Miskolc Mathematical Notes
Vol. 26 (2025), No. 2, pp. 727–748

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4990

STABILITY OF CONFORMABLE NON-AUTONOMOUS NON-INSTANTANEOUS IMPULSIVE DIFFERENTIAL EQUATIONS

YUANLIN DING

Received 07 March, 2024

Abstract. This paper focus on a type of conformable non-autonomous non-instantaneous impulsive equation. Firstly, we give a kind of non-autonomous conformable Cauchy matrix to present the solution of linear and nonlinear systems. Then, we investigate the asymptotic stability of linear homogeneous problem and the exponential stability of linear perturbed problem. Also, one presents the solution of nonlinear problem and verifies its Ulam-Hyers-Rassias stability.

2010 *Mathematics Subject Classification:* 34A37; 34D20

Keywords: asymptotical stability, non-autonomous, Ulam-Hyers-Rassias stability, conformable derivative

1. INTRODUCTION

Since the raise of impulsive equations, many papers based on the impulsive effects are published [5–7]. With the deepening of the research, Hernández and O'Regan firstly introduce the non-instantaneous impulsive differential equations [8] which can describe the processes in pharmacotherapy systems. Inspired by this work, the basic theory such as stability, periodicity, controllability are studied in the [3, 4, 9, 13, 15, 17, 18, 20].

There are many papers studying the non-instantaneous impulsive differential equations without non-autonomous effects. The behavior of the autonomous system depends only on the current state. However, in many systems, the state or behavior depends not only on the current state, but also on external factors. Hence, the non-autonomous non-instantaneous impulsive differential equations are worthy to consider. Also, motivated by the raise of conformable derivative in [10] and the relating works [1, 11, 12, 14, 19], we present the following problems:

This work is partially supported by the National Natural Science Foundation of China (11661016).

(i) Conformable linear homogeneous problem:

$$\left\{\begin{array}{ll} \mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) & = \mathcal{A}(t)\mathcal{B}(t),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\ \mathcal{B}(t_l^+) & = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-),\ l\in N,\\ \mathcal{B}(t) & = \mathcal{C}_l(t)\mathcal{B}(t_l^-),\ t\in(t_l,s_l],\ l\in N,\\ \mathcal{B}(s_l^+) & = \mathcal{B}(s_l^-),\ l\in N,\\ \mathcal{B}(u) & = \mathcal{B}_u\in\mathbb{R}^n, \end{array}\right. \tag{1.1}$$

for $N_0=\{0,1,2,\cdots,w\}$ and $N=\{1,2,\cdots,w\}$. One sets t_l as the impulsive point and $s_0=\mathcal{U}<t_1<s_1\cdots<s_i<t_{i+1}<\cdots<t_w=\mathcal{V}$. Suppose that $\mathcal{A}(\cdot):\bigcup_{l=0}^{\infty}(s_l,t_{l+1}]\to\mathbb{R}^{n\times n}$ and $\mathcal{C}_l(\cdot):\bigcup_{l=1}^{\infty}(t_l,s_l]\to\mathbb{R}^{n\times n}$ are $n\times n$ matrices.

(ii) Conformable linear nonhomogeneous problem:

$$\left\{\begin{array}{ll} \mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) & = \mathcal{A}(t)\mathcal{B}(t)+\mathcal{J}(t)\mathcal{B}(t),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\ \mathcal{B}(t_l^+) & = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-)+\mathcal{M}_l\mathcal{B}(t_l^-),\ l\in N,\\ \mathcal{B}(t) & = \mathcal{C}_l(t)\mathcal{B}(t_l^-)+\mathcal{M}_l\mathcal{B}(t_l^-),\ t\in(t_l,s_l],\ l\in N,\\ \mathcal{B}(s_l^+) & = \mathcal{B}(s_l^-),\ l\in N,\\ \mathcal{B}(u) & = \mathcal{B}_u\in\mathbb{R}^n, \end{array}\right. \tag{1.2}$$

where $\mathcal{J}(\cdot):\bigcup_{l=0}^{\infty}(s_l,t_{l+1}]\to\mathbb{R}^{n\times n}$ and $\mathcal{M}_l\in\mathbb{R}^{n\times n}$.

(iii) Conformable nonlinear non-autonomous problem:

$$\left\{\begin{array}{ll} \mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) & = \mathcal{A}(t)\mathcal{B}(t)+z(t,\mathcal{B}(t)),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\ \mathcal{B}(t_l^+) & = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-)+\mathcal{L}_l,\ l\in N,\\ \mathcal{B}(t) & = \mathcal{C}_l(t)\mathcal{B}(t_l^-)+\mathcal{L}_l,\ t\in(t_l,s_l],\ l\in N,\\ \mathcal{B}(s_l^+) & = \mathcal{B}(s_l^-),\ l\in N,\\ \mathcal{B}(u) & = \mathcal{B}_u\in\mathbb{R}^n, \end{array}\right. \tag{1.3}$$

where $z(\cdot,\mathcal{B}(\cdot)):\bigcup_{l=0}^{\infty}(s_l,t_{l+1}]\times\mathbb{R}^n\to\mathbb{R}^n$ and $\mathcal{L}_l\in\mathbb{R}^n$.

Section 2 presents the basic theories and introduces the notation of non-autonomous conformable Cauchy matrix to present the solution of linear homogeneous problem. Next, we derive the norm estimation about $\Gamma(t,\mathcal{U})$, even obtain many sufficient conditions about linear equations which are asymptotically stable. In Section 3, we use the method of constant variation to present the solution of (1.2) and obtain its asymptotical stability. Then on this basis, using the classical contracting mapping principle, we discuss existence and uniqueness of solution about (1.3). After

introducing the notation of non-instantaneous impulsive differential equations Ulam-Hyers-Rassias stability, the sufficient conditions of Ulam-Hyers-Rassias stability of nonlinear non-instantaneous impulsive equations are also derived.

2. PRELIMINARY

Set $\mathbb{U}=[\mathcal{U},\mathcal{V}]$ and $PC(\mathbb{U},\mathbb{R}^n):=\{\mathcal{B}:\mathbb{U}\to\mathbb{R}^n:\mathcal{B}\in C\big((t_l,t_{l+1}],\mathbb{R}^n\big),\ l=0,1,\cdots,$ there exists $\mathcal{B}(t_l^-)$ and $\mathcal{B}(t_l^+),\ l=1,2,\cdots,$ with $\mathcal{B}(t_l^-)=\mathcal{B}(t_l)\}$, where $C\big((t_l,t_{l+1}],\mathbb{R}^n\big)$ is the space of all continuous functions from $(t_l,t_{l+1}]$ into $\mathbb{R}^n$ endowed with the Chebyshev PC-norm $\|\mathcal{B}\|_{PC}:=\sup\{\|\mathcal{B}(t)\|:t\in\mathbb{U}\}$. For $\zeta=(\zeta_1,\cdots,\zeta_n)^\top\in\mathbb{R}^n$, its vector norm is $\|\zeta\|=\sum_{l=1}^{n}|\zeta_l|$. For $\alpha\in\mathbb{R}^{n\times n}$, its matrix norm is $\|\alpha\|=\max_{\|\mathcal{B}\|=1}\|\alpha\mathcal{B}\|$.

Definition 1. (see [2, Definition 2.1]) The conformable derivative of $\mathcal{B}:\mathbb{U}\to\mathbb{R}$ is

$$\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)=\lim_{\varepsilon\to0}\frac{\mathcal{B}(t+\varepsilon(t-\mathcal{U})^{1-\varsigma})-\mathcal{B}(t)}{\varepsilon},\ t\geq\mathcal{U},\ 0<\varsigma<1,$$
$$\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(\mathcal{U})=\lim_{t\to\mathcal{U}^+}\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t).$$

Remark 1. The conformable derivative $\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)$ exists iff $\mathcal{B}$ is differentiable at t and $\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)=(t-\mathcal{U})^{1-\varsigma}\mathcal{B}'(t)$ for $t>\mathcal{U}$.

Definition 2. (see [2, Notation]) The conformable integral of a function $\mathcal{B}:\mathbb{U}\to\mathbb{R}$ is

$$\mathfrak{I}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)=\int_{\mathcal{U}}^{t}\mathcal{B}(s)\mathcal{B}_{\varsigma}(s,\mathcal{U})=\int_{\mathcal{U}}^{t}(s-\mathcal{U})^{\varsigma-1}\mathcal{B}(s)ds,\ t\geq\mathcal{U},\ 0<\varsigma<1,$$

if $\mathcal{U}=0$, then we write $\mathcal{B}_{\varsigma}(s,\mathcal{U})$ as $\mathcal{B}_{\varsigma}(s)$.

