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ON COFINITELY (D^*_{12})-MODULES

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Abstract. In this paper, concepts of (cofinitely) (D^*_{12})-modules which are a proper generalization of concept of $\oplus_\delta$-supplemented modules are studied. We say that M is a (D^*_{12})-module if for every submodule A of M, there exists a direct summand B of M and an epimorphism $f: B \to \frac{M}{A}$ such that $\ker(f) \ll_\delta B$. The module M is called cofinitely (D^*_{12})-module if for every cofinite submodule A of M, there exists a direct summand B of M and an epimorphism $f: B \to \frac{M}{A}$ such that $\ker(f) \ll_\delta B$. In this paper, various properties of these modules are given. In addition, a new characterization of δ-semiperfect rings is given using cofinitely (D^*_{12})-modules.

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1. INTRODUCTION

Throughout this paper, all rings are associative with identity and all modules are unitary right modules, unless otherwise specified. Let R be such a ring and M be such a module. By the notation $X \leq M$, we mean that X is a submodule of M. A submodule X of M is called *small* in M if $M \neq X + Y$ for any proper submodule Y of M, denoted by $X \ll M$, and we denote that $\mathrm{Rad}(M)$, the sum of all small submodules of M. Dual to this concept, a submodule X of M is called *essential* in M, denoted by $X \trianglelefteq M$, if the intersection of X is non-zero with the other submodules of M, except for $\{0\}$. It is known that the set $Z(M) = \{m \in M \mid \mathrm{Ann}(m) \trianglelefteq R\}$ is the singular submodule of M. The module M is called *singular* in case $Z(M) = M$. A submodule X of M is called *cofinite* whenever $\frac{M}{X}$ is finitely generated. A supplement submodule T of X in M is minimal element of the set $\{Y \leq M \mid M = X + Y\}$ that equivalents $M = X + T$ and $X \cap T \ll T$. A module M is called *supplemented* if every submodule of M has a supplement in M [19]. A module M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M [3]. A generalization of supplement submodule is defined as a Rad-supplement submodule (according to [18], a generalized supplement submodule). For a module M and a submodule A of M, a submodule B of M is called a *Rad-supplement* of A if $M = A + B$ and $A \cap B \subseteq \mathrm{Rad}(B)$. An R-module M is called *GS-module* (or briefly Rad-supplemented) if each submodule of M has a Rad-supplement in M. A module M is called $\oplus$-*cofinitely radical supplemented* (according to [8] generalized $\oplus$-*cofinitely supplemented*) if every cofinite submodule of M has a Rad-supplement that is a direct summand of M. In [15] , it is used a cgs$^{\oplus}$-module.

Small submodules are generalized to δ-small submodules in [20]. By [20], a submodule A of M is called δ-*small* in M (denoted by $A \ll_{\delta} M$) if for any submodule B of M with $\frac{M}{B}$ is singular, $M = A + B$ implies that $M = B$. The sum of δ-small submodules of a module M is denoted by $\delta(M)$. It is easy to see that every small submodule of a module M is δ-small in M, so $\mathrm{Rad}(M) \subseteq \delta(M)$ and $\mathrm{Rad}(M) = \delta(M)$ if M is singular. Also any non-singular semisimple submodule of M is δ-small in M and any δ-small submodules of a singular module are small submodules. For more detailed discussion on δ-small submodules we refer to [20].

Let A be a submodule of a module M. A submodule B of M is called a δ-*supplement* of A in M provided that $M = A + B$ and $M \neq A + X$ for any proper submodule X of B with $\frac{B}{X}$ singular; or equivalently, $M = A + B$ and $A \cap B \ll_{\delta} B$ in [7].The module M is called δ-*supplemented* if every submodule of M has a δ-supplement in M by [7]. Some properties of this modules class are investigated in [16]. Also, M is called $\oplus - \delta$-*supplemented (or* $\oplus_{\delta}$-supplemented) if every submodule of M has a δ-supplement which is a direct summand of M in [12]. According to [13], an R-module M is called $\oplus$-*cofinitely* δ-*supplemented* (or $\oplus - cof_{\delta}$-supplemented) if every cofinite submodule of M has a δ-supplement that is a direct summand of M. A module M is called δ-*lifting*, if for every submodule A of M there exists a direct summand K of M with $K \subseteq A$ and $\frac{A}{K} \ll_{\delta} \frac{M}{K}$. Equivalently, for any $A \leq M$, there exists a decomposition $M = K \oplus B$ such that $K \leq A$ and $A \cap B \ll_{\delta} B$ by [7].

