Lambda_1 \chi_{[0,1]}\|_E}{(1-e^{-\alpha})^2} \|x\|,$$

for all $t_0 \in I$. □

In sufficient condition, we need to add a constraint of Banach function space.

Theorem 2. *Let E be a Banach function space such that* $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$. *If for each* $x \in X$, *the function* $\chi_{[t_0,\infty)} g_{t_0,x}$ *belongs to E and there exists a constant* $M(x) > 0$ *such that*

$$\|\chi_{[t_0,\infty)} g_{t_0,x}\|_E \le M(x),$$

for all $t_0 \in I$, *then the evolutionary family* $(U(t,s))_{t\ge s}$ *is exponentially stable on I.*

Remark 1. It is easy to see that the Lebesgue spaces $L^p(\mathbb{R})$ with $p \in [1,\infty)$ and the Lorentz spaces $L^{p,q}(\mathbb{R})$ with $p \in [1,\infty), q \in [1,\infty]$ satisfy the constraint above. Thus, Theorem 1 and Theorem 2 are an extension for Datko's result in the paper [4, Theorem 1]. This constraint is not to be missed; moreover, it also appears naturally in the proof of the theorem. On the other hand, we can give a simple example below to see that the condition $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$ can not be omitted.

In $\mathbb{R}^2$, consider $E = L^\infty(\mathbb{R})$ and the evolutionary family $(U(t,s))_{t\ge s}$ on I, in which

$$U(t,s) = \begin{pmatrix} \cos(t-s) & -\sin(t-s) \\ \sin(t-s) & \cos(t-s) \end{pmatrix}.$$

Obviously, the evolutionary family $(U(t,s))_{t\ge s}$ is not exponentially stable on I. For each $x \in \mathbb{R}^2$ and $t_0 \in I$, we have

$$g_{t_0,x}(t) = \|U(t,t_0)x\|_{\mathbb{R}^2} = \|x\|_{\mathbb{R}^2}, \quad \text{for} \quad t \ge t_0.$$

Therefore, $\|\chi_{[t_0,\infty)} g_{t_0,x}\|_{L^\infty(\mathbb{R})} = \|x\|_{\mathbb{R}^2}$ for all $t_0 \in I$.

Proof. For $t_0 \in I$ and $t > t_0$, for each $x \in X$ then mapping $\varphi_{t,t_0,x}$ is defined as follows:

$$\varphi_{t,t_0,x}(\xi) = \begin{cases} \|U(\xi,t_0)x\|, & \text{if } \xi \in [t_0,t], \\ 0, & \text{if } \xi \notin [t_0,t]. \end{cases}$$

Then, $\varphi_{t,t_0,x} \in E$ and

$$\|\varphi_{t,t_0,x}\|_E \le \|\chi_{[t_0,\infty)} g_{t_0,x}\|_E \le M(x). \tag{2.4}$$

We now construct a family of functions $\{\Phi_{t,t_0}\}$ determining by

$$\Phi_{t,t_0} : X \to \mathbb{R} \quad \text{with} \quad \Phi_{t,t_0}(x) = \|\varphi_{t,t_0,x}\|_E.$$

It is easy to see that Φ_{t,t_0} is a seminorm on X. On the other hand, we have

$$\varphi_{t,t_0,x}(\xi) \le K e^{c(t-t_0)} \chi_{[t_0,t]}(\xi) \|x\|,$$

for all $\xi \in \mathbb{R}$. Thus, $\Phi_{t,t_0}(x) \le Ke^{c(t-t_0)}\|\chi_{[t_0,t]}\|_E\|x\|$. So, Φ_{t,t_0} is a continuous seminorm on X. Moreover, by (2.4), the family of continuous seminorms $\{\Phi_{t,t_0} : t_0 \in I, t > t_0\}$ is pointwise bounded. Applying uniform boundedness principle (see Appendix), there exists a constant $C > 0$ such that

$$\Phi_{t,t_0}(x) \le C\|x\|, \tag{2.5}$$

for all $x \in X$ and $t_0 \in I, t > t_0$.

For $\xi \in [t_0,t]$, we have

$$e^{-c(t-\xi)}\|U(t,t_0)x\| = e^{-c(t-\xi)}\|U(t,\xi)U(\xi,t_0)x\| \le K\|U(\xi,t_0)x\|.$$

Therefore,

$$\chi_{[t_0,t]}(\xi)e^{-c(t-\xi)}\|U(t,t_0)x\| \le K\varphi_{t,t_0,x}(\xi),$$

for all $\xi \in \mathbb{R}$. So,

$$\|\chi_{[t_0,t]}e^{-c(t-\cdot)}\|_E\|U(t,t_0)x\| \le KC\|x\|.$$

By (2.2),

$$\|\chi_{[t_0,t]}e^{-c(t-\cdot)}\|_M\|U(t,t_0)x\| \le \frac{KCM}{\inf_{\tau\in\mathbb{R}}\|\chi_{[\tau,\tau+1]}\|_E}\|x\|,$$

for all $t_0 \in I$ and $t > t_0$.