Lemma 1. *A solution $\mathcal{B}\in C(\mathbb{U},\mathbb{R}^n)$ of*

$$\begin{cases}\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) &= \mathcal{A}(t)\mathcal{B}(t),\ 0<\varsigma<1,\\ \mathcal{B}(s) &= \mathcal{B}_s,\ t>s\geq\mathcal{U}.\end{cases}$$

is

$$\mathcal{B}(t)=\Xi(t,s)\mathcal{B}_s,$$

where $\Xi(\cdot,\cdot)$ is the Cauchy matrix for $s_l<s\leq t\leq t_{l+1}$.

Set $\|\Xi(t,s)\|\leq e^{\int_s^t\|\mathcal{A}(\theta)\|(\theta-\mathcal{U})^{\varsigma-1}d\theta},\ t,s\in(s_l,t_{l+1}]$ and $\xi_l=\max_{t\in(s_l,t_{l+1}]}\|\mathcal{A}(t)\|$.

Theorem 1. *Set $\mathcal{B}(t,s,\mathcal{B}_s)$ as the solution of (1.1) with $\mathcal{B}(s)=\mathcal{B}_s$. Then*

$$\mathcal{B}(t):=\mathcal{B}(t,s,\mathcal{B}_s)=\Gamma(t,s)\mathcal{B}_s,\ \mathcal{U}\leq s\leq t,$$

where $\Gamma(\cdot,\cdot)$ *is*

$$\Gamma(t,s)=\begin{cases}\Xi(t,s),\ t,s\in(s_l,t_{l+1}],\ l\in N_0;\\ C_l(t)I_l^{-1}(s),\ t,s\in(t_l,s_l],\ l\in N;\\ C_{\chi(\mathcal{U},t)}(t)\prod\limits_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_{\chi(\mathcal{U},s)+1},s),\\ s\in(s_{\chi(\mathcal{U},s)}\le t_{\chi(\mathcal{U},s)+1}],\ t\in(t_{\chi(\mathcal{U},t)},s_{\chi(\mathcal{U},t)}];\\ \Xi(t,s_{\chi(\mathcal{U},t)})C_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)}^-)\prod\limits_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_{\chi(\mathcal{U},s)+1},s),\\ s\in(s_{\chi(\mathcal{U},s)}\le t_{\chi(\mathcal{U},s)+1}],\ t\in(s_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}];\\ \Xi(t,s_{\chi(\mathcal{U},t)})C_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)}^-)\prod\limits_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]I_{\chi(\mathcal{U},s)}^{-1}(s),\\ s\in(t_{\chi(\mathcal{U},s)},s_{\chi(\mathcal{U},s)}],\ t\in(s_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}];\\ C_{\chi(\mathcal{U},t)}(t)\prod\limits_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]I_{\chi(\mathcal{U},s)}^{-1}(s),\\ s\in(t_{\chi(\mathcal{U},s)},s_{\chi(\mathcal{U},s)}],\ t\in(t_{\chi(\mathcal{U},t)},s_{\chi(\mathcal{U},t)}].\end{cases}$$

in which $\chi(\mathcal{U},t)$ *is the number of the impulsive points between* $\mathcal{U}$ *and* t. *When* $\chi(\mathcal{U},t)=\chi(\mathcal{U},s)$, *we have* $\prod\limits_{l=\chi(\mathcal{U},s)}^{\chi(\mathcal{U},t)-1}=E$, *where* E *is the identity matrix.*

In particular with $s=\mathcal{U}$,

$$\mathcal{B}(t):=\mathcal{B}(t,\mathcal{U},\mathcal{B}_{\mathcal{U}})=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}} \tag{2.1}$$

$$=\begin{cases}\Xi(t,\mathcal{U})\mathcal{B}_{\mathcal{U}},\ t\in(s_0,t_1];\\ C_{\chi(\mathcal{U},t)}(t)\prod\limits_{l=2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_1,\mathcal{U})\mathcal{B}_{\mathcal{U}},\\ t\in(t_{\chi(\mathcal{U},t)},s_{\chi(\mathcal{U},t)}];\\ \Xi(t,s_{\chi(\mathcal{U},t)})C_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)}^-)\prod\limits_{l=2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_1,\mathcal{U})\mathcal{B}_{\mathcal{U}},\\ t\in(s_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}].\end{cases}$$

Proof. Case 1: $\chi(\mathcal{U},t)=\chi(\mathcal{U},s)$.
(i) Consider any $t,s\in(s_l,t_{l+1}],\ l\in N_0$.
If $t\in(\mathcal{U},t_1]$,

$$\mathcal{B}(t)=\Xi(t,\mathcal{U})\mathcal{B}_{\mathcal{U}},$$
$$\mathcal{B}(s)=\Xi(s,\mathcal{U})\mathcal{B}_{\mathcal{U}}.$$

Next, for $t,s\in(s_1,t_2]$,

$$\mathcal{B}(t)=\Xi(t,s_1)\mathcal{B}_{s_1},$$

$$\mathcal{B}(s) = \Xi(s, s_1)\mathcal{B}_{s_1}.$$

Summarizing, we obtain that

$$\Gamma(t,s) = \Xi(t,s).$$

(ii) Consider any $t, s \in (t_l, s_l]$, $l \in N$,

$$\mathcal{B}(t) = \mathcal{C}_l(t)\mathcal{B}(t_l^-),$$
$$\mathcal{B}(s) = \mathcal{C}_l(s)\mathcal{B}(t_l^-),$$

so $\Gamma(t,s) = \mathcal{C}_l(t)\mathcal{C}_l^{-1}(s)$.

(iii) Consider any $s \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\begin{aligned}\mathcal{B}(t) &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{B}(s^+_{\chi(\mathcal{U},t)}) \\ &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)})\mathcal{C}^{-1}_{\chi(\mathcal{U},t)}(s)\mathcal{B}(s),\end{aligned}$$

so

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)})\mathcal{C}^{-1}_{\chi(\mathcal{U},t)}(s).$$

Case 2: $\chi(\mathcal{U},t) = \chi(\mathcal{U},s) + 1$.

(i) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\mathcal{B}(t) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\mathcal{B}(t^-_{\chi(\mathcal{U},t)}) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s)\mathcal{B}(s),$$

so

$$\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s).$$

(ii) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\begin{aligned}\mathcal{B}(t) &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{B}(s^+_{\chi(\mathcal{U},t)}) \\ &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\mathcal{B}(t^-_{\chi(\mathcal{U},t)}) \\ &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(s_{\chi(\mathcal{U},t)}, s)\mathcal{B}(s),\end{aligned}$$

so

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(s_{\chi(\mathcal{U},t)}, s).$$

(iii) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\begin{aligned}\mathcal{B}(t) &= \mathcal{C}_{\chi(\mathcal{U},t)}(t)\mathcal{B}(t^-_{\chi(\mathcal{U},t)}) \\ &= \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)})\mathcal{B}(s^+_{\chi(\mathcal{U},s)}) \\ &= \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)})\mathcal{C}_{\chi(\mathcal{U},s)}(s_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s)\mathcal{B}(s),\end{aligned}$$

so

$$\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)})\mathcal{C}_{\chi(\mathcal{U},s)}(s_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).$$

(iv) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\mathcal{B}(t) = \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{B}(s^+_{\chi(\mathcal{U},t)})$$

$$
\begin{aligned}
&= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \mathcal{B}(t^-_{\chi(\mathcal{U},t)}) \\
&= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)}) \mathcal{B}(s^+_{\chi(\mathcal{U},s)}) \\
&= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_l(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)}) \mathcal{C}_{\chi(\mathcal{U},s)}(s_{\chi(\mathcal{U},s)}) \mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s) \mathcal{B}(s),
\end{aligned}
$$

so

$$
\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)}) \mathcal{C}_{\chi(\mathcal{U},s)}(s_{\chi(\mathcal{U},s)}) \mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).
$$

Case 3: $\chi(\mathcal{U},t) = \chi(\mathcal{U},s) + 2$.

