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NEW QUANTUM BOUNDARIES FOR q-TRAPEZOIDAL AND q-MIDPOINT TYPES INEQUALITIES FOR (η_1,η_2)-CONVEX FUNCTIONS

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Abstract. Quantum calculus, also known as q-calculus, extends classical calculus with a deformation parameter q. The aim of this paper is to present new integral identities of q-trapezoidal and q-midpoint types for first-order q-differentiable functions. These identities are fundamental for establishing novel q-trapezoidal and q-midpoint type integral inequalities that can be applied to functions characterized by their first quantum derivatives and their absolute values being (η_1,η_2)-convex functions. To achieve this goal, the paper employs various mathematical tools, including the q-power mean inequality and q-Hölder's inequality. These tools are essential for deriving and demonstrating the proposed inequalities, which play a crucial role in understanding and characterizing the behavior of functions within the framework of quantum calculus.

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1. INTRODUCTION

Calculus, a branch of mathematics, plays a crucial role in the study of derivatives and integrals. However, the concept of classical calculus has been expanded and modified to create a new calculus called quantum calculus. Quantum calculus, also known as q-calculus, encompasses the study of calculus without limitations. It includes q-fractional calculus, q-integral calculus, and q-transform analysis, which have proven to be valuable tools in various fields of mathematics and physics. The development of quantum calculus can be attributed to early scholars like Euler and Jacobi. However, in recent decades, significant advancements and progress have been made, as evidenced in the works of Jackson and others [12]. Jackson was among the pioneers who defined the q-analogues of derivative and integral operators and explored their applications. Quantum integral inequalities, a vital aspect of this calculus, have been shown to be more valuable and informative than their classical counterparts.

The study of quantum calculus has led to the proposal and detailed discussion of new generalizations of classical concepts. For instance, Tariboon and Ntouyas [28] introduced quantum calculus ideas and derived various q-analogues of conventional mathematical objects on finite intervals. This inspiration has resulted in numerous innovative findings published in scientific literature, where quantum analogs of classical mathematical achievements have been explored. Overall, the emergence and development of quantum calculus have broadened the scope of calculus, providing powerful tools and concepts to tackle complex problems in mathematics and physics, for more details see in [4, 7, 13–16, 18] and the references cited therein.

Inequalities play a crucial role in the mathematical analysis of dynamic systems, control theory, and optimization problems. Sun et al. [21,30,32] employed inequalities in the sliding mode control of fuzzy Markov jump systems, ensuring stability and robustness in control mechanisms. Gong et al. [8] used inequalities to study boundedness and compactness of Cauchy-type integrals in weighted Morrey spaces, providing insights into integral operator behavior. Additionally, Bohner et al. [3] explored oscillation criteria for dynamic equations, applying inequalities to establish sufficient conditions for oscillatory or non-oscillatory behavior in nonlinear systems. Zhao et al. [42] and Zhang et al. [36] extended this to boundary value problems, demonstrating how inequalities can determine solvability and oscillation in fractional and neutral delay dynamic equations. Across these studies, inequalities are used as fundamental tools to analyze the behavior of complex systems, ensuring control, stability, and efficiency. Qiu et al. [19, 29] extend this analysis to neutral delay dynamic and fractional differential equations, offering solutions to boundary value problems and highlighting the impact of damping and perturbations. Zhang et al. [24, 37] address sliding mode control in the presence of stochastic disturbances and cyber-attacks, reinforcing the relevance of robust control strategies. Together, these works illustrate the deep interplay between inequalities, oscillation theory, and control methods in modern mathematical and applied research. The selected references explore various aspects of supply chain management, focusing on fuzzy control and robust strategies under uncertain conditions, which are areas where quantum calculus can offer valuable insights. Zhang et al. work covers models that address supply chain disruptions [41] and lead-time variations [38,39] by utilizing fuzzy and robust control approaches to ensure stable supply chain operations. These studies are particularly relevant to quantum calculus due to its capability to model non-linear dynamics and discrete-time phenomena in complex systems. Quantum calculus can contribute to refining these models by offering tools to analyze small-scale variations in system parameters, essential for precise control in dynamic and uncertain supply chains, as demonstrated in their discrete switched model for dynamic networks [40]. Integrating quantum calculus into these frameworks could enhance decision-making accuracy, potentially improving stability and resilience in supply chain networks [20, 23].

In recent times, the classical notion of convexity has undergone extensions and generalizations. Hanson, as cited in [10], introduced the concept of invexity, which stands as a significant generalization of convexity. Mohan and Weir, as mentioned in [27,34,35], have made significant contributions by establishing a novel class of functions called preinvexity. This introduction has opened up new avenues for exploring convexity from a different perspective. The research on invexity has considerably heightened its importance in optimization, as documented in [6, 17]. This concept serves as a further generalization of the traditional notion of convexity, allowing for more comprehensive and versatile mathematical frameworks. The development and study of these extended notions of convexity, such as invexity and preinvexity, have enhanced our understanding of mathematical inequalities and their practical applications in diverse fields, including optimization and scientific modeling. These advancements provide valuable tools for addressing complex problems and exploring new realms of mathematical analysis. Moreover, these studies highlight the importance of symmetry, integrability, and resonance in analyzing dynamical systems, employing advanced methods like center manifold analysis and quantum calculus for discrete modeling [22, 25, 26]. This collection of research contributes to a structured framework for understanding complex dynamical behaviors, with significant implications for mathematical physics and applied mathematics.

Suppose that function $\mathcal{H}\colon J \subset \mathbb{R} \to \mathbb{R}$ be convex on $[\delta_1, \delta_2]$, where $\delta_1, \delta_2 \in J$ with $\delta_1 < \delta_2$.