For $t \ge t_0 + 1$, we have

$$\|\chi_{[t_0,t]}e^{-c(t-\cdot)}\|_M \ge \int_{t-1}^{t} e^{-c(t-\xi)}\mathrm{d}\xi = \frac{1-e^{-c}}{c}.$$

Thus,

$$\|U(t,t_0)x\| \le \frac{KCMc}{(1-e^{-c})\inf_{\tau\in\mathbb{R}}\|\chi_{[\tau,\tau+1]}\|_E}\|x\|,$$

for all $t_0 \in I$ and $t \ge t_0 + 1$. Because the evolutionary family $(U(t,s))_{t\ge s}$ is exponentially bounded, there exists a constant $C_1 > 0$ such that

$$\|U(t,t_0)x\| \le C_1\|x\|,$$

for all $x \in X$ and $t_0 \in I, t \ge t_0$.

For $\xi \in [t_0,t]$, we have $\|U(t,t_0)x\| \le C_1\|U(\xi,t_0)x\|$. Therefore,

$$\chi_{[t_0,t]}(\xi)\|U(t,t_0)x\| \le C_1\varphi_{t,t_0,x}(\xi),$$

for all $\xi \in \mathbb{R}$. So,

$$\|\chi_{[t_0,t]}\|_E\|U(t,t_0)x\| \le C_1C\|x\|,$$

for all $t_0 \in I$ and $t > t_0$. On the other hand,

$$\chi_{[0,t-t_0]}(\xi) = \chi_{[t_0,t]}(\xi + t_0) = (T_{t_0}\chi_{[t_0,t]})(\xi).$$

Therefore, $\|\chi_{[0,t-t_0]}\|_E \le N\|\chi_{[t_0,t]}\|_E$. So,

$$\|U(t,t_0)x\| \le \frac{NC_1C}{\|\chi_{[0,t-t_0]}\|_E}\|x\|,$$

for all $t_0 \in I$ and $t > t_0$. Because of $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$, the evolutionary family $(U(t,s))_{t\geq s}$ is exponentially stable on I. □

The following corollaries are a minor weakening of Theorem 2 for special evolutionary families.

Corollary 1. *Let E be a Banach function space such that $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$ and $(T(t))_{t\geq 0}$ be a strongly continuous semigroup. If function $\chi_{[0,\infty)}g_{0,x}$ belongs to E for each $x \in X$ then $(T(t))_{t\geq 0}$ is exponentially stable, where $g_{0,x}(t) = \|T(t)x\|$ for $t \geq 0$.*

Remark 2. This corollary is more general than Pazy's result in [15, Chapter 4, Theorem 4.1].

Proof. For $t_0 \in \mathbb{R}$ and $x \in X$, we have $g_{t_0,x}(t) = \|T(t-t_0)x\|$ for $t \geq t_0$. Therefore,

$$(\chi_{[t_0,\infty)}g_{t_0,x})(t) = (\chi_{[0,\infty)}g_{0,x})(t-t_0) = (T_{-t_0}(\chi_{[0,\infty)}g_{0,x}))(t),$$

for $t \in \mathbb{R}$. By the item v. in Definition 3, we get $\chi_{[t_0,\infty)}g_{t_0,x} \in E$ and

$$\|\chi_{[t_0,\infty)}g_{t_0,x}\|_E \leq N\|\chi_{[0,\infty)}g_{0,x}\|_E, \quad \text{for all } t_0 \in \mathbb{R}.$$

By Theorem 2, $(T(t))_{t\geq 0}$ is an exponentially stable semigroup. □

Next, we will give the exponentially stable characterization for a periodic evolutionary family.

Definition 4. An evolutionary family $(U(t,s))_{t\geq s}$ is said to be periodic with a period $T > 0$ if $U(t+T,s+T) = U(t,s)$ for all $t \geq s$ and $t,s \in I$.

Corollary 2. *Let E be a Banach function space such that $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$ and $(U(t,s))_{t\geq s}$ be a periodic evolutionary family with period $T > 0$. If the function $\chi_{[T,\infty)}g_{T,x}$ belongs to E for each $x \in X$, then the evolutionary family $(U(t,s))_{t\geq s}$ is exponentially stable on I.*

Proof. By the periodicity of the evolutionary family $(U(t,s))_{t\geq s}$ on I so we just need to consider $t_0 \geq 0$. We will repeat manner of the proof in Theorem 2 to obtain (2.5) as follows.