(i) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$
\begin{aligned}
\mathcal{B}(t) &= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \mathcal{B}(t^-_{\chi(\mathcal{U},t)}) \\
&= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{B}(s^+_{\chi(\mathcal{U},s)+1}) \\
&= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{s_{\chi(\mathcal{U},s)+1}}(s^-_{\chi(\mathcal{U},s)+1}) \mathcal{B}(t^-_{\chi(\mathcal{U},s)+1}) \\
&= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{s_{\chi(\mathcal{U},s)+1}}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s) \mathcal{B}(s),
\end{aligned}
$$

so

$$
\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{s_{\chi(\mathcal{U},s)+1}}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s).
$$

(ii) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$
\begin{aligned}
\mathcal{B}(t) &= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{B}(s^+_{\chi(\mathcal{U},t)}) \\
&= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \mathcal{B}(t^-_{\chi(\mathcal{U},t)}) \\
&= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)-1}) \mathcal{B}(s^+_{\chi(\mathcal{U},t)-1}) \\
&= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)-1}) \\
&\qquad \times \mathcal{C}_{\chi(\mathcal{U},t)-1}(s^-_{\chi(\mathcal{U},s)+1}) \mathcal{B}(t^-_{\chi(\mathcal{U},s)+1}) \\
&= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)-1}) \\
&\qquad \times \mathcal{C}_{\chi(\mathcal{U},t)-1}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s) \mathcal{B}(s),
\end{aligned}
$$

so

$$
\begin{aligned}
\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)}) &\mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)-1}) \\
&\times \mathcal{C}_{\chi(\mathcal{U},t)-1}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s).
\end{aligned}
$$

(iii) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$
\begin{aligned}
\mathcal{B}(t) &= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \mathcal{B}(t^-_{\chi(\mathcal{U},t)}) \\
&= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{B}(s^+_{\chi(\mathcal{U},s)+1}) \\
&= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1}) \mathcal{B}(t^-_{\chi(\mathcal{U},s)+1}) \\
&= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})
\end{aligned}
$$

$$\begin{aligned}
&\times \mathcal{B}(s^+_{\chi(\mathcal{U},s)})\\
&= \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\\
&\quad \times \mathcal{C}_{\chi(\mathcal{U},s)}(s^-_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s)\mathcal{B}(s),
\end{aligned}$$

so

$$\begin{aligned}
\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\\
\times \mathcal{C}_{\chi(\mathcal{U},s)}(s^-_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).
\end{aligned}$$

(iv) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\begin{aligned}
\mathcal{B}(t) &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{B}(s^+_{\chi(\mathcal{U},t)})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\mathcal{B}(t^-_{\chi(\mathcal{U},t)})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{B}(s^+_{\chi(\mathcal{U},s)+1})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\\
&\quad \times \mathcal{B}(t^-_{\chi(\mathcal{U},s)+1})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\\
&\quad \times \Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\mathcal{B}(s^+_{\chi(\mathcal{U},s)})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\\
&\quad \times \Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\mathcal{C}_{\chi(\mathcal{U},s)}(s^-_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s)\mathcal{B}(s),
\end{aligned}$$

so

$$\begin{aligned}
\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\\
\times \Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\mathcal{C}_{\chi(\mathcal{U},s)}(s^-_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).
\end{aligned}$$

Case 4: General kinds of $\chi(\mathcal{U},t)$ and $\chi(\mathcal{U},s)$.

(i) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \prod_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1})\mathcal{C}_{l-1}(s^-_{l-1})]\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).$$

(ii) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t) \prod_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1})\mathcal{C}_{l-1}(s^-_{l-1})]\Xi(t_{\chi(\mathcal{U},s)+1}, s).$$

(iii) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)}) C_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \prod_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] \Xi(t_{\chi(\mathcal{U},s)+1}, s).$$

(iv) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\Gamma(t,s) = C_{\chi(\mathcal{U},t)}(t) \prod_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] C^{-1}_{\chi(\mathcal{U},s)}(s).$$

Thus $\Gamma(t,s)$ can be written in the form:

$$\Gamma(t,s) = \begin{cases} \Xi(t,s),\ t,s \in (s_l, t_{l+1}],\ l \in N_0; \\ C_l(t) C_l^{-1}(s),\ t,s \in (t_l, s_l],\ l \in N; \\ C_{\chi(\mathcal{U},t)}(t) \prod\limits_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] \Xi(t_{\chi(\mathcal{U},s)+1}, s), \\ s \in (s_{\chi(\mathcal{U},s)} \le t_{\chi(\mathcal{U},s)+1}],\ t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]; \\ \Xi(t, s_{\chi(\mathcal{U},t)}) C_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \prod\limits_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] \Xi(t_{\chi(\mathcal{U},s)+1}, s), \\ s \in (s_{\chi(\mathcal{U},s)} \le t_{\chi(\mathcal{U},s)+1}],\ t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]; \\ \Xi(t, s_{\chi(\mathcal{U},t)}) C_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \prod\limits_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] C^{-1}_{\chi(\mathcal{U},s)}(s), \\ s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}],\ t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]; \\ C_{\chi(\mathcal{U},t)}(t) \prod\limits_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] C^{-1}_{\chi(\mathcal{U},s)}(s), \\ s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}],\ t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}], \end{cases}$$

and $\Gamma(\cdot,\cdot)$ is called non-autonomous conformable Cauchy matrix. □

Definition 3. (1.1) is exponentially stable if there are $c_1 \ge 1$ and $c_2 > 0$ satisfying

$$\|\Gamma(t,s)\| \le c_1 e^{-c_2(t-s)},\ \mathcal{U} \le s \le t.$$

Lemma 2. *(see [16]) Let a nonnegative piecewise continuous function $\mathcal{B}(t)$ satisfies the inequality*

$$\mathcal{B}(t) \le B + \int_{\mathcal{U}}^{t} \nu(s)\mathcal{B}(s)ds + \sum_{\mathcal{U}<t_l<t} \alpha_l \mathcal{B}(t_l^-),\ l = 0,1,\cdots,m,$$

where $B \ge 0$, α_l, $\nu(t) > 0$, t_l are the first kind discontinuity points of the function $\mathcal{B}(t)$. Then the following estimate holds:

$$\mathcal{B}(t) \le B \prod_{\mathcal{U}<t_l<t} (1+\alpha_l) e^{\int_{\mathcal{U}}^{t} \nu(s)ds}.$$

3. Linear Homogeneous Problem

In this section, we present the stability of linear homogeneous problem.
Let $\xi = \max_{l\in N_0} \xi_l$.

Lemma 3. *For any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$, there is*

$$\|\Gamma(t,\mathcal{U})\| \le e^{\frac{\xi}{\varsigma}[((t-\mathcal{U})^\varsigma-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^\varsigma)+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^\varsigma-(s_l-\mathcal{U})^\varsigma)]}\prod_{l=1}^{\chi(\mathcal{U},t)}\|C_l(s_l^-)\|.$$

and for any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$, there is

$$\|\Gamma(t,\mathcal{U})\| \le e^{\frac{\xi}{\varsigma}\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^\varsigma-(s_l-\mathcal{U})^\varsigma)}\|\mathcal{C}_{\chi(\mathcal{U},t)}(t)\|\prod_{l=1}^{\chi(\mathcal{U},t)-1}\|C_l(s_l^-)\|.$$

Proof. By (2.1), for any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\begin{aligned}
\|\Gamma(t,\mathcal{U})\| &= \|\Xi(t,s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\prod_{l=2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_1,\mathcal{U})\| \\
&\le e^{\frac{\xi_{\chi(\mathcal{U},t)}}{\varsigma}((t-\mathcal{U})^\varsigma-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^\varsigma)}\|\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\|\prod_{l=2}^{\chi(\mathcal{U},t)}[e^{\frac{\xi_l}{\varsigma}((t_l-\mathcal{U})^\varsigma-(s_{l-1}-\mathcal{U})^\varsigma)} \\
&\quad\times\|C_{l-1}(s_{l-1}^-)\|]e^{\frac{\xi_0}{\varsigma}(t_1-\mathcal{U})^\varsigma} \\
&\le e^{\frac{\xi}{\varsigma}[((t-\mathcal{U})^\varsigma-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^\varsigma)+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^\varsigma-(s_l-\mathcal{U})^\varsigma)]}\prod_{l=1}^{\chi(\mathcal{U},t)}\|C_l(s_l^-)\|.
\end{aligned}$$

For any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\begin{aligned}
\|\Gamma(t,\mathcal{U})\| &= \|\mathcal{C}_{\chi(\mathcal{U},t)}(t)\prod_{l=2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_1,\mathcal{U})\| \\
&\le \|\mathcal{C}_{\chi(\mathcal{U},t)}(t)\|\prod_{l=2}^{\chi(\mathcal{U},t)}[e^{\frac{\xi_l}{\varsigma}((t_l-\mathcal{U})^\varsigma-(s_{l-1}-\mathcal{U})^\varsigma)}\|C_{l-1}(s_{l-1}^-)\|]e^{\frac{\xi_1}{\varsigma}(t_1-\mathcal{U})^\varsigma} \\
&\le e^{\xi\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^\varsigma-(s_l-\mathcal{U})^\varsigma)}\|\mathcal{C}_{\chi(\mathcal{U},t)}(t)\|\prod_{l=1}^{\chi(\mathcal{U},t)-1}\|C_l(s_l^-)\|.
\end{aligned}$$

□

Definition 4. The solution $\mathcal{B}(t)$ of (1.1) is locally asymptotically stable if there exists $\delta > 0$ such that $\|\mathcal{B}_\mathcal{U}\| < \delta$, it holds

$$\lim_{t\to\infty}\|\mathcal{B}(t,\mathcal{U},\mathcal{B}_\mathcal{U})\| = 0.$$

If δ is arbitrary, then $\mathcal{B}(t)$ is globally asymptotically stable.

For the next theorem, we assume that s_l and t_{l+1} satisfy

$$(t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma}\le\eta<\infty,\ l=0,1,2,\cdots. \tag{3.1}$$

and define $\zeta=\sup\limits_{l\ge1}\max\limits_{t\in(t_l,s_l]}\|C_l(t)\|$.