The inequality expressed in Equation (1.1) is defined as follows,

$$\mathcal{H}\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \mathcal{H}(\delta)\,\mathrm{d}\delta \leq \frac{\mathcal{H}(\delta_1)+\mathcal{H}(\delta_2)}{2}. \tag{1.1}$$

This inequality, denoted as Equation (1.1), was initially introduced by C. Hermite [11] and further explored by J. Hadamard [9] in 1893. It is worth noting that both of these inequalities are valid in the opposite direction if the function $\mathcal{H}$ is concave. The significance of the Hermit–Hadamard inequality cannot be overstated, as it plays a crucial role in various domains of modern mathematics, including numerical analysis, functional analysis, and mathematical analysis. Its validity and implications have been extensively studied and documented in the literature. Researchers and scholars in these fields consider this inequality to be of great importance due to its wide-ranging applications and its ability to provide valuable insights into the properties of functions and related mathematical phenomena.

2. Preliminaries

Mond introduced the notion of invex sets in his paper

Definition 1 ([27, Definition 2.1])**.** An invex set is a generalization of a convex set. Formally, a set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to a vector-valued function $\eta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if for every pair of points $\delta_1, \delta_2 \in S$, the following condition holds,

$$\delta_1 + \tau\eta(\delta_2, \delta_1) \in S \qquad \forall \tau \in [0,1].$$

This definition is a generalization of convexity, where the function η governs the direction of the combination of points. In the case where $\eta(\delta_2, \delta_1) = \delta_2 - \delta_1$, the invex set reduces to the usual convex set.

Weir has been introduced the definition of preinvex mapping

Definition 2 ([34, Definition 2.1])**.** A real-valued function $\mathcal{H}\colon S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$ is an invex set, is called preinvex with respect to a function $\eta\colon S \times S \to \mathbb{R}^n$ if for all $\delta_1, \delta_2 \in S$ and $\tau \in [0,1]$, the following inequality holds,

$$\mathcal{H}(\delta_1 + \tau\eta(\delta_2, \delta_1)) \leq (1-\tau)\mathcal{H}(\delta_1) + \tau\mathcal{H}(\delta_2).$$

This is a generalization of the convexity property. If $\eta(\delta_2, \delta_1) = \delta_2 - \delta_1$, then the preinvex function reduces to a convex function. The function η determines how the combination of points δ_1 and δ_2 is taken, and the inequality ensures that the function values respect this invex combination. In this paper, we employ continuous bifunctions denoted as $\eta(.,.)$ which operate on pairs of real vectors from $\mathbb{R}^n$ and produce vectors in $\mathbb{R}^n$.

Definition 3 ([5, Definition 1.2])**.** Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$. Consider $\mathcal{H}\colon I \to \mathbb{R}$ and $\eta_2 : \mathcal{H}(I) \times \mathcal{H}(I) \to \mathbb{R}$. The function $\mathcal{H}$ is said to be (η_1, η_2)-convax function

$$\mathcal{H}(\delta_2 + \tau\eta(\delta_1, \delta_2)) \leq \mathcal{H}(\delta_2) + \tau\eta(\mathcal{H}(\delta_1), \mathcal{H}(\delta_2))$$

for all $\delta_1, \delta_2 \in I$, $\tau \in [0,1]$.

2.1. *q-Derivative and q-Integral*

In this part, we provide a detailed presentation of the necessary definitions for q-Calculus, along with significant quantum integral inequalities for Hermite-Hadamard on both the left and right sides of the intervals.

Throughout this entire paper, we utilize the constants $0 < q < 1$.

The integers denoted as $[n]_q$ are referred to as q-integers and can be expressed as follows,

$$[n]_q = 1 + q + q^2 + \cdots + q^n - 1 = \frac{1-q^n}{1-q} \qquad \text{for } n = 1, 2, \ldots;$$
$$[n]_q = n \qquad \text{for } n = 1.$$

The symbols $[n]_q!$ and $\begin{bmatrix} n \\ i \end{bmatrix}_q !$ represent the q-factorial and q-series, respectively, and are expressed as follows,

$$[n]_q! = \prod_{i=1}^{n} [i]_q, \quad n \geq 1, \quad [0]_q! = 1,$$

$$\left[\begin{array}{c} n \\ i \end{array} \right]_q ! = \frac{[n]_q !}{[n-i]_q ! \, [i]_q !}.$$

Reverend Frank Hilton Jackson emerged as a prominent figure during the early twentieth century, leaving a lasting impact on the advancement of the classical concept of function derivatives at specific points. His invaluable contributions played a pivotal role in simplifying the exploration of ordinary calculus and number theory, making these subjects more accessible to scholars. Jackson's exceptional accomplishments encompassed groundbreaking studies, including the one documented in [12], as well as the formulation of q-analogues that extended the applicability of key principles in these domains.

To this end, Jackson introduced the notion of the q-derivative, symbolized as $D_q \mathcal{H}(\varpi)$. This derivative is defined through the following expression

$$D_q \mathcal{H}(\varpi) = \frac{\mathcal{H}(\varpi) - \mathcal{H}(q\varpi)}{(1-q)\varpi}, \qquad \varpi \neq 0. \tag{2.1}$$

Jackson's pioneering work led to the development of a classic approach, which we shall now outline.

$$\int_0^{\delta_2} \mathcal{H}(\varpi) \, \mathrm{d}_q \varpi = (1-q)\delta_2 \sum_{n=0}^{\infty} q^n \mathcal{H}(\delta_2 q^n),$$

subject to the condition that the corresponding series converges absolutely.

The q-Jackson integral, defined within the interval $[\delta_1, \delta_2]$, is expressed as follows,

$$\int_{\delta_1}^{\delta_2} \mathcal{H}(\varpi) \, \mathrm{d}_q \varpi = \int_0^{\delta_2} \mathcal{H}(\varpi) \, \mathrm{d}_q \varpi - \int_0^{\delta_1} \mathcal{H}(\varpi) \, \mathrm{d}_q \varpi.$$

As the value of q approaches 1^-, the outcomes in number theory, deduction, and ordinary integration can be represented by polynomial expressions in a real variable q.