(D_{12})-modules are generalized to $\oplus$-supplemented modules. To addition cofinitely (D_{12})-modules as a generalization of cofinitely $\oplus$- supplemented modules are introduced in [1,5] and [17], respectively. M is called a (D_{12})-module if for every submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B \to \frac{M}{A}$ such that $\ker(f) \ll B$. M is called a *cofinitely* (D_{12})-*module* if for every cofinite submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon \frac{M}{B} \to \frac{M}{A}$ such that $\ker(f) \ll \frac{M}{B}$. Similarly, (cofinitely) Rad$-D_{12}$-modules are studied and some features are obtained in [6] and [11].

In this paper, inspired from the definitions given above, we introduce the concept of (D_{12}^{*}) and cofinitely (D_{12}^{*})-modules, as follows. We say that M is a (D_{12}^{*})-module if for every submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B \to \frac{M}{A}$ such that $\ker(f) \ll_{\delta} B$ and M is a cofinitely (D_{12}^{*})-module if for every cofinite submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B \to \frac{M}{A}$ such that $\ker(f) \ll_{\delta} B$. We give some results related with these concepts. We give an example which is a cofinitely (D_{12}^{*})-module but not a cofinitely

(D_{12})-module. We have given a new characterization of δ-semiperfect rings using cofinitely (D_{12}^*)-modules and we have shown that every free right $R-$ module over a δ-perfect ring R is (D_{12}^*)-module. By the definitions given above, we can get the following implication on modules:

$$\begin{array}{ccc} \delta-\text{lifting module} & \Longrightarrow & \text{cofinitely } \delta-\text{lifting module} \\ \Downarrow & & \Downarrow \\ \oplus_\delta-\text{supplemented module} & \Longrightarrow & \text{cofinitely } \oplus_\delta-\text{supplemented module} \\ \Downarrow & & \Downarrow \\ (D_{12}^*)-\text{module} & \Longrightarrow & \text{cofinitely } (D_{12}^*)-\text{module} \end{array}$$

2. (COFINITELY) (D_{12}^*)-MODULES

Definition 1. *M is called a (D_{12}^*)-module if for every submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B \to \frac{M}{A}$ provided that $\ker(f) \ll_\delta B$.*

Example 1. For $n > 1$ consider the left $\mathbb{Z}$-module $M = \mathbb{Z}_{p^n}$ where p is an arbitrary prime integer. Since M is local, it is clear that M is a (D_{12}^*)-module. So $\mathbb{Z}$-module $\mathbb{Z}_4$, $\mathbb{Z}_8$ and $\mathbb{Z}_{p^\infty}$ are (D_{12}^*)-modules.

Proposition 1. *Let M be a $\oplus_\delta$-supplemented module. Then M is a (D_{12}^*)-module.*

Proof. Suppose that A be a submodule of M. There exist direct summands B and B_1 of M such that $M = A + B = B \oplus B_1$ and $A \cap B \ll_\delta B$ as M is a $\oplus_\delta$-supplemented module. From here, we have the epimorphism $\alpha\colon B \to \frac{M}{A}$, $\alpha(b) = b + A$ for every $b \in B$. Note that $\ker(\alpha) = B \cap A \ll_\delta B$. Finally M is a (D_{12}^*)-module. □

Corollary 1. *Let M be a $\oplus_\delta$-supplemented module and A be a submodule of M such that $\frac{M}{A}$ is projective. Then A is a (D_{12}^*)-module.*

Proof. By Theorem 2.7 in [12], A is a $\oplus_\delta$-supplemented module. If we use Proposition 1, then we get that A is a (D_{12}^*)-module. □

The notion of $I-\oplus$-supplemented modules are introduced in [14], where I is an ideal of R. A module M is called $I-\oplus-$*supplemented* if for every submodule A of M, there exists a direct summand B of M such that $M = A + B$, $A \cap B \subseteq IB$ and $A \cap B \ll_\delta B$. $I-\oplus$-supplemented modules are characterized in [14]. It is clear that every $I-\oplus$-supplemented module is $\oplus_\delta$-supplemented, for every ideal I of R.

