Miskolc Mathematical Notes
Vol. 26 (2025), No. 2, pp. 941–953

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4985

EXISTENCE AND STABILITY OF SOLUTIONS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATION THROUGH BIELECKI METRIC

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Received 05 March, 2024

Abstract. In this paper, we have shown that, by considering the Caputo fractional differential equation and Bielecki norm, the existence and uniqueness of solution for Cauchy problem is established using Banach fixed point theorem and further demonstrate the existence of solution without the Lipschitz condition utilizing the Schauder fixed point theorem. Furthermore, under the condition of existence and uniqueness of solution for Caputo fractional differential equation, we present some results about the error estimation, local approximation of solutions and the Ulam-Hyers-Rassias stability. Finally, we provide a numerical example to verify the theoretical results about error estimation.

2010 *Mathematics Subject Classification:* 34A08; 34A12; 34D20

Keywords: Fractional differential equation, Caputo derivative, Cauchy problem, Bielecki norm, existence, stability

1. INTRODUCTION

Fractional calculus has significantly attracted attention in theories [7, 12, 19, 21, 22] and applications [10, 13, 14], owing to their accuracy and effectiveness in describing memory processes, power law phenomena and others. However, solving fractional differential equations still faces substantial challenges. Therefore, many scholars have focused on two main aspects of solving these equations. On the one hand, authors have focused on utilizing numerical and analytical approaches to derive approximate solutions for fractional differential equations, which include Picard's iterative method [2], Adams-type predictor–corrector method [5], radial basis functions [6], variational iteration method [20], wavelet method [1] and others. On the other hand,some scholars have applied fixed point theorem to analyze diverse types

The first author was supported in part by by Guizhou University introduced talent research project (2022) 70 and Basic research project of Guizhou University [2023] 39.

of fractional differential equations and demonstrate the existence and uniqueness of solutions under different definition of fractional derivative [3, 4, 8, 18].

Ulam-Hyers stability, being one of the most active research topics in differential equations, has acquired a series of results. Li et al. [8] demonstrate the existence and uniqueness of solution and Ulam-Hyers and Ulam-Hyers-Rassias stability for conformable fractional differential equation on finite and infinite time intervals. Liu et al. [9] studied the Ulam-Hyers stability for the linear Caputo-Fabrizio fractional differential equations with the Mittag-Leffler kernel. Wang et al. [16] prove the Ulam-Hyers stability and Ulam-Hyers-Rassias stability using Laplace transform method in terms of Caputo linear fractional differential equations.

Most papers are concerned with the existence and uniqueness of solution for the Caputo fractional differential equation in Banach space, where the solution is given in the sense of all continuous functions with the maximum norm. Thus, motivated by [8, 11, 17], the objective of this paper is to consider the following Cauchy problem of Caputo type fractional differential equation:

$$\begin{cases} {}^c\mathrm{D}^{\alpha} y = f(x,y), \\ y(0) = y_0, \end{cases} x \in I = [0,a], \tag{1.1}$$

where $a > 0$, $y \in \mathbb{R}^m$, the operator ${}^c\mathrm{D}^{\alpha}$ represents the Caputo fractional derivative of order α, $\alpha \in (0,1)$. The function $f : I \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous.

In the following, by using the Bielecki norm, we show the existence and uniqueness of solution to Caputo fractional differential equation via the Banach fixed point theorem and Schauder fixed point theorem. Further, under the condition of the existence and uniqueness of solution, we present the results about error estimation, local approximation of solutions and Ulam-Hyers-Rassias stability.

2. PRELIMINARIES

Throughout this paper, we use $\mathbb{R}^m$ to denote m-dimensional Euclidean space, where the norm is defined as $\|\xi\| = \left(\sum_{i=1}^{m} |\xi_i|^2\right)^{\frac{1}{2}}$ and $\xi \in \mathbb{R}^m$. Moreover, let $C([a,b])$ be Banach space consisting of all continuous functions from $[a,b]$ into $\mathbb{R}$ endowed with the Bielecki norm $\|u\|_{\lambda} = \max_{t \in [a,b]} e^{-\lambda t} \|u(t)\|$, where $0 < a < b < \infty$ and $\lambda > 0$.