Theorem 2. *Assume that* (3.1) *holds. If*

$$\frac{\eta}{\varsigma}+\frac{1}{\xi}\ln\zeta<0, \tag{3.2}$$

then (1.1) *is asymptotically stable.*

Proof. Combining Lemma 3 with (3.1), one has

$$\begin{aligned}\|\Gamma(t,\mathcal{U})\|&\le e^{\frac{\xi}{\varsigma}[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]}\zeta^{\chi(\mathcal{U},t)}\\&\le e^{\frac{\xi}{\varsigma}[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}\eta]}\zeta^{\chi(\mathcal{U},t)}\\&\le e^{\frac{\xi\eta}{\varsigma}}e^{\chi(\mathcal{U},t)(\frac{\xi\eta}{\varsigma}+\ln\zeta)}.\end{aligned}$$

By (3.2), there is

$$e^{\xi(\frac{\eta}{\varsigma}+\frac{1}{\xi}\ln\zeta)}<1,$$

so

$$\lim_{t\to\infty}\|\Gamma(t,\mathcal{U})\|\le e^{\frac{\xi\eta}{\varsigma}}e^{\chi(\mathcal{U},t)(\frac{\xi\eta}{\varsigma}+\ln\zeta)}\to0.$$

□

Theorem 3. *Suppose that*

$$\lim_{t\to\infty}\frac{\chi(\mathcal{U},t)}{((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})}:=\rho<\infty. \tag{3.3}$$

If $\vartheta=\frac{\xi}{\varsigma}+\rho\ln\zeta<0$, *then* (1.1) *is asymptotically stable.*

Proof.

$$\begin{aligned}\|\Gamma(t,\mathcal{U})\|&\le e^{\frac{\xi}{\varsigma}[(t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]}\zeta^{\chi(\mathcal{U},t)}\\&\le e^{\vartheta[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]}\\&\quad\times e^{\ln\zeta\left(\chi(\mathcal{U},t)-\rho[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]\right)}.\end{aligned}$$

Using (3.3), for any enough large t, one has

$$\left|\frac{\chi(\mathcal{U},t)}{((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})}-\rho\right|\le-\frac{\vartheta}{2\ln\zeta}.$$

Then

$$
\begin{aligned}
&e^{\ln\zeta\left(\chi(\mathcal{U},t)-\rho[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]\right)}\\
&=e^{\ln\zeta[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]\left(\frac{\chi(\mathcal{U},t)}{((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})}-\rho\right)}\\
&\le e^{-\frac{\vartheta}{2}[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]}.
\end{aligned}
$$

Hence,

$$
\lim_{t\to\infty}\|\Gamma(t,\mathcal{U})\|\le e^{\frac{\vartheta}{2}[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]}\to 0.
$$

□

4. Linear Perturbed Problem

This section considers the stability of (1.2) by $\Gamma(\cdot,\cdot)$.

We firstly study the following problem:

$$
\begin{cases}
\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)=\mathcal{A}(t)\mathcal{B}(t)+j(t),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\
\mathcal{B}(t_l^{+})=\mathcal{C}_l(t)\mathcal{B}(t_l^{-})+m_l,\ l\in N,\\
\mathcal{B}(t)=\mathcal{C}_l(t)\mathcal{B}(t_l^{-})+m_l,\ t\in(t_l,s_l],\ l\in N,\\
\mathcal{B}(s_l^{+})=\mathcal{B}(s_l^{-}),\ l\in N,
\end{cases}
\tag{4.1}
$$

where $\mathcal{A}(t),\ \mathcal{C}_l(t),\ s_l,\ t_l$ are defined in (1.1), $m_l\in\mathbb{R}^n$ and $j\in C(\mathbb{U},\mathbb{R}^n)$.

Theorem 4. *A solution $\mathcal{B}\in PC(\mathbb{U},\mathbb{R}^n)$ of (4.1) with $\mathcal{B}(\mathcal{U})=\mathcal{B}_{\mathcal{U}}\in\mathbb{R}^n$ is*

$$
\begin{aligned}
\mathcal{B}(t)=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}&+\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\
&+\int_{s_{\chi(\mathcal{U},t)}}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)m_l.
\end{aligned}
$$

Proof. When $t\in[s_0,t_1]$, we set

$$
\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)=\mathcal{A}(t)\mathcal{B}(t),
\tag{4.2}
$$

and Lemma 1 implies that the solution of (4.2) is

$$
\mathcal{B}(t)=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}.
$$

Let $\mathcal{B}_{\mathcal{U}}=\mathcal{B}_{\mathcal{U}}(t)$,

$$
\begin{aligned}
\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)&=\mathfrak{D}_{\varsigma}^{\mathcal{U}}\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}(t)+\Gamma(t,\mathcal{U})\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}_{\mathcal{U}}(t)\\
&=\mathcal{A}(t)\mathcal{B}(t)+\Gamma(t,\mathcal{U})(t-\mathcal{U})^{1-\varsigma}\mathcal{B}'_{\mathcal{U}}(t)\\
&=\mathcal{A}(t)\mathcal{B}(t)+j(t),
\end{aligned}
$$

and then

$$
\mathcal{B}'_{\mathcal{U}}(t)=\Gamma^{-1}(t,\mathcal{U})j(t)(t-\mathcal{U})^{\varsigma-1},
$$

$$\mathcal{B}_{\mathcal{U}}(t)=\int_{\mathcal{U}}^{t}\Gamma^{-1}(s,\mathcal{U})j(s)(s-\mathcal{U})^{\varsigma-1}ds+\mathcal{B}_{\mathcal{U}}.$$

Then

$$\begin{aligned}\mathcal{B}(t)&=\Gamma(t,\mathcal{U})\left[\int_{\mathcal{U}}^{t}\Gamma^{-1}(s,\mathcal{U})j(s)(s-\mathcal{U})^{\varsigma-1}ds+\mathcal{B}_{\mathcal{U}}\right]\\&=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\int_{\mathcal{U}}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds.\end{aligned}$$

When $t\in(t_1,s_1]$,

$$\begin{aligned}\mathcal{B}(t)&=\mathcal{C}_1(t)\mathcal{B}(t_1^-)+m_1\\&=\mathcal{C}_1(t)\Gamma(t_1,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\mathcal{C}_1(t)\int_{\mathcal{U}}^{t_1}\Gamma(t_1,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds+m_1.\end{aligned}$$

When $t\in(s_1,t_2]$,

$$\begin{aligned}\mathcal{B}(t)&=\Gamma(t,s_1)\mathcal{B}(s_1)+\int_{s_1}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&=\Gamma(t,s_1)\mathcal{C}_1(s_1)\Gamma(t_1,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\Gamma(t,s_1)\mathcal{C}_1(s_1)\int_{\mathcal{U}}^{t_1}\Gamma(t_1,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&\quad+\Gamma(t,s_1)m_1+\int_{s_1}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\int_{\mathcal{U}}^{t_1}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds+\int_{s_1}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&\quad+\Gamma(t,s_1)m_1.\end{aligned}$$

If the following equality holds:

$$\begin{aligned}\mathcal{B}(t)=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}&+\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&+\int_{s_{\chi(\mathcal{U},t)}}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)m_l,\end{aligned}$$

for $t\in(s_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}]$, then

$$\begin{aligned}\mathcal{B}(t)&=\mathcal{C}_l(t)\mathcal{B}(t_{\chi(\mathcal{U},t)+1}^-)+m_{\chi(\mathcal{U},t)+1}\\&=\mathcal{C}_l(t)\left[\Gamma(t_{\chi(\mathcal{U},t)+1},\mathcal{U})\mathcal{B}_{\mathcal{U}}+\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t_{\chi(\mathcal{U},t)+1},s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\right.\\&\quad\left.+\int_{s_{\chi(\mathcal{U},t)}}^{t_{\chi(\mathcal{U},t)+1}}\Gamma(t_{\chi(\mathcal{U},t)+1},s)j(s)(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t_{\chi(\mathcal{U},t)+1},s_l)m_l\right]\\&\quad+m_{\chi(\mathcal{U},t)+1},\qquad t\in(t_{\chi(\mathcal{U},t)+1},s_{\chi(\mathcal{U},t)+1}].\end{aligned}$$