Corollary 2. *Let M be an $I-\oplus$-supplemented module. Then M is a (D_{12}^*)-module.*

Example 2. Let R be a discrete valuation ring with maximal ideal m and any ideal I of R. By Proposition 3.7 in [14], ${}_RR$ is $I-\oplus$-supplemented module if and only if $I = m$ and $I = R$. So ${}_RR$ is a (D_{12}^*)-module.

Recall from [10] that a submodule $A \leq M$ is called *weak* δ-*supplement of a submodule B* of M if $M = A + B$ and $A \cap B \ll_\delta M$. The module M is called *weakly* δ-*supplemented if every submodule A* of M has a *weak* δ-*supplement*.

Recall from [19] that a module M is called *refinable* if for, every submodules A, $B \leq M$ with $M = A + B$, there exists a direct summand B_1 of M with $B_1 \leq B$ and $M = B_1 + B$.

Proposition 2. *Let M be a weakly* δ*-supplemented refinable module. Then M is a* (D^*_{12})*-module.*

Proof. Suppose that $A \leq M$. Since M is *weakly* δ- *supplemented*, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll_\delta M$. Since M is a refinable module, then there is a direct summand A_1 of M such that $M = A_1 + A$ and $A_1 \leq B$. If we consider the natural epimorphism $\psi\colon A_1 \to \frac{A_1}{A_1 \cap A}$, we have $\ker(\psi) = A_1 \cap A$. As $A_1 \leq B$, $A_1 \cap A \leq B \cap A \ll_\delta M$. Since there exists an isomorphism $\theta\colon \frac{A_1}{A_1 \cap A} \to \frac{M}{A}$, say $f = \theta\psi\colon A_1 \to \frac{M}{A}$. Here $\ker(f) = \ker(\theta\psi) = \psi^{-1}(\ker\theta) = \psi^{-1}(0) = \ker\psi = A_1 \cap A$. From here $A_1 \cap A \ll_\delta A_1$ because A_1 is a direct summand of M. Therefore M is a (D^*_{12})-module. □

Recall from [12] that a module M is called δ-*radical* if $\delta(M) = M$ and the sum of all δ-radical submodules of the module M is denoted by $P_\delta(M)$, that is, $P_\delta(M) = \{U \leq M \mid \delta(U) = U\}$. It is clear that, for any submodule M, $P_\delta(M)$ is the largest δ-*radical* submodule of M.

Proposition 3. *Let M be a* (D^*_{12})*-module. If* $P_\delta(M)$ *is a direct summand of M, then* $P_\delta(M)$ *is a* D^*_{12}*-module.*

Proof. Since $P_\delta(M)$ is a direct summand of M, there exists a submodule A of M such that $M = P_\delta(M) \oplus A$. By the hypothesis, there exists a direct summand B of M and an epimorphism $\varphi\colon B \to \frac{M}{T \oplus A}$ such that $\ker(\varphi) \ll_\delta B$ for any submodule T of $P_\delta(M)$. Note that $\frac{M}{T \oplus A} \cong \frac{P_\delta(M)}{T}$. Hence $\delta\left(\frac{B}{\ker(\varphi)}\right) = \frac{B}{\ker(\varphi)}$. We have $\delta(B) = B$ because $\ker(\varphi) \ll_\delta B$ and so $B \leq P_\delta(M)$. □

Theorem 1. *Let* $M = M_1 \oplus M_2$*. Then* M_2 *is a* (D^*_{12})*-module if and only if for every submodule A of M containing* M_1*, there exists a direct summand B of* M_2 *and an epimorphism* $f\colon M \to \frac{M}{A}$ *such that B is a* δ*-supplement of* $\ker(f)$ *in M.*

Proof.

(⇒)**:** Assume that M_2 is a (D^*_{12})-module and A is a submodule of M with $M_1 \leq A$. Consider the submodule $A \cap M_2$ of M_2. Since M_2 is a (D^*_{12})-module, there exists a direct summand B of M_2 and an epimorphism $g\colon B \to \frac{M_2}{A \cap M_2}$ such that $\ker(g) \ll_\delta B$. On the other hand, we have $M = A + M_2$ and for any submodule B_1 of M, $M = B \oplus B_1$ because B is a direct summand of M_2. Consider the projection map $h\colon M \to B$ and the isomorphism $\mu\colon \frac{M_2}{A \cap M_2} \to \frac{M}{A}$ defined by

$\mu(m_2+A\cap M_2)=m_2+A$. Thus $f=\mu\circ g\circ h\colon M\to\frac{M}{A}$ is an epimorphism. It is clear that $\ker(f)=\ker(g)+B_1=A+B_1$. Hence $M=B+\ker(f)$. From here, $B\cap\ker(f)=B\cap A=\ker(g)\ll_{\delta}B$. Finally, B is a δ-supplement of $\ker(f)$ in M which is a direct summand of M.