Definition 1 (see [7]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function y is defined as

$$\mathrm{I}^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} y(s) ds,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2 (see [7]). For $\alpha > 0$ and $y : (0,\infty) \to \mathbb{R}$, the Caputo fractional derivative of y is given by

$$ {}^{c}\mathrm{D}^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} y^{(n)}(s) ds, $$

where $n \in \mathbb{N}$ such that $n-1 < \alpha < n$.

Lemma 1 (see [7]). *The solution of Cauchy problem* (1.1) *is equivalent to the solution of the second kind of Volterra integral equation*

$$ y(x) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, y(s)) ds. \tag{2.1} $$

Definition 3 (see [15]). If for each function y satisfying

$$ |{}^{c}\mathrm{D}^{\alpha} y(x) - f(x,y)| \le \varphi(x), \quad x \in [a,b], $$

there is a solution $\hat{y}(x)$ of the Caputo fractional differential equation (1.1) and a constant $C > 0$ such that

$$ |y(x) - \hat{y}(x)| \le C\varphi(x), \quad x \in [a,b], $$

where $\varphi : [a,b] \to (0,+\infty)$ is a nonnegative continuous function, then we say that the Caputo fractional differential equation (1.1) has the Ulam-Hyers-Rassias stability.

3. Results for existence of solution

To get our conclusions, we need to introduce the following assumptions.

A1 Let $f : G_0 \subset I \times \mathbb{R}^m \to \mathbb{R}^m$ be continuous function and satisfy equation (1.1), where

$$ G_0 = \{(x,y) \in I \times \mathbb{R}^m : 0 \le x \le a, \|y - y_0\| \le b\}, $$

$a > 0$, $b > 0$ and $y_0 \in \mathbb{R}^m$. Then there exists a $L > 0$ such that

$$ \|f(x,y_1) - f(x,y_2)\| \le L\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^m. $$

A2 Let $g : G_0 \to \mathbb{R}^m$ be continuous function. Then there exists a monotone function $p : I \to \mathbb{R}^+$ such that

$$ \|f(x,y) - g(x,y)\| \le p(x), \quad \forall (x,y) \in I \times \mathbb{R}^m. $$

A3 Let $\Phi, \Psi_n : G_0 \to \mathbb{R}^m$ be continuous functions and $n \in \mathbb{N}^+$. Φ and Ψ_n are Lipschitz with respect to second argument, i.e.

$$ \|\Phi(x,y_1) - \Phi(x,y_2)\| \le L_1\|y_1 - y_2\|, $$

and

$$ \|\Psi_n(x,y_1) - \Psi_n(x,y_2)\| \le L_2\|y_1 - y_2\|, $$

where $L_1, L_2 > 0$.

A4 The sequence $(\Psi_n)_{n\ge 1}$ uniformly converges to Ψ over the domain G_0.

Let IVP_n and IVP be notations of following Caputo fractional differential equations, i.e.

$$(IVP_n)\begin{cases} {}^c\mathrm{D}^{\alpha}y=\Psi_n(x,y)+\Phi(x,y), \\ y(0)=y_0, \end{cases} n\in\mathbb{N}^+,$$

and

$$(IVP)\begin{cases} {}^c\mathrm{D}^{\alpha}y=\Psi(x,y)+\Phi(x,y), \\ y(0)=y_0. \end{cases}$$

Next, we first present the existence and uniqueness of solution to the Cauchy problem for Caputo fractional differential equation in the sense of Bielecki norm.

Theorem 1. *Assume that* $[A1]$ *is true. Then the equation* (1.1) *possess a unique solution defined on* $|x|\le h_0$*, where* $h_0=\min\left\{a,\left[\frac{b\Gamma(\alpha+1)}{M}\right]^{\frac{1}{\alpha}}\right\}$ *and* $M=\sup_{(x,y)\in G_0}\|f(x,y)\|$.