Definition 4 ([28, Definition 2.1])**.** We consider an arbitrary function

$$\mathcal{H}: \ [\delta_1, \delta_2] \to \mathbb{R}.$$

Then q_{δ_1}-derivative of $\mathcal{H}$ at $\varpi \in [\delta_1, \delta_2]$ is defined as follows,

$${}_{\delta_1} D_q \mathcal{H}(\varpi) = \frac{\mathcal{H}(\varpi) - \mathcal{H}(q\varpi + (1-q)\delta_1)}{(1-q)(\varpi - \delta_1)}, \qquad \varpi \neq \delta_1. \tag{2.2}$$

As arbitrary function $\mathcal{H}$ from $[\delta_1, \delta_2]$ to $\mathbb{R}$, so for $\varpi = \delta_1$, we define ${}_{\delta_1} D_q \mathcal{H}(\delta_1) = \lim_{\varpi \to \delta_1} {}_{\delta_1} D_q \mathcal{H}(\varpi)$ if it exists and is finite.

Remark 1. An important observation to make is that when $\delta_1 = 0$ in equation (2.2), we arrive at a similar form of the q-derivative as defined in equation (2.1).

The subsequent lemma assumes a pivotal role in the computation of q_{δ_1}-derivatives.

Lemma 1 ([28, Lemma 2.1]). *Taking* $\xi \in \mathbb{R}$, *we have*

$$ {}_{\delta_1}D_q (x-\delta_1)^{\xi} = \left(\frac{1-q^{\xi}}{1-q} \right) (x-\delta_1)^{\xi-1}. $$

Definition 5 ([28, Definition 2.3]). We consider an arbitrary function

$$ \mathcal{H}\colon [\delta_1, \delta_2] \to \mathbb{R}, $$

then the q_{δ_1}-definite integral on $\varpi \in [\delta_1, \delta_2]$ is described as follows,

$$ \int_{\delta_1}^{\varpi} \mathcal{H}(\varpi)\ {}_{\delta_1}\mathrm{d}_q\varpi = (1-q)(\varpi-\delta_1) \sum_{n=0}^{\infty} q^n \mathcal{H}\left(q^n\varpi + (1-q^n)\delta_1\right), \qquad \varpi \in [\delta_1, \delta_2]. \tag{2.3} $$

If $\delta_1 = 0$, then (2.3) becomes

$$ \int_{0}^{\varpi} \mathcal{H}(\varpi)\ {}_{0}\mathrm{d}_q\varpi = (1-q)\varpi \sum_{n=0}^{\infty} q^n \mathcal{H}(q^n\varpi), \qquad \varpi \in [\delta_1, \delta_2]. $$

The following properties are of great importance in the field of quantum calculus

Theorem 1 ([28, Theorem 2.1]). *Let* $\mathcal{H}\colon I \to \mathbb{R}$ *be a continuous function. Then,*

(1) ${}_{\delta_1}D_q \int_{\delta_1}^{x} \mathcal{H}(\tau)\ {}_{\delta_1}\mathrm{d}_q\tau = \mathcal{H}(x)$;

(2) $\int_{\chi}^{x} {}_{\delta_1}D_q\mathcal{H}(\tau)\ {}_{\delta_1}\mathrm{d}_q\tau = \mathcal{H}(x) - \mathcal{H}(\chi)$, $\chi \in (\delta_1, x)$.

The following results are valuable when it comes to evaluating q_{δ_1}-integrals

Lemma 2 ([28, Lemma 2.2]). *Suppose that* $\zeta \in \mathbb{R}\setminus\{-1\}$, *then*

$$ \int_{\delta_1}^{\delta} (\tau-\delta_1)^{\zeta}\ {}_{\delta_1}\mathrm{d}_q\tau = \left(\frac{1-q}{1-q^{\zeta+1}} \right) (\delta-\delta_1)^{\zeta+1}. $$

Alp et al. introduced and established the q_{δ_1}-Hermite-Hadamard inequalities for convexity, which can be defined as follows,

Theorem 2 ([1, Theorem 6]). *Suppose that differentiable function* $\mathcal{H}\colon [\delta_1, \delta_2] \to \mathbb{R}$ *be convex on the interval* $[\delta_1, \delta_2]$. *In this case, the* q_{δ_1}*-Hermite-Hadamard inequalities can be stated as follows,*

$$ \mathcal{H}\left(\frac{q\delta_1 + \delta_2}{[2]_q} \right) \leq \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \mathcal{H}(\varpi)\ {}_{\delta_1}\mathrm{d}_q\varpi \leq \frac{q\mathcal{H}(\delta_1) + \mathcal{H}(\delta_2)}{[2]_q}. $$

In contrast, Bermudo et al. presented a novel characterization of q^{δ_2}-derivatives, q^{δ_2}-integrals, and the associated q^{δ_2}-Hermite-Hadamard inequalities in their study [2].

Definition 6 ([2, Definition 4]). Let $\mathcal{H}:\ [\delta_1,\delta_2]\to\mathbb{R}$ be an arbitrary function. The q^{δ_2}-derivative of $\mathcal{H}$ at $\varpi\in[\delta_1,\delta_2]$ is defined as follows,

$$ {}^{\delta_2}D_q\mathcal{H}(\varpi)=\frac{\mathcal{H}(q\varpi+(1-q)\delta_2)-\mathcal{H}(\varpi)}{(1-q)(\delta_2-\varpi)},\qquad \varpi\neq\delta_2. $$

Taking $\delta_2=1$ then we get

$$ {}^{1}D_q\mathcal{H}(\varpi)=\frac{\mathcal{H}(q\varpi+(1-q))-\mathcal{H}(\varpi)}{(1-q)(1-\varpi)},\qquad \varpi\neq 1. $$

As $\mathcal{H}$ is an arbitrary function from $[\delta_1,\delta_2]$ to $\mathbb{R}$, so for $\varpi=\delta_2$, we define

$$ {}^{\delta_2}D_q\mathcal{H}(\delta_2)=\lim_{\varpi\to\delta_2}{}^{\delta_2}D_q\mathcal{H}(\varpi) $$

if it exists and is finite.