Hence, for $t \in (s_{\chi(\mathcal{U},t)+1}, t_{\chi(\mathcal{U},t)+2}]$,

$$\begin{aligned}
\mathcal{B}(t) &= \Gamma(t, s_{\chi(\mathcal{U},t)+1})\mathcal{B}(s_{\chi(\mathcal{U},t)+1}) + \int_{s_{\chi(\mathcal{U},t)+1}}^{t} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&= \Gamma(t, s_{\chi(\mathcal{U},t)+1})\mathcal{C}_l(s_{\chi(\mathcal{U},t)+1})\Gamma(t_{\chi(\mathcal{U},t)+1}, \mathcal{U})\mathcal{B}_{\mathcal{U}} \\
&\quad + \Gamma(t, s_{\chi(\mathcal{U},t)+1})\mathcal{C}_l(s_{\chi(\mathcal{U},t)+1}) \sum_{l=0}^{\chi(\mathcal{U},t)-1} \int_{s_l}^{t_{l+1}} \Gamma(t_{\chi(\mathcal{U},t)+1}, s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&\quad + \Gamma(t, s_{\chi(\mathcal{U},t)+1})\mathcal{C}_l(s_{\chi(\mathcal{U},t)+1}) \int_{s_{\chi(\mathcal{U},t)}}^{t_{\chi(\mathcal{U},t)+1}} \Gamma(t_{\chi(\mathcal{U},t)+1}, s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&\quad + \Gamma(t, s_{\chi(\mathcal{U},t)+1})\mathcal{C}_l(s_{\chi(\mathcal{U},t)+1}) \sum_{l=1}^{\chi(\mathcal{U},t)} \Gamma(t_{\chi(\mathcal{U},t)+1}, s_l) m_l \\
&\quad + \Gamma(t, s_{\chi(\mathcal{U},t)+1}) m_{\chi(\mathcal{U},t)+1} + \int_{s_{\chi(\mathcal{U},t)+1}}^{t} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&= \Gamma(t, \mathcal{U})\mathcal{B}_{\mathcal{U}} + \sum_{l=0}^{\chi(\mathcal{U},t)} \int_{s_l}^{t_{l+1}} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&\quad + \int_{s_{\chi(\mathcal{U},t)+1}}^{t} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds + \sum_{l=1}^{\chi(\mathcal{U},t)+1} \Gamma(t, s_l) m_l.
\end{aligned}$$

Using mathematical induction, the solution of (4.1) is:

$$\begin{aligned}
\mathcal{B}(t) = \Gamma(t, \mathcal{U})\mathcal{B}_{\mathcal{U}} &+ \sum_{l=0}^{\chi(\mathcal{U},t)-1} \int_{s_l}^{t_{l+1}} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&+ \int_{s_{\chi(\mathcal{U},t)}}^{t} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds + \sum_{l=1}^{\chi(\mathcal{U},t)} \Gamma(t, s_l) m_l.
\end{aligned}$$

□

Then, the solution of (1.2) is

$$\begin{aligned}
\mathcal{B}(t) = \Gamma(t, \mathcal{U})\mathcal{B}_{\mathcal{U}} &+ \sum_{l=0}^{\chi(\mathcal{U},t)-1} \int_{s_l}^{t_{l+1}} \Gamma(t,s) \mathcal{J}(s)\mathcal{B}(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&+ \int_{s_{\chi(\mathcal{U},t)}}^{t} \Gamma(t,s) \mathcal{J}(s)\mathcal{B}(s)(s-\mathcal{U})^{\varsigma-1} ds + \sum_{l=1}^{\chi(\mathcal{U},t)} \Gamma(t, s_l)\mathcal{M}_l \mathcal{B}(t_l^-).
\end{aligned} \tag{4.3}$$

Theorem 5. *Assume that* (1.1) *is exponentially stable. If*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\chi(\mathcal{U},t)}{t} = \varpi < \infty, \ \sup_{t\in\mathbb{U}} \frac{(t-\mathcal{U})^{\varsigma}}{\varsigma t} = \rho < \infty, \\
&\limsup_{t\to\infty} \|\mathcal{J}(t)\| < \delta, \ \limsup_{l\to\infty} \|\mathcal{M}_l\| < \delta,
\end{aligned}$$

in which ϖ , ρ *and* δ *satisfy*

$$Upsilon = c_2 - c_1\delta\rho - \varpi\ln(1+J\delta) > 0,$$

then (1.2) *is exponentially stable.*

Proof. Since the exponential stability of (1.1), one has $c_1 \geq 1$ and $c_2 > 0$ with

$$\|\Gamma(t,s)\| \leq c_1 e^{-c_2(t-s)}, \mathcal{U} \leq s \leq t.$$

When $t \in [\mathcal{U}, t_{l_{\mathcal{U}}+1}]$, $l_{\mathcal{U}} \in \mathbb{N}$, $J_{\mathcal{U}} = \max\limits_{t\in[\mathcal{U},t_{l_{\mathcal{U}}+1}]} \|\mathcal{J}(t)\|$, $\mathcal{M}_{\mathcal{U}} = \max\limits_{l\in\{1,2,\cdots l_{\mathcal{U}}\}} \|\mathcal{M}_l\|$, (4.3) gives

$$\|\mathcal{B}(t)\| \leq c_1\|\mathcal{B}_{\mathcal{U}}\| + c_1 J_{\mathcal{U}} \int_{\mathcal{U}}^{t} \|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1\mathcal{M}_{\mathcal{U}} \sum_{l=1}^{\chi(\mathcal{U},t)} \|\mathcal{B}(t_l^-)\|.$$

With Lemma 2, we yield

$$\begin{aligned}\|\mathcal{B}(t)\| &\leq c_1\|\mathcal{B}_{\mathcal{U}}\|(1+c_1\mathcal{M}_{\mathcal{U}})^{\chi(\mathcal{U},t)} e^{c_1 J_{\mathcal{U}} \frac{(t-\mathcal{U})^{\varsigma}}{\varsigma}} \\ &\leq c_1\|\mathcal{B}_{\mathcal{U}}\|(1+c_1\mathcal{M}_{\mathcal{U}})^{\chi(\mathcal{U},t)} e^{c_1 J_{\mathcal{U}} \frac{(t_{l_{\mathcal{U}}+1}-\mathcal{U})^{\varsigma}}{\varsigma}} = C_{\mathcal{U}}\|\mathcal{B}_{\mathcal{U}}\|,\end{aligned}$$

in which

$$C_{\mathcal{U}} = c_1(1+c_1\mathcal{M}_{\mathcal{U}})^{\chi(\mathcal{U},t)} e^{c_1 J_{\mathcal{U}} \frac{(t_{l_{\mathcal{U}}+1}-\mathcal{U})^{\varsigma}}{\varsigma}}.$$

For $\mathcal{J}(t)\| \leq \delta$ and $\|\mathcal{M}_l\| \leq \delta$ with any $t > t_{l_{\mathcal{U}}}$ and $\chi(\mathcal{U},t) > l_{\mathcal{U}}$, taking the norm of (4.3), there is

$$\begin{aligned}e^{c_2 t}\|\mathcal{B}(t)\| &\leq (c_1+\bar{c}_1)\|\mathcal{B}_{\mathcal{U}}\| + c_1\delta \sum_{l=l_{\mathcal{U}}}^{\chi(\mathcal{U},t)-1} \int_{s_l}^{t_{l+1}} e^{c_2 s}\|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds \\ &\quad + c_1\delta\int_{s_{\chi(\mathcal{U},t)}}^{t} e^{c_2 s}\|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1\delta\sum_{l=l_{\mathcal{U}}}^{\chi(\mathcal{U},t)} e^{c_2 s_l}\|\mathcal{B}(t_l^-)\| \\ &\leq \widetilde{c}_1\|\mathcal{B}_{\mathcal{U}}\| + c_1\delta\int_{\mathcal{U}}^{t} e^{c_2 s}\|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1\delta\sum_{l=1}^{\chi(\mathcal{U},t)} e^{c_2 s_l}\|\mathcal{B}(t_l^-)\|,\end{aligned}$$

where $\bar{c}_1 = c_1 C_{\mathcal{U}} \sum\limits_{l=0}^{l_{\mathcal{U}}-1} \int_{s_l}^{t_{l+1}} e^{c_2 s}\|\mathcal{J}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1 C_{\mathcal{U}} \sum\limits_{l=0}^{l_{\mathcal{U}}} e^{c_2 s_l}\|\mathcal{M}_l\|$, $\widetilde{c}_1 = c_1 + \bar{c}_1$. Summarizing, we have

$$e^{c_2 t}\|\mathcal{B}(t)\| \leq c\|\mathcal{B}_{\mathcal{U}}\| + c_1\delta\int_{\mathcal{U}}^{t} e^{c_2 s}\|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1\delta\sum_{l=1}^{\chi(\mathcal{U},t)} e^{c_2 s_l}\|\mathcal{B}(t_l^-)\|,$$

for any $t \geq \mathcal{U}$, in which $c = \max\{\widetilde{c}_1, C_{\mathcal{U}} e^{c_2 t_{l_{\mathcal{U}}}}\}$.