Proof. According to Lemma 1, we know that the solution of equation (1.1) is equivalent to the solution of following Volterra integral equation

$$y(x)=y_0+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}f(s,y(s))ds,\quad |x|\le h_0.$$

Here, we choose the Banach space consisting of all continuous functions on $|x|\le h_0$. The initial value problem equation (1.1) for the case where $x\in[0,h_0]:=I_0$ are discussed. Then we define the operator $T:C(I_0)\to C(I_0)$,

$$(Ty)(x):=y_0+\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}f(s,y(s))ds.$$

By direct calculation, we have

$$\begin{aligned}\|Ty-y_0\|_\lambda&\le\frac{1}{\Gamma(\alpha)}\max_{x\in I_0}e^{-\lambda x}\left\|\int_0^x(x-s)^{\alpha-1}f(s,y(s))ds\right\|\\&\le\frac{M}{\Gamma(\alpha)}\max_{x\in I_0}e^{-\lambda x}\left|\int_0^x(x-s)^{\alpha-1}ds\right|\\&\le\frac{Mh_0^\alpha}{\Gamma(\alpha+1)}\max_{x\in I_0}e^{-\lambda x}\le\frac{Mh_0^\alpha}{\Gamma(\alpha+1)}\le b.\end{aligned}$$

For all $y_1,y_2\in C(I_0)$, we obtain

$$\begin{aligned}\|Ty_1(x)-Ty_2(x)\|&=\left\|\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}(f(s,y_1(s))-f(s,y_2(s))ds\right\|\\&\le\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\|f(s,y_1(s))-f(s,y_2(s)\|\,ds\\&\le\frac{1}{\Gamma(\alpha)}\int_0^x L(x-s)^{\alpha-1}\|y_1(s)-y_2(s)\|\,ds.\end{aligned}$$

Further, we get

$$\begin{aligned}\|Ty_1 - Ty_2\|_\lambda &= \frac{L}{\Gamma(\alpha)} \max_{x\in I_0} e^{-\lambda x} \int_0^x e^{\lambda s}(x-s)^{\alpha-1} e^{-\lambda s} \|y_1(s) - y_2(s)\| \, ds \\ &\le \frac{L\|y_1 - y_2\|_\lambda}{\Gamma(\alpha)} \max_{x\in I_0} e^{-\lambda x} \int_0^x e^{\lambda s}(x-s)^{\alpha-1} ds \qquad (3.1) \\ &\le \frac{L\|y_1 - y_2\|_\lambda}{\lambda^\alpha} := \beta \|y_1 - y_2\|_\lambda , \qquad (3.2)\end{aligned}$$

since

$$\begin{aligned}\int_0^x e^{\lambda s}(x-s)^{\alpha-1} ds &= e^{\lambda x} \int_0^x t^{\alpha-1} e^{-\lambda t} dt = e^{\lambda x} \lambda^{-\alpha} \int_0^{\lambda x} r^{\alpha-1} e^{-r} dr \\ &\le e^{\lambda x} \lambda^{-\alpha} \int_0^\infty r^{\alpha-1} e^{-r} dr = \frac{e^{\lambda x}\Gamma(\alpha)}{\lambda^\alpha}.\end{aligned}$$

Here, we choose the appropriate λ such that $0 < \beta < 1$. Thus the operator T is a contraction via the Bielecki norm $\|\cdot\|_\lambda$ on $C(I_0)$, which demonstrate the equation (1.1) possess a unique solution defined on I_0. $\square$

In the following, we shall use the Schauder fixed point theorem to demonstrate the existence of solution for Caputo fractional differential equation without the Lipschitz condition.

Theorem 2. *Assume that $f(\cdot,\cdot)$ is a continuous function defined on G_0. Then, the equation (1.1) has at least one solution.*

Proof. Similar operator T can be established in Theorem 1 and we will divide our demonstration into several procedures.

Step 1: T is continuous.

Let u_n be a sequence such that $u_n \to u$ in $C(I)$. For all $x \in I$, we obtain that

$$\begin{aligned}\|Tu_n(x) - Tu(x)\| &= \frac{1}{\Gamma(\alpha)} \left\| \int_0^x (x-s)^{\alpha-1} (f(s,u_n(s)) - f(s,u(s))) ds \right\| \\ &\le \frac{1}{\Gamma(\alpha)} \left\| \int_0^x (x-s)^{\alpha-1} e^{\lambda s} e^{-\lambda s} (f(s,u_n(s)) - f(s,u(s))) ds \right\| \\ &\le \frac{L e^{\lambda x} \|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_\lambda}{\lambda^\alpha}.\end{aligned}$$

Further, we have

$$\|Tu_n - Tu\|_\lambda \le \frac{L\|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_\lambda}{\lambda^\alpha} \to 0$$

since $\|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_\lambda \to 0$. This shows the fact that operator T is continuous.