Definition 7 ([2, Definition 6]). Let $\mathcal{H}:\ [\delta_1,\delta_2]\to\mathbb{R}$ be an arbitrary function. The q^{δ_2}-definite integral on $[\delta_1,\delta_2]$ is defined as follows,

$$ \int_{\varpi}^{\delta_2}\mathcal{H}(\varpi)\ {}^{\delta_2}\mathrm{d}_q\varpi=(1-q)(\delta_2-\varpi)\sum_{n=0}^{\infty}q^n\mathcal{H}(q^n\varpi+(1-q^n)\delta_2), \tag{2.4} $$

$$ \varpi\in[\delta_1,\delta_2]. $$

If $\delta_2=1$, then (2.4) becomes

$$ \int_{\varpi}^{1}\mathcal{H}(\varpi)\ {}^{1}\mathrm{d}_q\varpi=(1-q)(1-\varpi)\sum_{n=0}^{\infty}q^n\mathcal{H}(q^n\varpi+(1-q^n)),\qquad \varpi\in[\delta_1,\delta_2]. $$

Theorem 3 ([2, Theorem 12]). *Let $\mathcal{H}:\ [\delta_1,\delta_2]\to\mathbb{R}$ be a convex functions on $[\delta_1,\delta_2]$. In this case, the q^{δ_2}-Hermite-Hadamard inequalities can be stated as follows,*

$$ \mathcal{H}\left(\frac{\delta_1+q\delta_2}{[2]_q}\right)\leq\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\mathcal{H}(\varpi)\ {}^{\delta_2}\mathrm{d}_q\varpi\leq\frac{\mathcal{H}(\delta_1)+q\mathcal{H}(\delta_2)}{[2]_q}. $$

Based on Theorem 2 and Theorem 3, we can derive the following inequalities

Lemma 3 ([31, Lemma 1]). *Suppose we have two continuous functions, $\mathcal{H}$ and G, defined on the interval $[\delta_1,\delta_2]$ and whose values are real numbers. Under these conditions, the following equality is valid,*

$$ \int_0^c G(\varpi)\ {}^{\delta_2}D_q\mathcal{H}(\varpi\delta_1+(1-\varpi)\delta_2)\,\mathrm{d}_q\varpi $$

$$= \frac{1}{\delta_2-\delta_1}\int_0^c D_q\, G(\varpi)\mathcal{H}\left(q\varpi\delta_1+(1-q\varpi)\delta_2\right)\mathrm{d}_q\varpi - \left.\frac{G(\varpi)\mathcal{H}\left(\varpi\delta_1+(1-\varpi)\delta_2\right)}{\delta_2-\delta_1}\right|_0^c.$$

Lemma 4 ([33, Lemma 2.1])**.** *If we examine continuous functions $\mathcal{H}$ and G that are defined on the interval $[\delta_1,\delta_2]$ and whose outputs are real numbers, we can conclude that the following equality holds true,*

$$\begin{aligned}&\int_0^c G(\varpi)\ {}_{\delta_1}D_q\mathcal{H}\left(\varpi\delta_2+(1-\varpi)\delta_1\right)\mathrm{d}_q\varpi\\ &= \left.\frac{G(\varpi)\mathcal{H}\left(\varpi\delta_2+(1-\varpi)\delta_1\right)}{\delta_2-\delta_1}\right|_0^c - \frac{1}{\delta_2-\delta_1}\int_0^c D_qG(\varpi)\mathcal{H}\left(q\varpi\delta_2+(1-q\varpi)\delta_1\right)\mathrm{d}_q\varpi.\end{aligned}$$

3. q-TRAPEZOIDAL INEQUALITIES

Lemma 5. *Suppose that $I\subset\mathbb{R}$ is an invex set with respect to η_1. If for $v,\delta\in I$ with $\eta_1>0$ If $\mathcal{H}\colon [v,v+\eta_1(\delta,v)],[\delta,\delta+\eta_1(v,\delta)]\subset\mathbb{R}\to\mathbb{R}$ is q-differentiable such that ${}_vD_q\mathcal{H}$ and ${}^{\delta}D_q\mathcal{H}$ are integrable and continuous on $[v,v+\eta_1(\delta,v)],[\delta,\delta+\eta_1(v,\delta)]$, then*

$$\begin{aligned}\Phi(v,\delta) =:{}& \frac{\mathcal{H}(v+\eta_1(\delta,v))+\mathcal{H}(\delta+\eta_1(v,\delta))}{2}\\ &-\frac{1}{2\eta_1(\delta,v)}\left[\int_v^{v+\eta_1(\delta,v)}\mathcal{H}(x)\ {}_v\mathrm{d}_qx+\int_{\delta+\eta_1(v,\delta)}^{\delta}\mathcal{H}(x)\ {}^{\delta}\mathrm{d}_qx\right]\\ &=\frac{\eta_1(\delta,v)}{2}\left[\int_0^1 q\tau\ {}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau\right.\\ &\qquad\left.+\int_0^1 q\tau\ {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau\right].\end{aligned}\tag{3.1}$$