Replace t_0 with T and discuss the same as in the first paragraph in the proof of Theorem 2, there exists a constant $D > 0$ such that

$$\Phi_{t,T}(x) \leq D\|x\|,$$

for all $x \in X$ and $t > T$. For $t_0 \in [0,T)$ and $t \in (t_0,T]$, we have

$$\varphi_{t,t_0,x}(\xi) \leq \chi_{[0,T]}(\xi)Ke^{cT}\|x\|,$$

for all $\xi \in \mathbb{R}$. Therefore, $\varphi_{t,t_0,x} \in E$ and $\|\varphi_{t,t_0,x}\|_E \leq Ke^{cT}\|\chi_{[0,T]}\|_E\|x\|$. For $t > T$,

$$\varphi_{t,t_0,x}(\xi) \leq \chi_{[0,T]}(\xi)Ke^{cT}\|x\| + \chi_{[T,t]}(\xi)\|U(\xi,T)U(T,t_0)x\|,$$

for all $\xi \in \mathbb{R}$. Thus, $\varphi_{t_0,x} \in E$ and

$$\|\varphi_{t,t_0,x}\|_E \leq Ke^{cT}\|\chi_{[0,T]}\|_E\|x\| + \Phi_{t,T}(U(T,t_0)x) \leq Ke^{cT}(\|\chi_{[0,T]}\|_E + D)\|x\|.$$

For $t_0 \geq T$ and $t > t_0$, we can write $t_0 = nT + \tau$ with $n \in \mathbb{N}$ and $\tau \in [0,T)$. Then,

$$\|U(\xi,t_0)x\| = \|U(\xi,nT+\tau)x\| = \|U(\xi - nT,\tau)x\| = \|U(\xi - t_0 + \tau,\tau)x\|,$$

for $\xi \in [t_0,t]$. Put

$$\psi_{\tau,x}(\xi) = \begin{cases} \|U(\xi,\tau)x\|, & \text{if } \xi \in [\tau, t - t_0 + \tau], \\ 0, & \text{if } \xi \notin [\tau, t - t_0 + \tau]. \end{cases}$$

Then, $\varphi_{t,t_0,x}(\xi) = (T_{-t_0+\tau}\psi_{\tau,x})(\xi)$ for all $\xi \in \mathbb{R}$. Because of $\tau \in [0,T)$ so $\psi_{\tau,x} \in E$, hence $\varphi_{t,t_0,x} \in E$ and

$$\|\varphi_{t,t_0,x}\|_E \leq NKe^{cT}(\|\chi_{[0,T]}\|_E + D)\|x\|.$$

So, for all $t_0 \geq 0$ and $t > t_0$ then $\varphi_{t,t_0,x} \in E$ and

$$\|\varphi_{t,t_0,x}\|_E \leq NKe^{cT}(\|\chi_{[0,T]}\|_E + D)\|x\|.$$

Therefore,

$$\Phi_{t,t_0}(x) \leq NKe^{cT}(\|\chi_{[0,T]}\|_E + D)\|x\|,$$

for all $x \in X$ and $t_0 \geq 0, t > t_0$. The next step, by the same discussions as in the proof of Theorem 2 we deduce that the evolutionary family $(U(t,s))_{t \geq s}$ is exponentially stable on I. □

APPENDIX

For completeness, we restate here the boundedness principle for a family of continuous seminorms.

Let X be a Banach space over the field K ($K = \mathbb{R}$ or $\mathbb{C}$). A mapping $p\colon X \to \mathbb{R}_+$ is a seminorm on X if $p(\theta x) = |\theta| p(x)$ and $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$ and $\theta \in K$. As is known, a seminorm p is continuous on X if and only if p is continuous at 0. Moreover, if A is a bounded linear operator on X then $p_A(x) = \|Ax\|$ for $x \in X$ is a continuous seminorm on X.

Let Λ be an index set. A family of continuous seminorms $\{p_\lambda\colon \lambda \in \Lambda\}$ on X is called

- pointwise bounded if for each $x \in X$ there exists a constant $M(x) > 0$ such that $p_\lambda(x) \leq M(x)$ for all $\lambda \in \Lambda$;
- uniformly bounded if there is a constant $M > 0$ such that $p_\lambda(x) \leq M\|x\|$ for all $\lambda \in \Lambda$ and $x \in X$.

A similar proof to the uniform boundedness principle for a family of bounded linear operators (see Kreyszig [8]), we get a version of the uniform boundedness principle for a family of continuous seminorms.

Uniform boundedness principle. *Let* $\{p_\lambda : \lambda \in \Lambda\}$ *be a family of continuous seminorms on* X. *If this family is pointwise bounded then it is also uniformly bounded.*

Obviously, this version is more general than the old version in functional analysis.

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