$(\Leftarrow)$**:** Conversely, suppose that every submodule of M containing M_1 has the stated property. Suppose that X be a submodule of M_2. Note that $X\oplus M_1\leq M$. By the hypothesis, there exists a direct summand B of M_2 and an epimorphism $f\colon M\to\frac{M}{X\oplus M_1}$ such that $M=B+\ker(f)$ and $B\cap\ker(f)\ll_{\delta}B$. Assume that $g\colon B\to\frac{M}{X\oplus M_1}$ be the restriction of f to B. We can take the isomorphism $h\colon\frac{M}{X\oplus M_1}\to\frac{M_2}{X}$ defined by $h(m_1+m_2+(X\oplus M_1))=m_2+X$. From here, $\mu=hog\colon B\to\frac{M_2}{X}$ is an epimorphism. Clearly,

$$\begin{aligned}\ker(\mu)&=\ker(h\circ g)=\{b\in B|\,h(g(b))=\{X\}\}\\&=\{b\in B|\,g(b)=\frac{X\oplus M_1}{X\oplus M_1}\}=\{b\in B|\,b\in\ker(g)\}.\end{aligned}$$

Since $\ker(g)=\ker(\mu)=B\cap\ker(f)\ll_{\delta}B$. Finally, M_2 is a (D_{12}^{*})-module.

□

Recall from [19] that a submodule X of M is called *fully invariant* if $\varphi(X)$ is contained in X for every R-endomorphism φ of M. In [4], a module M is called a *duo module* if every submodule of M is fully invariant.

Theorem 2. *Let M_i be a (D_{12}^{*})-module for every $i\in I$ and $M=\oplus_{i\in I}M_i$. If M is a duo module, then M is a (D_{12}^{*})-module.*

Proof. Suppose that $A\leq M$. Since M is a duo module, we have $A=\oplus_{i\in I}(A\cap M_i)$ by Lemma 2.1 in [4]. Now, we consider the submodule $A\cap M_i$ of M_i for every $i\in I$. As M_i is a (D_{12}^{*})-module, we have a direct summand B_i of M_i and an epimorphism $\varphi_i\colon\oplus_{i\in I}B_i\to\oplus_{i\in I}\left(\frac{M_i}{A\cap M_i}\right)\cong\frac{\oplus_{i\in I}M_i}{\oplus_{i\in I}(A\cap M_i)}$ by $b_{i_1}+b_{i_2}+\cdots+b_{i_n}\to\varphi_{i_1}(b_{i_1})+\varphi_{i_2}(b_{i_2})+\cdots+\varphi_{i_n}(b_{i_n})$ with $b_{i_j}\in B_{i_j}$ for every $j=1,2,\ldots,n$. It is so easy to check that φ is an epimorphism with $\ker(\oplus_{i\in I}\varphi_i)=\oplus_{i\in I}\ker(\varphi_i)$ and $\oplus_{i\in I}B_i$ is a direct summand of M. For every $i\in I$, since $\ker(\varphi_i)\ll_{\delta}B_i$, we can get that $\oplus_{i\in I}\ker(\varphi_i)\ll_{\delta}\oplus_{i\in I}B_i$. Hence, M is a (D_{12}^{*})-module. □

Definition 2. M is called a cofinitely (D_{12}^{*})-module if for every cofinite submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B\to\frac{M}{A}$ such that $\ker(f)\ll_{\delta}B$.

Now, we give examples which are (D_{12}^{*})-modules but not a (D_{12})-modules. Since ${}_RR$ is finitely generated, the module in the following example is (cofinitely) (D_{12}^{*})-module but not (cofinitely) (D_{12})-module.

Example 3.