Proof. According to Lemma 4, we perform the computation

$$\begin{aligned}I_1 &= \int_0^1 q\tau\ {}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau\\ &= q\tau\frac{\mathcal{H}(v+\tau\eta_1(\delta,v))}{\eta_1(\delta,v)}\Big|_0^1-\frac{q}{\eta_1(\delta,v)}\int_0^1\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau\\ &= q\frac{\mathcal{H}(v+\eta_1(\delta,v))}{\eta_1(\delta,v)}-\frac{q}{\eta_1(\delta,v)}\\ &\quad\times\left\{\frac{1-q}{q}\sum_{n=0}^{\infty}q^n\mathcal{H}\left(v+\tau\eta_1q^n(\delta,v)\right)-\frac{(1-q)}{q}\mathcal{H}(v+\eta_1(\delta,v))\right\}\end{aligned}\tag{3.2}$$

$$\eta_1(\delta,v)I_1=\mathcal{H}(v+\eta_1(\delta,v))-\frac{1}{\eta_1(\delta,v)}\int_v^{v+\eta_1(\delta,v)}\mathcal{H}(x)\ {}_v\mathrm{d}_qx.$$

Likewise, by applying Lemma 3, we obtain

$$\begin{aligned} I_2 &= \int_0^1 q\tau\ {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau \\ &= -q\tau\frac{\mathcal{H}(\delta+\tau\eta_1(v,\delta))}{\eta_1(\delta,v)}\Big|_0^1+\frac{q}{\eta_1(\delta,v)}\int_0^1\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau \\ &= -q\frac{\mathcal{H}(\delta+\eta_1(v,\delta))}{\eta_1(\delta,v)}+\frac{q}{\eta_1(\delta,v)} \\ &\quad\times\left\{\frac{1-q}{q}\sum_{n=0}^{\infty}q^n\mathcal{H}\left(\delta+\tau\eta_1q^n(v,\delta)\right)-\frac{(1-q)}{q}\mathcal{H}(\delta+\eta_1(v,\delta))\right\} \end{aligned} \tag{3.3}$$

$$\eta_1(\delta,v)I_2=\frac{1}{\eta_1(\delta,v)}\int_{\delta+\eta_1(v,\delta)}^{\delta}\mathcal{H}(x)\ {}^{\delta}\mathrm{d}_qx-\mathcal{H}(\delta+\eta_1(v,\delta)).$$

To obtain the desired identity (3.1), we need to combine equations (3.2) and (3.3). □

Theorem 4. *Based on the assumptions in Lemma 5, the following inequality holds if both* $\left|{}_vD_q\mathcal{H}\right|$ *and* $\left|{}^{\delta}D_q\mathcal{H}\right|$ *are* (η_1,η_2)*-convex functions*

$$\begin{aligned} |\Phi(v,\delta)|\le\frac{q\eta_1(\delta,v)}{2}\Bigg[&\frac{\left|{}_vD_q\mathcal{H}(v)\right|+\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|}{[2]_q} \\ &+\frac{\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)}{[3]_q} \\ &+\frac{\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)}{[3]_q}\Bigg]. \end{aligned}$$

Proof. Using Lemma 5 and the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|$ and $\left|{}^{\delta}D_q\mathcal{H}\right|$, we obtain the following result:

$$\begin{aligned} |\Phi(v,\delta)| &\le \frac{\eta_1(\delta,v)}{2}\Bigg[\int_0^1 q\tau\ \left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|\ \mathrm{d}_q\tau \\ &\qquad+\int_0^1 q\tau\ \left|{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\right|\ \mathrm{d}_q\tau\Bigg] \\ &\le \frac{\eta_1(\delta,v)}{2}\Bigg[\int_0^1 q\tau\left\{(|{}_vD_q\mathcal{H}(v)|+\tau\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|))\right\}\ \mathrm{d}_q\tau \\ &\qquad+\int_0^1 q\tau\left\{(|{}^{\delta}D_q\mathcal{H}(\delta)|+\tau\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|))\right\}\ \mathrm{d}_q\tau \\ &= \frac{\eta_1(\delta,v)}{2}\Bigg[\left|{}_vD_q\mathcal{H}(v)\right|\int_0^1 q\tau\ \mathrm{d}_q\tau+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\int_0^1 q\tau^2\ \mathrm{d}_q\tau \end{aligned}$$

$$
\begin{aligned}
&+\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|\int_0^1 q\tau\,\mathrm{d}_q\tau \\
&\quad+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\int_0^1 q\tau^2\,\mathrm{d}_q\tau\Bigg] \\
&=\frac{\eta_1(\delta,v)}{2}\left[\left|{}_vD_q\mathcal{H}(v)\right|\frac{q}{[2]_q}+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\frac{q}{[3]_q}\right. \\
&\quad\left.+\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|\frac{q}{[2]_q}+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\frac{q}{[3]_q}\right] \\
&=\frac{\eta_1(\delta,v)}{2}\left[\left(\left|{}_vD_q\mathcal{H}(v)\right|+\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|\right)\frac{q}{[2]_q}\right. \\
&\quad+((\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|) \\
&\quad\left.+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\right)\frac{q}{[3]_q}\Bigg)\Bigg].
\end{aligned}
$$

This concludes the proof. □

Theorem 5. *In the context of Lemma 5 assumptions, the following inequality holds if* $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ *and* $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$ *with* $p_1\geq 1$ *are* (η_1,η_2)*-convex functions*

$$
\begin{aligned}
|\Phi(v,\delta)|\leq\frac{q\eta_1(\delta,v)}{2[2]_q}&\left[\left(\left|{}_vD_q\mathcal{H}(v)\right|^{p_1}\right.\right. \\
&\left.+\frac{[2]_q\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})}{[3]_q}\right)^{\frac{1}{p_1}} \\
&+\left(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\right. \\
&\left.\left.+\frac{[2]_q\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})}{[3]_q}\right)^{\frac{1}{p_1}}\right].
\end{aligned}
$$