Step 2: T maps bounded sets into bounded sets of $C(I)$.

$$\begin{aligned}\|Tu(x)\| &\leq \|y_0\| + \frac{1}{\Gamma(\alpha)}\left\|\int_0^x (x-s)^{\alpha-1} f(s,u(s))ds\right\| \\ &\leq \|y_0\| + \frac{M}{\Gamma(\alpha)}\left|\int_0^x (x-s)^{\alpha-1} ds\right| \leq \|y_0\| + \frac{Ma^{\alpha}}{\Gamma(\alpha+1)},\end{aligned}$$

where $M = \sup_{(x,y)\in G_0} \|f(x,y(x))\|$. Further, we have

$$\|Tu\|_{\lambda} \leq \|y_0\| + \frac{Ma^{\alpha}}{\Gamma(\alpha+1)}.$$

Step 3: T maps bounded sets into equicontinuous sets in $C(I)$.

Let $x_1, x_2 \in I$, with $0 < x_1 < x_2 < h$. Thus, we have following results

$$\begin{aligned}\|Tu(x_1) - Tu(x_2)\| &= \frac{1}{\Gamma(\alpha)}\left\|\int_0^{x_1} (x_1-s)^{\alpha-1} f(s,u(s))ds \right. \\ &\quad \left. - \int_0^{x_2} (x_2-s)^{\alpha-1} f(s,u(s))ds\right\| \\ &\leq \frac{M}{\Gamma(\alpha+1)} \left|x_2^{\alpha} - x_1^{\alpha}\right|.\end{aligned}$$

It is obvious that the right-hand side of above inequality tends to zero as $x_1 \to x_2$. Thus, the operator T is equicontinuous.

By combined with Step $1-3$, the operator T is completely continuous with the Arzelà-Ascoli theorem. Further, based on the Schauder fixed point theorem, the operator T has a fixed point corresponding to a solution for (1.1). The proof is finished. □

Under the condition of the Bielecki metric, we present the corollary about the prior and posterior error estimate by using the Picard iteration.

Corollary 1. *Let $(C(I), \|\cdot\|_{\lambda})$ be a complete metric space and assume that $[A1]$ holds. Then the prior and posterior error estimate are present as follows*

$$d(y_n, y^*) = \frac{L^n}{\lambda^{n\alpha-1}(\lambda^{\alpha} - L)} d(y_0, y_1), \quad \forall n \geq 0,\ y \in C(I),$$

and

$$d(y_n, y^*) = \frac{L}{\lambda^{\alpha} - L} d(y_{n-1}, y_n),$$

where $d(x,y) = \|x-y\|_{\lambda}$, $\forall x, y \in C(I)$, and y^* is a unique fixed point.

Proof. By Lemma 1, we construct the Picard iteration

$$y_n(x) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, y_{n-1}(s))ds,$$

and define the operator $S : C(I) \to C(I)$ by

$$(Sy)(x) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, y(s)) ds.$$

Notice that, by (3.1), $y_n = S(y_{n-1})$, we have

$$d(y_1, y_2) = d(Sy_0, Sy_1) \leq \beta d(y_0, y_1).$$

Thus, we deduce that

$$d(y_n, y_{n+1}) \leq \beta^n d(y_0, y_1),$$

and

$$\begin{aligned} d(y_n, y_{n+p}) &\leq \sum_{k=n}^{n+p-1} d(y_k, y_{k+1}) \leq \sum_{k=n}^{n+p-1} \beta^k d(y_0, y_1) \\ &\leq \frac{\beta^n}{1-\beta} d(y_0, y_1), \quad \forall n, p \in \mathbb{N}, p > 0. \end{aligned}$$

Finally, as $p \to \infty$, we obtain that

$$d(y_n, y^*) = \lim_{p \to \infty} d(y_n, y_{n+p}) \leq \frac{\beta^n}{1-\beta} d(y_0, y_1) = \frac{L^n}{\lambda^{n\alpha-1}(\lambda^\alpha - L)} d(y_0, y_1), \quad \forall n \geq 0.$$