(i) (See [2, Example 3.10]) Let S be a field, $U = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$ and

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N}, x_i \in M_2(S)\},$$

with component-wise operations. The Jacobson radical $\mathrm{Rad}(R) = 0$ and R is not regular ring, hence R is not semiperfect. Since

$$\delta(R) = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N}, x_i \in M_2(S), x \in V\},$$

where $V = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$, R is δ-semiperfect. From here, R as a right R-module is (D^*_{12})-module by Theorem 8. But ${}_RR$ is not (D_{12})-module by Theorem 4.7 in [5].

(ii) (See [15, Example 2.2]) Let M be a uniform module and $S = \mathrm{End}(M)$. Consider the projective S-module P with $\dim(P) = (1,0)$. So P is a indecomposable w-local module. Since $\dim(P) = (1,0)$, P is not finitely generated. By [15], P is a cgs$^{\oplus}$-module but not cofinitely supplemented. Therefore P is a $\oplus$-*cofinitely* δ-*supplemented* module. It follows from that P is a cofinitely (D^*_{12})-module. Since P is not $\oplus$-cofinitely supplemented, P is not a cofinitely (D_{12})-module by [6, Example 2.4].

Proposition 4. *Let M be a cofinitely $\oplus_\delta$-supplemented module. Then M is a cofinitely (D^*_{12})-module.*

Proof. The proof can be made similar to Proposition 1. □

Theorem 3. *Let M be a quasi-projective module.*

(i) *If M is a (D^*_{12})-module, then M is $\oplus_\delta$-supplemented.*
(ii) *If M is a cofinitely (D^*_{12})-module, then M is cofinitely $\oplus_\delta$-supplemented.*

Proof.

(i) For a submodule A of M, by the hypothesis there exists a direct summand B of M and an epimorphism $\varphi\colon B \to \frac{M}{A}$ such that $\ker(\varphi) \ll_\delta B$. Now, we consider $\alpha\colon M \to \frac{M}{A}$ be the natural epimorphism. Since M is quasi-projective, we get that the following commutative diagram for the homomorphism $\beta\colon M \to B$ with $\alpha = \varphi \circ \beta$:

$$\begin{array}{ccccc} & & M & & \\ & \overset{\beta}{\swarrow} & \downarrow\alpha & & \\ B & \longrightarrow & \frac{M}{A} & \longrightarrow & 0 \end{array}$$

From here $\beta(M) = B$ because $(\varphi \circ \beta)(M) = \alpha(M) = \frac{M}{A}$ and $\ker(\varphi) \ll_\delta B$. And so β is an epimorphism. On the other hand, β splits as B is M-projective. Therefore, there is a direct summand of B_1 of M such that $\beta\downarrow_{B_1}\colon B_1 \cong B$ and so $\alpha\downarrow_{B_1}$ is an epimorphism. Hence $M = B_1 + A$ and $B_1 \cap A = \ker(\alpha\downarrow_{B_1}) \ll_\delta$

M. It follows from $B_1 \cap A \ll_\delta B_1$ because B_1 is a direct summand of M. Finally M is $\oplus_\delta$-supplemented.

(ii) The proof can be made similar to (i).

□

Recall from [20] that an epimorphism $f\colon P \to M$ is called δ-*cover* if $\ker(f) \ll_\delta P$ and a cover f is called a *projective* δ-*cover* if P is a projective module. A module M is called δ-*semiperfect* if every factor module of M has a projective δ-cover. A ring R is called δ-*perfect* (δ-semiperfect) if every (finitely generated) right (or left) R-module has a projective δ-cover.

Theorem 4. *Let M be a projective module. Then M is δ-semiperfect if and only if M is a (D_{12}^{*})-module.*

Proof.

(⇒)**:** Assume that M is a projective δ-*semiperfect* module. By Lemma 2.4 in [9] and Proposition 1, M is $\oplus_\delta$-supplemented and so M is a (D_{12}^{*})-module.

(⇐)**:** This proof is made using Theorem 3 (ii) and Lemma 2.4 in [9], respectively.