Proof. Using the power mean inequality along with Lemma 5, we get:

$$
\begin{aligned}
|\Phi(v,\delta)|&\leq\frac{\eta_1(\delta,v)}{2}\left[\int_0^1 q\tau\,\left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|\,\mathrm{d}_q\tau\right. \\
&\quad\left.+\int_0^1 q\tau\,\left|{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\right|\,\mathrm{d}_q\tau\right] \\
&\leq\frac{\eta_1(\delta,v)}{2}\left[\left(\int_0^1 q\tau\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(\int_0^1 q\tau\,\left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|^{p_1}\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right. \\
&\quad\left.+\left(\int_0^1 q\tau\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\right.
\end{aligned}
$$

$$\times \left(\int_0^1 q\tau \left| {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta)) \right|^{p_1} \mathrm{d}_q\tau \right)^{\frac{1}{p_1}} \Bigg].$$

By using Lemma 5 and the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ and $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$, we derive:

$$\begin{aligned}
|\Phi(v,\delta)| \leq \frac{\eta_1(\delta,v)}{2} \Bigg[& \left(\int_0^1 q\tau \, \mathrm{d}_q\tau \right)^{1-\frac{1}{p_1}} \left(\int_0^1 q\tau \left\{ (|{}_vD_q\mathcal{H}(v)|^{p_1} \right.\right. \\
& \left.\left. +\tau\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1}, |{}_vD_q\mathcal{H}(v)|^{p_1})) \right\} \mathrm{d}_q\tau \right)^{\frac{1}{p_1}} \\
& + \left(\int_0^1 q\tau \, \mathrm{d}_q\tau \right)^{1-\frac{1}{p_1}} \left(\int_0^1 q\tau \left\{ (|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1} \right.\right. \\
& \left.\left. +\tau\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1}, |{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})) \right\} \mathrm{d}_q\tau \right)^{\frac{1}{p_1}} \Bigg] \\
= \frac{q\eta_1(\delta,v)}{2[2]_q} \Bigg[& \left(\left|{}_vD_q\mathcal{H}(v)\right|^{p_1} \right. \\
& \left. + \frac{[2]_q\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1}, |{}_vD_q\mathcal{H}(v)|^{p_1})}{[3]_q} \right)^{\frac{1}{p_1}} \\
& + \left(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1} \right. \\
& \left. + \frac{[2]_q\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1}, |{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})}{[3]_q} \right)^{\frac{1}{p_1}} \Bigg].
\end{aligned}$$

This concludes the proof. □

Theorem 6. *In the context of Lemma 5 assumptions, the following inequality holds if* $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ *and* $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$ *with* $p_1 > 1$ *are* (η_1,η_2)*-convex functions*

$$\begin{aligned}
|\Phi(v,\delta)| \leq \frac{q\eta_1(\delta,v)}{2} \left(\frac{1}{[r_1+1]_q} \right)^{\frac{1}{r_1}} \Bigg[& \left(\left|{}_vD_q\mathcal{H}(v)\right|^{p_1} \right. \\
& \left. + \frac{\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1}, |{}_vD_q\mathcal{H}(v)|^{p_1})}{[2]_q} \right)^{\frac{1}{p_1}} \\
& + \left(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1} \right. \\
& \left. + \frac{\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1}, |{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})}{[2]_q} \right)^{\frac{1}{p_1}} \Bigg],
\end{aligned}$$

where $r_1^{-1} + p_1^{-1} = 1$.

Proof. The Hölder inequality, combined with Lemma 5, provides

$$\begin{aligned}|\Phi(v,\delta)| &\leq \frac{\eta_1(\delta,v)}{2}\left[\int_0^1 q\tau\ \left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|\ \mathrm{d}_q\tau\right.\\ &\qquad\left.+\int_0^1 q\tau\ \left|{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\right|\ \mathrm{d}_q\tau\right]\\ &\leq \frac{\eta_1(\delta,v)}{2}\left[\left(\int_0^1 (q\tau)^{r_1}\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^1 \left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|^{p_1}\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right.\\ &\qquad\left.+\left(\int_0^1 (q\tau)^{r_1}\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^1 \left|{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\right|^{p_1}\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

By applying Lemma 5 and utilizing the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ and $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$, we derive

$$\begin{aligned}|\Phi(v,\delta)| \leq \frac{q\eta_1(\delta,v)}{2}&\left[\left(\int_0^1 (\tau)^{r_1}\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^1 \left\{(|{}_vD_q\mathcal{H}(v)|^{p_1}\right.\right.\right.\\ &\left.+\tau\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1}))\right\}\ \mathrm{d}_q\tau\Big)^{\frac{1}{p_1}}\\ &+\left(\int_0^1 (\tau)^{r_1}\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^1 \left\{(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\right.\right.\\ &\left.\left.\left.+\tau\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}))\right\}\ \mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

After applying the definition of the q-integral, we obtain the desired result. □

4. q-MIDPOINT INEQUALITIES

This section contains the proof of many correct estimates for the inequality (1.1), which is relevant to (η_1,η_2)-convex functions that are differentiable.

Lemma 6. *Suppose that $I\subset\mathbb{R}$ is an invex set with respect to η_1. If for $v,\delta\in I$ with $\eta_1>0$ If $\mathcal{H}\colon [v,v+\eta_1(\delta,v)],[\delta,\delta+\eta_1(v,\delta)]\subset\mathbb{R}\to\mathbb{R}$ is q-differentiable such that ${}_vD_q\mathcal{H}$ and ${}^{\delta}D_q\mathcal{H}$ are integrable and continuous on $[v,v+\eta_1(\delta,v)],[\delta,\delta+\eta_1(v,\delta)]$, then*

$$\begin{aligned}\Psi(v,\delta) =:\ &\mathcal{H}\left(\frac{(v+\eta_1(\delta,v))+(\delta+\eta_1(v,\delta))}{2}\right)\\ &-\frac{1}{2\eta_1(\delta,v)}\left[\int_v^{v+\eta_1(\delta,v)}\mathcal{H}(x)\ {}_v\mathrm{d}_qx\right.\\ &\left.+\int_{\delta+\eta_1(v,\delta)}^{\delta}\mathcal{H}(x)\ {}^{\delta}\mathrm{d}_qx\right]\end{aligned}$$

$$= \frac{\eta_1(\delta,v)}{2}\left[\int_0^{\frac{1}{2}} q\tau\ {}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau \right.$$
$$+\int_{\frac{1}{2}}^1 (q\tau-1)\ {}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau$$
$$+\int_0^{\frac{1}{2}} (-q\tau)\ {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau$$
$$\left.+\int_{\frac{1}{2}}^1 (1-q\tau)\ {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau\right].$$