Let us notice that by (3.1), we have

$$d(y_{n+1}, y_n) \leq \beta d(y_n, y_{n-1}),$$

and by induction,

$$d(y_{n+k-1}, y_{n+k}) \leq \beta^k d(y_n, y_{n-1}).$$

Then we get

$$d(y_n, y_{n+p}) \leq (\beta + \beta^2 + \cdots + \beta^p) d(y_{n-1}, y_n) \leq \frac{\beta}{1-\beta} d(y_{n-1}, y_n).$$

By letting $p \to \infty$, we get

$$d(y_n, y_{n+p}) \leq \frac{\beta}{1-\beta} d(y_{n-1}, y_n) = \frac{L}{\lambda^\alpha - L} d(y_{n-1}, y_n).$$

□

Remark 1.

(i) The result of prior and posterior estimate indicate that the approximation error of the nth iteration starting from the initial value y_0 is determined by the Lipschitz coefficient, the parameter λ, and the initial displacement.

(ii) The posterior estimate provides a direct stopping criterion for Picard iteration approximation of the fixed point, while the prior estimate indirectly gives a stopping criterion.

(iii) The Picard iteration shows the rate of convergence is linear.

(iv) In most cases, the definition of (3.1) is locally satisfied on the whole space $C(I)$.

Then we present the sufficient condition to the error estimation of solutions for Cauchy problem of Caputo fractional differential equations.

Theorem 3. *Assume that* $[A1]$ *and* $[A2]$ *are satisfied, then we have the error estimation*

$$\|u-v\|_\lambda \leq \frac{h_1^\alpha \lambda^\alpha P}{\Gamma(\alpha+1)(\lambda^\alpha - L)}, \quad 0 \leq x \leq h_1,$$

where $0 < h_1 \leq \min\left\{a, \left[\frac{b\Gamma(\alpha+1)}{M_1}\right]^{\frac{1}{\alpha}}\right\}$, $P = \max_{x\in[0,h_1]} |p(x)|$, $M_1 = \max\{\|f(x,y)\|, \|g(x,y)\|\}$ *and* u,v *are solutions satisfying equation* (1.1) *with respect to* f,g.

Proof. By Lemma 1, by using the equivalent Volterra integral equations (2.1), we define the operators $A,B : C(I) \to C(I)$ by

$$(Au)(x) := y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s,u(s))ds,$$
$$(Bv)(x) := y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s,v(s))ds.$$

Draw on the Theorem 1, the existence of a solution is guaranteed by the continuous conditions on f and g. The domain G_0 turns into G_1, i.e.

$$G_1 = \{(x,y) \in I_1 \times \mathbb{R}^m : 0 \leq x \leq h_1, \|y-y_0\| \leq b\},$$

where $h_1 = \min\left\{a, \left[\frac{b\Gamma(\alpha+1)}{M_1}\right]^{\frac{1}{\alpha}}\right\}$. We can write the initial value problems of Caputo fractional differential equations as fixed points $u = Au, v = Bv, u,v \in C(I_1)$. By using the assumption [A2], we have following inequalities

$$\begin{aligned}
\|Ay(x) - By(x)\| &= \left\|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}(f(s,y(s)) - g(s,y(s)))ds\right\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\|f(s,y(s)) - g(s,y(s))\| ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} p(s)ds.
\end{aligned}$$

Further, we get

$$\begin{aligned}
\|Ay - By\|_\lambda &= \frac{1}{\Gamma(\alpha)} \max_{x\in I_1} e^{-\lambda x}\int_0^x (x-s)^{\alpha-1} p(s)ds \\
&\leq \frac{P}{\Gamma(\alpha+1)} \max_{x\in I_1} e^{-\lambda x} x^\alpha \leq \frac{h_1^\alpha P}{\Gamma(\alpha+1)},
\end{aligned} \tag{3.3}$$

where $P = \max_{x \in I_1} |p(x)|$. By (3.1), we have

$$\|Ay_1 - Ay_2\|_\lambda \le \frac{L\|y_1 - y_2\|_\lambda}{\lambda^\alpha} := \beta \|y_1 - y_2\|_\lambda. \tag{3.4}$$