□

Corollary 3. *Let R be a δ-perfect ring. Then every free right R-module is a (D_{12}^{*})-module.*

Theorem 5. *Let $M = M_1 \oplus M_2$. Then M_2 is a cofinitely (D_{12}^{*})-module if and only if for every cofinite submodule A of M containing M_1, there exists a direct summand B of M_2 and an epimorphism $f\colon M \to \frac{M}{A}$ such that B is a δ-supplement of* $\ker(f)$ *in M.*

Proof. The proof can be made similar to Theorem 1. □

Theorem 6. *Let $\{M_i\}_{i\in I}$ be any family cofinitely (D_{12}^{*})-module and $M = \oplus_{i\in I} M_i$. If every cofinite submodule of M is fully invariant, then M is a cofinitely (D_{12}^{*})-module.*

Proof. Let A be a cofinite submodule of M. As A is fully invariant, we can write $A = \oplus_{i\in I}(A \cap M_i)$. Since $\frac{M}{A} \cong \oplus_{i\in I}\frac{M_i}{A\cap M_i}$, $A \cap M_i$ is a cofinite submodule of M_i for every $i \in I$. The rest of the proof by Theorem 2. □

Recall from [19] that an R-module M has the *summand sum property* (SSP) if the sum of two direct of M is again a direct summand of M.

Proposition 5. *Let M be a (cofinitely) (D_{12}^{*})-module with the property (SSP) and A be direct summand of M. Then $\frac{M}{A}$ is a (cofinitely) (D_{12}^{*})-module.*

Proof. Suppose that $\frac{B}{A}$ be a submodule of $\frac{M}{A}$. Then B is a submodule of M. As M is (cofinitely) (D^*_{12})-module, then there exists a direct summand T of M and an epimorphism $f: T \to \frac{M}{B}$ with $\ker(f) \ll_\delta T$. As M has the property (SSP), both A and T are direct summand of M . Hence, there exists a submodule Y of M such that $M = (A+T) \oplus Y$. Since the property $\frac{T+A}{A} \cap \frac{Y+A}{A} \subseteq \frac{[Y\cap(T+A)]+[A\cap(T+A+Y)]}{A} = \frac{A}{A}$, we have $\frac{M}{A} = \frac{T+A}{A} \oplus \frac{Y+A}{A}$. From here, we obtain that $\frac{\frac{M}{A}}{\frac{B}{A}} \cong \frac{M}{B}$. So we can define the homomorphism $g\colon \frac{T+A}{A} \to \frac{M}{B}$ by $t+a+A = t+A \to g(t)$ with $t \in T$, $a \in A$. It is clear that g is an epimorphism with $\ker(g) \ll_\delta \frac{T+A}{A}$ and $\frac{T+A}{A}$ is a direct summand of $\frac{M}{A}$. Finally, $\frac{M}{A}$ is a (cofinitely) (D^*_{12})-module. □

Theorem 7. *Let M be a (cofinitely) (D^*_{12})-module. If A is a fully invariant submodule of M, then $\frac{M}{A}$ is a (cofinitely) (D^*_{12})-module.*

Proof. Let $\frac{B}{A}$ be a (cofinite) submodule of $\frac{M}{A}$. Then B is a (cofinite) submodule of M. Since M is a (cofinitely) (D^*_{12})-module, there exists a direct summand T of M and an epimorphism $f\colon T \to \frac{M}{B}$ with $\ker(f) \ll_\delta T$. From here, we can write $M = T \oplus T_1$ for every submodule T_1 of M. As A is a fully invariant submodule of M, $A = (A\cap T) \oplus (A \cap T_1)$. Note that $\frac{M}{A} = \frac{A+T}{A} \oplus \frac{A+T_1}{A}$. It follows from $\frac{\frac{M}{A}}{\frac{B}{A}} \cong \frac{M}{B}$ that, we can define the homomorphism $g\colon \frac{T+A}{A} \to \frac{M}{B}$ by $t+A \to g(t+A) = f(t)$ with $t \in T$. Therefore g is an epimorphism with $\ker(g) \ll_\delta \frac{T+A}{A}$. Therefore, $\frac{M}{A}$ is a (cofinitely) (D^*_{12})-module. □

Theorem 8. *R is a δ-semiperfect ring if and only if every free right R-module is cofinitely (D^*_{12})-module.*

Proof.

($\Rightarrow$): Let R be a δ-semiperfect ring. By Lemma 3.5 in [12], every free right R-module is $\oplus_\delta$-cofinitely supplemented. Then every free right R-module is cofinitely (D^*_{12})-module by Proposition 4.

($\Leftarrow$): As every free right R-module is a cofinitely (D^*_{12})-module, it is $\oplus_\delta$-cofinitely supplemented by Theorem 3 (ii). If we use Lemma 3.5 in [12], then we get that R is δ-semiperfect.

□

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