According to Lemma 2, we obtain

$$e^{c_2 t}\|\mathcal{B}(t)\| \leq c(1+c_1\delta)^{\chi(\mathcal{U},t)} e^{c_1\delta\frac{(t-\mathcal{U})^{\varsigma}}{\varsigma}}\|\mathcal{B}_{\mathcal{U}}\|,$$

so

$$\|\mathcal{B}(t)\| \leq ce^{-\left(c_2-c_1\delta\frac{(t-\mathcal{U})^{\varsigma}}{\varsigma t}-\frac{\chi(\mathcal{U},t)}{t}\ln(1+c_1\delta)\right)t}\|\mathcal{B}_{\mathcal{U}}\|.$$

Due to

$$\begin{aligned}\liminf_{t\to\infty}\left(c_2-c_1\delta\frac{(t-\mathcal{U})^{\varsigma}}{\varsigma t}-\frac{\chi(\mathcal{U},t)}{t}\ln(1+c_1\delta)\right)\\ \geq c_2-c_1\delta\rho-\varpi\ln(1+c_1\delta)=\Upsilon>0,\end{aligned}$$

we can find $t_1>\mathcal{U}$ and

$$c_2-c_1\delta\frac{(t-\mathcal{U})^{\varsigma}}{\varsigma t}-\frac{\chi(\mathcal{U},t)}{t}\ln(1+c_1\delta)\geq\frac{\Upsilon}{2},$$

for $t\geq t_1$. Hence, when $t\geq t_1$, we obtain

$$\|\mathcal{B}(t)\| \leq ce^{-\frac{\Upsilon}{2}t}\|\mathcal{B}_{\mathcal{U}}\|.$$

□

5. Nonlinear Problem

This section presents the Ulam-Hyers-Rassias stability of (1.3).
The solution of (1.3) is:

$$\begin{aligned}\mathcal{B}(t)=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)z(s,\mathcal{B}(s))(s-\mathcal{U})^{\varsigma-1}ds\\ +\int_{s_{\chi(\mathcal{U},t)}}^{t}\Gamma(t,s)z(s,\mathcal{B}(s))(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)\mathcal{L}_l.\end{aligned}\tag{5.1}$$

Then, we present the condition: (H_1) $z\in C(\mathbb{U}\times\mathbb{R}^n,\mathbb{R}^n)$ and there is a $\mathcal{L}_z>0$ satisfying:

$$\|z(t,\mathcal{B}_1)-z(t,\mathcal{B}_2)\|\leq\mathcal{L}_z\|\mathcal{B}_1-\mathcal{B}_2\|,\ t\in\mathbb{U},\ \mathcal{B}_1,\ \mathcal{B}_2\in\mathbb{R}^n.$$

Theorem 6. *Assume that (H_1) holds. If system (1.1) is exponentially stable and*

$$\mathcal{L}_z c_1\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}<1,$$

then (1.3) has a unique solution.

Proof. We set a mapping $\mathcal{P}:PC(\mathbb{U},\mathbb{R}^n)\to PC(\mathbb{U},\mathbb{R}^n)$,

$$\begin{aligned}(\mathcal{P}\mathcal{B})(t)=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)z(s,\mathcal{B}(s))(s-\mathcal{U})^{\varsigma-1}ds\\ +\int_{s_{\chi(\mathcal{U},t)}}^{t}\Gamma(t,s)z(s,\mathcal{B}(s))(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)\mathcal{L}_l,\end{aligned}$$

and

$$\begin{aligned}
&\|(\mathcal{P}\mathcal{B})(t)-(\mathcal{P}\overline{\mathcal{B}})(t)\| \\
&\qquad\le \sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\left\|\Gamma(t,s)(s-\mathcal{U})^{\varsigma-1}\left[z(s,\mathcal{B}(s))-z(s,\overline{\mathcal{B}}(s))\right]\right\|ds \\
&\qquad\quad +\int_{s_{\chi(\mathcal{U},t)}}^{t}\left\|\Gamma(t,s)(s-\mathcal{U})^{\varsigma-1}\left[z(s,\mathcal{B}(s))-z(s,\overline{\mathcal{B}}(s))\right]\right\|ds \\
&\qquad\le \mathcal{L}_z\Bigg(\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}c_1e^{-c_2(t-s)}(s-\mathcal{U})^{\varsigma-1}ds \\
&\qquad\quad +\int_{s_{\chi(\mathcal{U},t)}}^{t}c_1e^{-c_2(t-s)}(s-\mathcal{U})^{\varsigma-1}ds\Bigg)\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC} \\
&\qquad\le \mathcal{L}_z\left(\int_{\mathcal{U}}^{t}c_1(s-\mathcal{U})^{\varsigma-1}ds\right)\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC} \\
&\qquad\le \mathcal{L}_zc_1\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC}, \qquad (5.2)
\end{aligned}$$

for $\forall\ \mathcal{B},\ \overline{\mathcal{B}}\in PC(\mathbb{U},\mathbb{R}^n)$.

Then (5.2) implies that

$$\|(\mathcal{P}\mathcal{B})(t)-(\mathcal{P}\overline{\mathcal{B}})(t)\|\le\mathcal{L}_{\mathcal{P}}\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC},$$

in which

$$\mathcal{L}_{\mathcal{P}}:=\mathcal{L}_zc_1\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma},$$

so

$$\|(\mathcal{P}\mathcal{B})-(\mathcal{P}\overline{\mathcal{B}})\|_{PC}\le\mathcal{L}_{\mathcal{P}}\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC}.$$

Hence, (1.3) has a unique solution. □

Next, we refer to Ulam-Hyers-Rassias stability of (1.3).

Set $\varepsilon>0$, $\kappa>0$ and $\mu\in PC(\mathbb{U},\mathbb{R}_+)$ be a nondecreasing function, and

$$\begin{cases}
\|\mathfrak{D}_{\varsigma}^{\mathcal{U}}\widetilde{\mathcal{B}}(t)-\mathcal{A}(t)\widetilde{\mathcal{B}}(t)-z(t,\widetilde{\mathcal{B}}(t))\|\le\varepsilon\mu(t),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\
\|\widetilde{\mathcal{B}}(t_l^+)-\mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-)-\mathcal{L}_l\|\le\varepsilon\kappa,\ l\in N,\\
\|\widetilde{\mathcal{B}}(t)-\mathcal{C}_l(t)\mathcal{B}(t_l^-)-\mathcal{L}_l\|\le\varepsilon\kappa,\ t\in(t_l,s_l],\ l\in N.
\end{cases}\qquad (5.3)$$

Set

$$D:=PC^1(\mathbb{U},\mathbb{R}^n)\bigcap_{l\in\mathbb{N}}C^1\left((s_l,t_{l+1}],\mathbb{R}^n\right).$$

Definition 5. We set a solution $\mathcal{B}\in D$ of (1.3) is Ulam-Hyers-Rassias stable if for each $\varepsilon>0$ and for each function $\widetilde{\mathcal{B}}\in D$ satisfying (5.3) with $\widetilde{\mathcal{B}}(\mathcal{U})=\mathcal{B}_{\mathcal{U}}$, there is a

$c_{z,N,\mu} > 0$ satisfying

$$\|\widetilde{\mathcal{B}}(t) - \mathcal{B}(t)\| \le c_{z,N,\mu}\varepsilon\big(\mu(t)+\kappa\big),\ t \in \mathbb{U}.$$

Remark 2. (5.3) has a solution $\widetilde{\mathcal{B}} \in D$ iff there exists $G \in \bigcup_{l\in N_0} C^1\big((s_l,t_{l+1}],\mathbb{R}^n\big)$ and $g \in \bigcup_{l\in N} C\big((t_l,s_l],\mathbb{R}^n\big)$ satisfying

(i) $\|G(t)\| \le \varepsilon\mu(t)$, $t \in \bigcup_{l\in N_0}(s_l,t_{l+1}]$, $\|g(t)\| \le \varepsilon\kappa$, $t \in \bigcup_{l\in N}(t_l,s_l]$;

(ii) $\mathfrak{D}_{\varsigma}^{\mathcal{U}}\widetilde{\mathcal{B}}(t) = \mathcal{A}(t)\widetilde{\mathcal{B}}(t) + z(t,\widetilde{\mathcal{B}}(t)) + G(t)$, $t \in (s_l,t_{l+1}]$, $l \in N_0$;

(iii) $\widetilde{\mathcal{B}}(t_l^+) = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-) + \mathcal{L}_l + g(t_l)$, $l \in N$;

(iv) $\widetilde{\mathcal{B}}(t) = \mathcal{C}_l(t)\mathcal{B}(t_l^-) + \mathcal{L}_l + g(t)$, $t \in (t_l,s_l]$, $l \in N$.

We suppose that the following conditions hold:

(H_2) $\omega := \sup_{t\in\mathbb{U}} \sum_{l=1}^{\chi(\mathcal{U},t)} e^{-c_2(t_{\chi(\mathcal{U},t)+1}-s_l)} < \infty$ holds.