Proof. The proof can be easily shown using the same method as in Lemma 5. □

Theorem 7. *In the context of Lemma 6 assumptions, the following inequality holds if* $\left|{}^{\delta}D_q\mathcal{H}\right|$ *and* $\left|{}_vD_q\mathcal{H}\right|$ *are* (η_1,η_2)*-convex functions*

$$|\Psi(v,\delta)| \le \frac{\eta_1(\delta,v)}{2}\left[|{}_vD_q\mathcal{H}(v)|\frac{q}{4[2]_q}\right.$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\frac{q}{8[3]_q}$$
$$+|{}_vD_q\mathcal{H}(v)|\left(\frac{q-2}{4[2]_q}\right)$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\left(\frac{-6+q^2+q}{8[2]_q[3]_q}\right)$$
$$+|{}^{\delta}D_q\mathcal{H}(\delta)|\left(\frac{q^2}{4[2]_q}\right)$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\left(\frac{q^3}{8[3]_q}\right)$$
$$+|{}^{\delta}D_q\mathcal{H}(\delta)|\left(\frac{6-3q^2-2q}{4[2]_q}\right)$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)$$
$$\left.\times\left(\frac{7q+9q^3-2q^2-8q^4-2}{8[2]_q[3]_q}\right)\right].$$

Proof. By applying Lemma 6

$$\Psi(v,\delta) \le \frac{\eta_1(\delta,v)}{2}\left[\int_0^{\frac{1}{2}} q\tau\ |{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))|\ \mathrm{d}_q\tau\right.$$
$$\left.+\int_{\frac{1}{2}}^1 (q\tau-1)\ |{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))|\ \mathrm{d}_q\tau\right.$$

$$+\int_0^{\frac{1}{2}}(-q\tau)\ |{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))|\ \mathrm{d}_q\tau$$
$$+\int_{\frac{1}{2}}^{1}(1-q\tau)\ |{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))|\ \mathrm{d}_q\tau\Bigg].$$

Utilizing the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|$ and $\left|{}^{\delta}D_q\mathcal{H}\right|$, we derive

$$\leq \frac{\eta_1(\delta,v)}{2}\Bigg[|{}_vD_q\mathcal{H}(v)|\int_0^{\frac{1}{2}}q\tau\ \mathrm{d}_q\tau$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\int_0^{\frac{1}{2}}q\tau^2\ \mathrm{d}_q\tau$$
$$+|{}_vD_q\mathcal{H}(v)|\int_{\frac{1}{2}}^{1}(q\tau-1)\ \mathrm{d}_q\tau$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\int_{\frac{1}{2}}^{1}(q\tau-1)\tau\ \mathrm{d}_q\tau$$
$$+|{}^{\delta}D_q\mathcal{H}(\delta)|\int_0^{\frac{1}{2}}(-q\tau)\mathrm{d}_q\tau$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\int_0^{\frac{1}{2}}(-q\tau^2)\mathrm{d}_q\tau$$
$$+|{}^{\delta}D_q\mathcal{H}(\delta)|\int_{\frac{1}{2}}^{1}(1-q\tau)\ \mathrm{d}_q\tau$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\int_{\frac{1}{2}}^{1}(1-q\tau)\tau\ \mathrm{d}_q\tau\Bigg].$$

After applying the definition of the q-integral, we obtain the desired result. □

Theorem 8. *In the context of Lemma 6 assumptions, the following inequality holds if* $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$ *and* $\left|{}_vD_q\mathcal{H}\right|^{p_1}, p_1\geq 1$ *are* (η_1,η_2)*-convex functions*

$$|\Psi(v,\delta)|\leq\frac{\eta_1(\delta,v)}{2}\Bigg[\left(\frac{q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\Bigg(|{}_vD_q\mathcal{H}(v)|^{p_1}\frac{q}{4[2]_q}$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})\frac{q}{8[3]_q}\Bigg)^{\frac{1}{p_1}}$$
$$+\left(\frac{q-2}{4[2]_q}\right)^{1-\frac{1}{p_1}}\Bigg(|{}_vD_q\mathcal{H}(v)|^{p_1}\left(\frac{q-2}{4[2]_q}\right)$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})\left(\frac{-6+q^2+q}{8[2]_q[3]_q}\right)\Bigg)^{\frac{1}{p_1}}$$

$$+\left(\frac{q^2}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\frac{q^2}{4[2]_q}\right.$$
$$\left.+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\frac{q^3}{8[3]_q}\right)^{\frac{1}{p_1}}$$
$$+\left(\frac{6-3q^2-2q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\left(\frac{6-3q^2-2q}{4[2]_q}\right)\right.$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})$$
$$\left.\left.\times\left(\frac{7q+9q^3-2q^2-8q^4-2}{8[2]_q[3]_q}\right)\right)^{\frac{1}{p_1}}\right].$$