The choice of β is similar to Theorem 1. Then we combine (3.3) and (3.4) to obtain following results

$$\begin{aligned}\|u - v\|_\lambda &= \|Au - Bv\|_\lambda \le \|Au - Av\|_\lambda + \|Av - Bv\|_\lambda \\ &\le \frac{L\|u - v\|_\lambda}{\lambda^\alpha} + \frac{h_1^\alpha P}{\Gamma(\alpha+1)} \le \frac{h_1^\alpha \lambda^\alpha P}{\Gamma(\alpha+1)(\lambda^\alpha - L)}.\end{aligned}$$

□

By using above approach, we investigate the local approximation of solutions for a class of Cauchy problems based on IVP_n and IVP.

Theorem 4. *Assume that $[A3]$ and $[A4]$ are satisfied, then the Cauchy problem of Caputo fractional differential equation (IVP) has (locally) a unique solution.*

Proof. Under the condition of Theorem 1, the Cauchy problem (IVP) has at least one solution. Further, we can deduce from $[A4]$ that sequence $(\Psi_n)_{n\ge1}$ is uniformly bounded such that

$$\|\Psi_n\| \le M_2, \quad \|\Psi\| \le M_2, \quad \forall n \ge 1,$$

for some $M_2 > 0$. Let u_n and u be solutions with respect to IVP_n and IVP. It is shown that the problem (IVP_n) possess a unique solution u_n, which defined on

$$u_n : (0, h_2) \to \mathbb{R},$$

where $h_2 = \min\left\{a, \left[\frac{b\Gamma(\alpha+1)}{M_2}\right]^{\frac{1}{\alpha}}\right\}$. The domain G_0 turns into G_2, and we keep the same notations in Theorem 3, i.e.

$$G_2 = \{(x,y) \in I_2 \times \mathbb{R}^m : 0 \le x \le h_2,\ \|y - y_0\| \le b\},$$

and

$$f(x,y) = \Psi_n(x,y) + \Phi(x,y), \quad g(x,y) = \Psi(x,y) + \Phi(x,y).$$

Further, by using the assumption $[A3]$, we derive the results from f and g, i.e.

$$\begin{aligned}\|f(x,y_1) - f(x,y_2)\| &= \|\Phi(x,y_1) + \Psi_n(x,y_1) - \Phi(x,y_2) - \Psi_n(x,y_2)\| \\ &\le (L_1 + L_2)\|y_1 - y_2\|,\end{aligned}$$

where $(x,y_1), (x,y_2) \in G_2$, and

$$\|f(x,y) - g(x,y)\| = \|\Psi_n(x,y) - \Phi(x,y)\|.$$

Within the framework of Theorem 3, we obtain following results between IVP_n and IVP

$$\begin{aligned}\|Au_n - Au\|_\lambda &= \frac{(L_1+L_2)}{\Gamma(\alpha)} \max_{x\in I_2} e^{-\lambda x} \int_0^x e^{\lambda s}(x-s)^{\alpha-1} e^{-\lambda s}\|u_n(s)-u(s)\|\,ds \\ &\leq \frac{(L_1+L_2)\,\|u_n-u\|_\lambda}{\Gamma(\alpha)} \max_{x\in I_2} e^{-\lambda x}\int_0^x e^{\lambda s}(x-s)^{\alpha-1}ds \\ &\leq \frac{(L_1+L_2)\,\|u_n-u\|_\lambda}{\lambda^\alpha},\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\|Au - Bu\|_\lambda &= \frac{1}{\Gamma(\alpha)} \max_{x\in I_2} e^{-\lambda x}\int_0^x e^{\lambda s}(x-s)^{\alpha-1}e^{-\lambda s}\|\Psi_n - \Psi\|\,ds \\ &\leq \frac{\|\Psi_n-\Psi\|_\lambda}{\Gamma(\alpha)} \max_{x\in I_2} e^{-\lambda x}\int_0^x e^{\lambda s}(x-s)^{\alpha-1}ds \leq \frac{\|\Psi_n-\Psi\|_\lambda}{\lambda^\alpha}.\end{aligned} \tag{3.6}$$