(H_3) There is a $\psi > 0$ satisfying

$$\int_{\mathcal{U}}^{t} e^{-c_2(t-s)}\mu(s)(s-\mathcal{U})^{\varsigma-1}ds \le \psi\mu(t),\ t \in \mathbb{U}.$$

Theorem 7. *Suppose that* (H_1), (H_2) *and* (H_3) *hold. If* (1.1) *is exponentially stable and*

$$c_1\mathcal{L}_z \max\Big\{\psi, \kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\Big\} < 1 \tag{5.4}$$

is satisfied, then (1.3) *is Ulam-Hyers-Rassias stable.*

Proof. Let $\mathcal{B}(t)$ be a solution of (1.3) and $\widetilde{\mathcal{B}} \in D$ be a solution of (5.3) with $\widetilde{\mathcal{B}}(\mathcal{U}) = \mathcal{B}_{\mathcal{U}}$.

Consider

$$\begin{cases} \mathfrak{D}_{\varsigma}^{\mathcal{U}}\widetilde{\mathcal{B}}(t) = \mathcal{A}(t)\widetilde{\mathcal{B}}(t) + z(t,\widetilde{\mathcal{B}}(t)) + G(t),\ t \in (s_l,t_{l+1}],\ l \in N_0, \\ \widetilde{\mathcal{B}}(t_l^+) = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-) + \mathcal{L}_l + g(t_l),\ l \in N, \\ \widetilde{\mathcal{B}}(t) = \mathcal{C}_l(t)\mathcal{B}(t_l^-) + \mathcal{L}_l + g(t),\ t \in (t_l,s_l],\ l \in N. \end{cases} \tag{5.5}$$

The solution $\widetilde{\mathcal{B}}(t)$ of (5.5) with $\widetilde{\mathcal{B}}(\mathcal{U}) = \mathcal{B}_{\mathcal{U}}$ has the following expression:

$$\begin{aligned} \widetilde{\mathcal{B}}(t) = \Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}} &+ \sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}} \Gamma(t,s)\big[z(s,\widetilde{\mathcal{B}}(s)) + G(s)\big](s-\mathcal{U})^{\varsigma-1}ds \\ &+ \int_{s_{\chi(\mathcal{U},t)}}^{t} \Gamma(t,s)\big[z(s,\widetilde{\mathcal{B}}(s)) + G(s)\big](s-\mathcal{U})^{\varsigma-1}ds \\ &+ \sum_{l=1}^{\chi(\mathcal{U},t)} \Gamma(t,s_l)\big(\mathcal{L}_l + g(s_l)\big) = (\mathcal{P}\widetilde{\mathcal{B}})(t) + \widetilde{f}(t), \end{aligned}$$

in which

$$\widetilde{f}(t)=\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)G(s)(s-\mathcal{U})^{\varsigma-1}ds+\int_{s_{(\mathcal{U},t)}}^{t}\Gamma(t,s)G(s)(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)g(s_l).$$

Combining (H_2) with (H_3), there is

$$\begin{aligned}\|\widetilde{f}(t)\|&\le\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\|\Gamma(t,s)\|\|G(s)\|(s-\mathcal{U})^{\varsigma-1}ds\\&\quad+\int_{s_{(\mathcal{U},t)}}^{t}\|\Gamma(t,s)\|\|G(s)\|(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\|\Gamma(t,s_l)\|\|g(s_l)\|\\&\le\varepsilon c_1\left(\int_{\mathcal{U}}^{t}e^{-c_2(t-s)}\mu(s)(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}e^{-c_2(t-s_l)}\kappa\right)\\&\le\varepsilon c_1\left(\psi\mu(t)+\sum_{l=1}^{\chi(\mathcal{U},t)-1}e^{-c_2(t_{\chi(\mathcal{U},t)}-s_l)}\kappa+\kappa\right)\\&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big),\ t\in(t_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}].\end{aligned}\tag{5.6}$$

Using (5.1) and (5.6), we get

$$\begin{aligned}\|\widetilde{\mathcal{B}}(t)-\mathcal{B}(t)\|&\le\|\widetilde{f}(t)\|+\|(\mathcal{P}\widetilde{\mathcal{B}})(t)-(\mathcal{P}\mathcal{B})(t)\|\\&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\\&\quad+c_1\mathcal{L}_z\int_{\mathcal{U}}^{t}e^{-c_2(t-s)}\|\widetilde{\mathcal{B}}(s)-\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds.\end{aligned}\tag{5.7}$$

Set

$$h(\mathcal{V}):=\sup_{t\in\mathbb{U}}\frac{\|\widetilde{\mathcal{B}}(t)-\mathcal{B}(t)\|}{\mu(t)+\kappa},$$

and combining (5.7) with (H_3), one has

$$\begin{aligned}\|\widetilde{\mathcal{B}}(t)-\mathcal{B}(t)\|&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\\&\quad+c_1\mathcal{L}_z h(\mathcal{V})\int_{\mathcal{U}}^{t}e^{-c_2(t-s)}\big(\mu(t)+\kappa\big)(s-\mathcal{U})^{\varsigma-1}ds\\&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\\&\quad+c_1\mathcal{L}_z h(\mathcal{V})\left[\psi\mu(t)+\int_{\mathcal{U}}^{t}\kappa e^{-c_2(t-s)}(s-\mathcal{U})^{\varsigma-1}ds\right]\\&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\end{aligned}$$

$$
\begin{aligned}
&+c_1\mathcal{L}_z h(\mathcal{V})\left[\psi\mu(t)+\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right]\\
&\le \varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\\
&\quad +c_1\mathcal{L}_z h(\mathcal{V})\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}\big(\mu(t)+\kappa\big),\ t\in\mathbb{U}. \qquad (5.8)
\end{aligned}
$$

(5.8) implies

$$
h(\mathcal{V})\le \varepsilon c_1\max\{1+\omega,\psi\}+c_1\mathcal{L}_z h(\mathcal{V})\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}.
$$

Then,

$$
h(\mathcal{V})\le \varepsilon\frac{c_1\max\{1+\omega,\psi\}}{1-c_1\mathcal{L}_z\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}},
$$

so

$$
\begin{aligned}
\|\widetilde{\mathcal{B}}(t)-\mathcal{B}(t)\|\le & \quad \varepsilon\frac{c_1\max\{1+\omega,\psi\}}{1-c_1\mathcal{L}_z\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}}\big(\mu(t)+\kappa\big)\\
&\le c_{z,N,\mu}\varepsilon\big(\mu(t)+\kappa\big),\ t\in\mathbb{U}, \qquad (5.9)
\end{aligned}
$$

and $c_{z,N,\mu}=\frac{c_1\max\{1+\omega,\psi\}}{1-c_1\mathcal{L}_z\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}}>0$.

Since (5.9) is satisfied for any $\mathcal{V}>0$, (1.3) is Ulam-Hyers-Rassias stable. □

Example 1. We set $s_0=\mathcal{U}=\frac{3}{2}$, $t_1=2$, $\mathcal{V}=20$, $\varsigma=\frac{1}{2}$, $s_l=l+\frac{3}{2}$, $t_{l+1}=l+2$, $l=1,2,\cdots$, and

$$
\mathcal{A}(t)=\begin{pmatrix}\frac{1}{2}(t-s_l) & 0\\ 0 & \frac{1}{4}(t-s_l)\end{pmatrix},\qquad \mathcal{C}_l(t)=\begin{pmatrix}\frac{1}{20}+\frac{1}{20}\frac{t-t_l}{s_l-t_l} & 0\\ 0 & \frac{1}{20}+\frac{1}{20}\frac{t-t_l}{s_l-t_l}\end{pmatrix},
$$

$$
\mathcal{B}_{\mathcal{U}}=\begin{pmatrix}1\\0\end{pmatrix},\qquad z(t,\mathcal{B}(t))=\begin{pmatrix}\frac{\cos t}{e^7}(\mathcal{B}_1(t)+1)\\ \frac{\cos t}{e^7}\mathcal{B}_2(t)\end{pmatrix},\qquad \mathcal{L}_l=\begin{pmatrix}1\\0\end{pmatrix}.
$$

Now $\chi(\mathcal{U},t)=\lceil t\rceil-2$. For $s_{\chi(\mathcal{U},t)}<t\le t_{\chi(\mathcal{U},t)+1}$ and $s_{\chi(\mathcal{U},s)}<s\le t_{\chi(\mathcal{U},s)+1}$, $\chi(\mathcal{U},t)>\chi(\mathcal{U},s)$, there is

$$
\begin{aligned}
\Gamma(t,s)\le & \begin{pmatrix}e^{(t-1)^{\frac{1}{2}}-(s_{\chi(\mathcal{U},t)}-1)^{\frac{1}{2}}} & 0\\ 0 & e^{\frac{1}{2}(t-1)^{\frac{1}{2}}-\frac{1}{2}(s_{\chi(\mathcal{U},t)}-1)^{\frac{1}{2}}}\end{pmatrix}\begin{pmatrix}\frac{1}{10} & 0\\ 0 & \frac{1}{10}\end{pmatrix}\\
&\times\prod_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)}\begin{pmatrix}\frac{1}{10}e^{\frac{1}{4}} & 0\\ 0 & \frac{1}{10}e^{\frac{1}{8}}\end{pmatrix}\\
&\times\begin{pmatrix}e^{(t_{\chi(\mathcal{U},s)+1}-1)^{\frac{1}{2}}-(s-1)^{\frac{1}{2}}} & 0\\ 0 & e^{\frac{1}{2}(t_{\chi(\mathcal{U},s)+1}-1)^{\frac{1}{2}}-\frac{1}{2}(s-1)^{\frac{1}{2}}}\end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} e^{\frac{1}{2}(t-\lceil t\rceil+\frac{1}{2})} & 0 \\ 0 & e^{\frac{1}{4}(t-\lceil t\rceil+\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \frac{1}{10}e^{\frac{1}{4}} & 0 \\ 0 & \frac{1}{10}e^{\frac{1}{8}} \end{pmatrix}^{\lceil t\rceil-\lceil s\rceil-1} \times \begin{pmatrix} e^{\frac{1}{2}(\lceil s\rceil-s)} & 0 \\ 0 & e^{\frac{1}{4}(\lceil s\rceil-s)} \end{pmatrix}.$$