Proof. The power mean inequality, in conjunction with Lemma 6 and utilizing the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ and $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$, we derive

$$=\frac{\eta_1(\delta,v)}{2}\left[\left(\int_0^{\frac{1}{2}}q\tau\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(|{}_vD_q\mathcal{H}(v)|^{p_1}\int_0^{\frac{1}{2}}q\tau\,\mathrm{d}_q\tau\right.\right.$$
$$\left.+\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})\int_0^{\frac{1}{2}}q\tau^2\,\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}$$
$$+\left(\int_{\frac{1}{2}}^1(q\tau-1)\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(|{}_vD_q\mathcal{H}(v)|^{p_1}\int_{\frac{1}{2}}^1(q\tau-1)\,\mathrm{d}_q\tau\right.$$
$$\left.+\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})\int_{\frac{1}{2}}^1(q\tau-1)\tau\,\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}$$
$$+\left(\int_0^{\frac{1}{2}}(-q\tau)\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\int_0^{\frac{1}{2}}(-q\tau)\mathrm{d}_q\tau\right.$$
$$\left.+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\int_0^{\frac{1}{2}}(-q\tau^2)\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}$$
$$+\left(\int_{\frac{1}{2}}^1(1-q\tau)\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\int_{\frac{1}{2}}^1(1-q\tau)\,\mathrm{d}_q\tau\right.$$
$$\left.\left.+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\int_{\frac{1}{2}}^1(1-q\tau)\tau\,\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right].$$

After applying the definition of the q-integral, we obtain the desired result. □

Theorem 9. *In the context of Lemma 6 assumptions, the following inequality holds if* $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$ *and* $\left|{}_{v}D_q\mathcal{H}\right|^{p_1}, p_1>1$ *are* (η_1,η_2)*-convex functions*

$$
\begin{aligned}
|\Psi(v,\delta)| \leq \frac{\eta_1(\delta,v)}{2}\Bigg[& q\left(\frac{1}{2^{r_1+1}\,[r_1+1]_q}\right)^{\frac{1}{r_1}}\Bigg(\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\left(\frac{q}{4[2]_q}\right)\\
&+\eta_2\left(\left|{}_{v}D_q\mathcal{H}(\delta)\right|^{p_1},\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\right)\left(\frac{q}{8[3]_q}\right)\Bigg)^{\frac{1}{p_1}}\\
&+\left(\int_{\frac{1}{2}}^{1}(1-q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\Bigg(\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\left(\frac{q-2}{4[2]_q}\right)\\
&+\eta_2\left(\left|{}_{v}D_q\mathcal{H}(\delta)\right|^{p_1},\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\right)\left(\frac{-6+q^2+q}{8[2]_q[3]_q}\right)\Bigg)^{\frac{1}{p_1}}\\
&+q\left(\frac{1}{2^{r_1+1}\,[r_1+1]_q}\right)^{\frac{1}{r_1}}\Bigg(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\left(\frac{q^2}{4[2]_q}\right)\\
&+\eta_2\left(\left|{}^{\delta}D_q\mathcal{H}(v)\right|^{p_1},\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\right)\left(\frac{q^3}{8[3]_q}\right)\Bigg)^{\frac{1}{p_1}}\\
&+\left(\int_{\frac{1}{2}}^{1}(1-q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\Bigg(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\left(\frac{6-3q^2-2q}{4[2]_q}\right)\\
&+\eta_2\left(\left|{}^{\delta}D_q\mathcal{H}(v)\right|^{p_1},\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\right)\\
&\times\left(\frac{7q+9q^3-2q^2-8q^4-2}{8[2]_q[3]_q}\right)\Bigg)^{\frac{1}{p_1}}\Bigg],
\end{aligned}
$$

where $p_1^{-1}+r_1^{-1}=1$.

Proof. The Hölder inequality, combined with Lemma 6 and utilizing the (η_1,η_2)-convexity of $\left|{}_{v}D_q\mathcal{H}\right|^{p_1}$ and $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$, we derive

$$
\begin{aligned}
|\Psi(v,\delta)| \leq \frac{\eta_1(\delta,v)}{2}\Bigg[& \left(\int_0^{\frac{1}{2}}(q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\Bigg(\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\int_0^{\frac{1}{2}}q\tau\,\mathrm{d}_q\tau\\
&+\eta_2\left(\left|{}_{v}D_q\mathcal{H}(\delta)\right|^{p_1},\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\right)\int_0^{\frac{1}{2}}q\tau^2\,\mathrm{d}_q\tau\Bigg)^{\frac{1}{p_1}}\\
&+\left(\int_{\frac{1}{2}}^{1}(q\tau-1)^{r_1}\,\mathrm{d}_q\tau\right)^{1-\frac{1}{r_1}}\Bigg(\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\int_{\frac{1}{2}}^{1}(q\tau-1)\,\mathrm{d}_q\tau
\end{aligned}
$$

$$+\eta_2(|_vD_q\mathcal{H}(\delta)|^{p_1},|_vD_q\mathcal{H}(v)|^{p_1})\int_{\frac{1}{2}}^{1}(q\tau-1)\tau\,\mathrm{d}_q\tau\Bigg)^{\frac{1}{p_1}}$$

$$+\left(\int_0^{\frac{1}{2}}(-q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{1-\frac{1}{r_1}}\Bigg(|^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\int_0^{\frac{1}{2}}(-q\tau)\mathrm{d}_q\tau$$

$$+\eta_2(|^{\delta}D_q\mathcal{H}(v)|^{p_1},|^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\int_0^{\frac{1}{2}}(-q\tau^2)\mathrm{d}_q\tau\Bigg)^{\frac{1}{p_1}}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{1-\frac{1}{r_1}}\Bigg(|^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\int_{\frac{1}{2}}^{1}(1-q\tau)\,\mathrm{d}_q\tau$$

$$+\eta_2(|^{\delta}D_q\mathcal{H}(v)|^{p_1},|^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\int_{\frac{1}{2}}^{1}(1-q\tau)\tau\,\mathrm{d}_q\tau\Bigg)^{\frac{1}{p_1}}\Bigg].$$

After applying the definition of the q-integral, we obtain the desired result. □

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