By virtue of (3.5) and (3.6), we obtain that

$$\|u_n - u\|_\lambda \leq \frac{\|\Psi_n-\Psi\|_\lambda}{\lambda^\alpha - (L_1+L_2)}, \quad \forall x\in(0,h_2).$$

Finally, as $n\to\infty$, we obtain uniform convergence that

$$u = \lim_{n\to\infty} u_n,$$

which demonstrates the uniqueness of u. □

4. Results for Ulam-Hyers-Rassias stability

Theorem 5. *Assume that [A1] and [A2] are satisfied. If $y\in C(I)$ is such that*

$$\|{}^c\mathrm{D}^\alpha y(x) - f(x,y)\| \leq \varphi(x).$$

Then there is a unique solution $\hat{y}\in C(I)$ satisfying equation (1.1) and

$$\|y(x)-\hat{y}(x)\|_\lambda \leq \frac{\lambda^\alpha}{\lambda^\alpha - L}\varphi(x), \quad x\in I. \tag{4.1}$$

This implies that equation (1.1) exhibits Hyers-Ulam-Rassias stability under the aforementioned condition.

Proof. Applying the Lemma 1, the equivalent integral equation for (1.1) is constructed and define the same operator T in Theorem 1. Further, for any $u,v\in C(I)$, we have

$$\|Tu - Tv\|_\lambda \leq \frac{L}{\lambda^\alpha}\|u-v\|_\lambda.$$

By choosing large enough parameter λ, then the operator T is strictly contractive in $C(I)$. Thus, the equation (1.1) is Ulam-Hyers-Rassis stability. Further, by the Bielecki metric in $C(I)$, we have

$$d(Tu,Tv) \leq \frac{L}{\lambda^{\alpha}} d(u,v),$$

for any $u.v \in C(I)$. Based on the above inequality, we obtain

$$d(y,\hat{y}) \leq d(y,Ty) + d(Ty,\hat{y}) = d(y,Ty) + \frac{L}{\lambda^{\alpha}} d(y,\hat{y}).$$

Further, we have following results

$$d(y,\hat{y}) \leq \frac{\lambda^{\alpha}}{\lambda^{\alpha} - L} d(y,Ty).$$

Thus, the result of (4.1) is obtained. $\square$

Remark 2. If $\varphi(\cdot)$ is a constant function in Theorem 5, we say that the equation (1.1) has the Ulam-Hyers stability.

5. An example

We consider the Cauchy problems of following Caputo fractional differential equations

$$\begin{cases} {}^{c}\mathrm{D}^{\alpha} y = x, \\ y(0) = 0, \end{cases} \quad \textit{and} \quad \begin{cases} {}^{c}\mathrm{D}^{\alpha} y = x - y, \\ y(0) = 0, \end{cases} \tag{5.1}$$

where the domain is defined by

$$G = \left\{ (x,y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2} \right\}.$$

We denote

$$f(x,y) = x, \quad g(x,y) = x - y,$$

and let u and v be solutions with respect to f and g. The fractional differential equations (5.1) satisfy the following conditions

$$\lambda = 10000, \quad \alpha = \frac{1}{2}, \quad L = \frac{1}{2}, \quad h = \min\left\{1, \frac{\pi}{16}\right\}.$$

Then we obtain the difference with respect to f and g,

$$|f(x,y) - g(x,y)| = |y| \leq \frac{1}{2}.$$

Setting $p(x) = \frac{1}{2}$ and combining with Theorem 3, we observe that

$$\|u - v\|_{\lambda} \leq 0.2512562.$$

The numerical results for the equations (5.1) are presented in Figure 1.

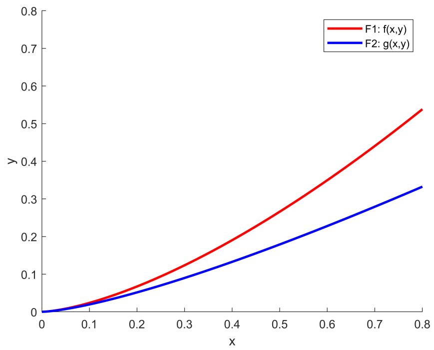

FIGURE 1. Error estimation of numerical result by (5.1).

ACKNOWLEDGEMENTS

The authors would like to thank the associate editor and reviewers for their constructive comments, which helped us to improve the paper.

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