Hence

$$\|\Gamma(t,s)\| \le 10e^{\frac{1}{2}}(\frac{1}{10}e^{\frac{1}{4}})^{\lceil t\rceil-\lceil s\rceil} = 10e^{\frac{1}{2}}(e^{\frac{1}{4}-\ln 10})^{\lceil t\rceil-\lceil s\rceil} \le e^{\ln 10+\frac{1}{2}}e^{-(\ln 10-\frac{1}{4})(t-s)}.$$

The system is exponentially stable with $c_2 = \ln 10 - \frac{1}{4}$ and $c_1 = e^{\ln 10+\frac{1}{2}}$.

Note

$$\sup_{t\in\mathbb{U}} \sum_{l=1}^{\chi(\mathcal{U},t)} e^{-(\ln 10-\frac{1}{4})(\chi(\mathcal{U},t)+2-l-\frac{3}{2})} \le \frac{e^{\frac{\frac{1}{4}-\ln 10}{2}}}{1-e^{\frac{1}{4}-\ln 10}},$$

so (H_2) holds and $\omega \le \frac{e^{\frac{\frac{1}{4}-\ln 10}{2}}}{1-e^{\frac{1}{4}-\ln 10}}$.

Let $\mu(t) = t^{\frac{3}{4}}, \kappa = 1$, and we have

$$\int_{\frac{3}{2}}^{t} e^{-(\ln 10-\frac{1}{4})(t-s)}s^{\frac{3}{2}}s^{-\frac{1}{2}}ds = \int_{\frac{3}{2}}^{t} e^{(\ln 10-\frac{1}{4})(s-t)}sds \le \frac{t^{\frac{1}{4}}}{\ln 10-\frac{1}{4}} \le \frac{t^{\frac{3}{4}}}{\ln 10-\frac{1}{4}},$$

so (H_3) holds with $\psi = \frac{1}{\ln 10-\frac{1}{4}}$.

Then,

$$\mathcal{L}_z\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma} = \frac{2\sqrt{20}}{e^7} < 1,$$

so (1.3) has a unique solution from Theorem 6.

Also,

$$\mathcal{L}_z \max\left\{\psi, \kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\} < 1,$$

and (5.4) are satisfied.

Theorem 7 implies that (1.3) is Ulam-Hyers-Rassias stable with

$$c_{z,N,\mu} = \frac{c_1\max\{1+\omega,\psi\}}{1-c_1\mathcal{L}_z\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}} = \frac{e^{\ln 10+\frac{1}{2}}(\frac{e^{\frac{\frac{1}{4}-\ln 10}{2}}}{1-e^{\frac{1}{4}-\ln 10}}+1)}{1-\frac{2\sqrt{20}e^{\ln 10+\frac{1}{2}}}{e^7}} = 26.87.$$

ACKNOWLEDGEMENTS

The authors would like to thank the associate editor and reviewers for their constructive comments, which helped us to improve the paper.

REFERENCES

[1] A. A. Abdelhakim and J. A. T. Machado, "A critical analysis of the conformable derivative," *Nonlinear Dynamics*, vol. 95, no. 4, pp. 3063–3073, 2019, doi: 10.1007/s11071-018-04741-5.

[2] T. Abdeljawad, "On conformable fractional calculus," *Journal of Computational and Applied Mathematics*, vol. 279, pp. 57–66, 2015, doi: 10.1016/j.cam.2014.10.016.

[3] R. Agarwal, D. O'Regan, and S. Hristova, "Monotone iterative technique for the initial value problem for differential equations with non-instantaneous impulses," *Applied Mathematics and Computation*, vol. 298, pp. 45–56, 2017, doi: 10.1016/j.amc.2016.10.009.

[4] R. Agarwal, D. O'Regan, and S. Hristova, "Noninstantaneous impulses in Caputo fractional differential equations and practical stability via Lyapunov functions," *Journal of the Franklin Institute*, vol. 354, no. 7, pp. 3097–3119, 2017, doi: 10.1016/j.jfranklin.2017.02.002.

[5] M. U. Akhmet, J. Alzabut, and A. Zafer, "Perron's theorem for linear impulsive differential equations with distributed delay," *Journal of Computational and Applied Mathematics*, vol. 193, no. 1, pp. 204–218, 2006, doi: 10.1016/j.cam.2005.06.004.

[6] Z. Fan and G. Li, "Existence results for semilinear differential equations with nonlocal and impulsive conditions," *Journal of Functional Analysis*, vol. 258, no. 5, pp. 1709–1727, 2010, doi: 10.1016/j.jfa.2009.10.023.

[7] J. Henderson and A. Ouahab, "Impulsive differential inclusions with fractional order," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1191–1226, 2010, doi: 10.1016/j.camwa.2009.05.011.

[8] E. Hernández and D. O'Regan, "On a new class of abstract impulsive differential equations," *Proceedings of the American Mathematical Society*, vol. 141, no. 5, pp. 1641–1649, 2013, doi: 10.1090/S0002-9939-2012-11613-2.

[9] E. Hernández, M. Pierri, and D. O'Regan, "On abstract differential equations with non instantaneous impulses," *Topological Methods in Nonlinear Analysis*, vol. 46, no. 2, pp. 1067–1088, 2015, doi: 10.12775/TMNA.2015.080.

[10] R. Khalil, M. A. Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014, doi: 10.1016/j.cam.2014.01.002.

[11] Z. Korpinar, F. Tchier, M. Inc, F. Bousbahi, F. M. Tawfiq, and M. A. Akinlar, "Applicability of time conformable derivative to Wick-fractional-stochastic PDEs," *Alexandria Engineering Journal*, vol. 59, no. 3, pp. 1485–1493, 2020, doi: 10.1016/j.aej.2020.05.001.

[12] M. Li, J. Wang, and D. O'Regan, "Existence and Ulam's stability for conformable fractional differential equations with constant coefficients," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 4, pp. 1791–1812, 2019, doi: 10.1007/s40840-017-0576-7.

[13] S. Liu, J. Wang, and Y. Zhou, "Optimal control of noninstantaneous impulsive differential equations," *Journal of the Franklin Institute*, vol. 354, no. 17, pp. 7668–7698, 2017, doi: 10.1016/j.jfranklin.2017.09.010.

[14] S. Mehmood, F. Zafar, and N. Yasmin, "Hermite–Hadamard–Fejér inequalities for generalized conformable fractional integrals," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 5, pp. 3746–3758, 2021, doi: 10.1002/mma.6978.

[15] M. Pierri, R. Hernan, and A. Prokopczyk, "Global solutions for abstract differential equations with non-instantaneous impulses," *Mediterranean Journal of Mathematics*, vol. 13, no. 4, pp. 1685–1708, 2016, doi: 10.1007/s00009-015-0609-0.

[16] A. M. Samoilenko, N. A. Perestyuk, and Y. Chapovsky, *Impulsive differential equations*. World Scientific, 1995.

[17] J. Wang, "Stability of noninstantaneous impulsive evolution equations," *Applied Mathematics Letters*, vol. 73, pp. 157–162, 2017, doi: 10.1016/j.aml.2017.04.010.

[18] J. Wang, M. Li, and D. O'Regan, "Lyapunov regularity and stability of linear non-instantaneous impulsive differential systems," *IMA Journal of Applied Mathematics*, vol. 84, no. 4, pp. 712–747, 2019, doi: 10.1093/imamat/hxz012.

[19] G. Xiao and J. Wang, "On the stability of solutions to conformable stochastic differential equations," *Miskolc Mathematical Notes*, vol. 21, no. 1, pp. 509–523, 2020, doi: 10.18514/mmn.2020.3257.

[20] P. Yang, J. Wang, and M. Fečkan, "Periodic nonautonomous differential equations with noninstantaneous impulsive effects," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 10, pp. 3700–3720, 2019, doi: 10.1002/mma.5606.

Author's address

Yuanlin Ding
Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China
E-mail address: yldingmath@126.com