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ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} \varphi_i \left(y^{(i)}\right)$

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Abstract. The asymptotic representations for solutions of differential equations of the type $y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} \varphi_i(y^{(i)})$, where φ_i are regularly varying in some one-sided neighborhoods of critical points, are established.

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1. INTRODUCTION

Consider the differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} \varphi_i(y^{(i)}), \tag{1.1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[$[1] $\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, ..., n)$ are continuous functions, $Y_i \in \{0, \pm\infty\}$ and Δ_{Y_i} is either the interval $[y_i^0, Y_i[$[2] or the interval $]Y_i, y_i^0]$.

We also suppose that every $\varphi_i(z)$ is regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index σ_i and $\sum_{i=0}^{n-1} \sigma_i \ne 1$.

We say that the measurable function $\varphi : \Delta_Y \to]0, +\infty[$ is regularly varying as $z \to Y$ of index σ if for every $\lambda > 0$ we have

$$\lim_{\substack{z \to Y \\ z \in \Delta_Y}} \frac{\varphi(\lambda z)}{\varphi(z)} = \lambda^\sigma. \tag{1.2}$$

[1] If $\omega > 0$ we will take $a > 0$.

[2] If $Y_i = +\infty (Y_i = -\infty)$ we take $y_i^0 > 0$ $(y_i^0 < 0)$ correspondingly.

Here, $Y \in \{0, \pm\infty\}$, Δ_Y is some one-sided neighbourhood of Y. If $\sigma = 0$, such function is called slowly varying.

It follows from the results of the monograph [11] that regularly varying functions have the next properties.

M_1: The function $\varphi(z)$ is regularly varying of index σ as $z \to Y$ if and only if the next representation takes place

$$\varphi(z) = z^{\sigma}\theta(z),$$

where $\theta(z)$ is a slowly varying function as $z \to Y$.

M_2: If the function $L : \Delta_{Y^0} \to]0, +\infty[$ is slowly varying as $z \to Y_0$, the function $\varphi : \Delta_Y \to \Delta_{Y^0}$ is regularly varying as $z \to Y$, then the function $L(\varphi) : \Delta_Y \to]0, +\infty[$ is slowly varying as $z \to Y$.

M_3: If the function $\varphi : \Delta_Y \to]0, +\infty[$ satisfies the condition

$$\lim_{\substack{z \to Y \\ z \in \Delta}} \frac{z\varphi'(z)}{\varphi(z)} = \sigma \in \mathbb{R},$$

then φ is regularly varying as $z \to Y$ of index σ.

M_4: For every regularly varying function φ as $z \to Y$ the property (1.2) takes place uniformly as $\lambda \in [c, d]$ for every segment $[c, d] \subset]0, +\infty[$.

According to M_1, it is clear that for every solution y of the equation (1.1) defined on $[t_0, \omega[\subset [a, \omega[$ such that

$$y^{(i)} : [t_0, \omega[\longrightarrow \Delta_{Y_i}, \qquad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \quad (i = 0, \dots, n-1), \tag{1.3}$$

the representations $\varphi_i(y^{(i)}(t)) = |y^{(i)}(t)|^{\sigma_i + o(1)}$ take place as $t \uparrow \omega$. Therefore the equation (1.1) is in some sense similar to the well known differential equation

$$y^{(n)} = \alpha_0 p(t) \prod_{i=0}^{n-1} |y^{(i)}|^{\sigma_i}. \tag{1.4}$$

We call the solution y of the equation (1.1), that satisfies (1.3), the $P_\omega(\lambda^0_{n-1})$-solution ($-\infty \le \lambda^0_{n-1} \le +\infty$) if the next condition takes place

$$\lim_{t \uparrow \omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t)\, y^{(n-2)}(t)} = \lambda^0_{n-1}. \tag{1.5}$$

If p is regularly varying as $t \uparrow \omega$ it is easy to show, using Proposition 9 in [10, p. 116], that every regularly varying as $t \uparrow \omega$ of index $\gamma \in R \setminus \{0, 1, \dots, n-1\}$ solution of Equation (1.1), that satisfies (1.3), is the $P_\omega\left(\frac{\gamma-n+2}{\gamma-n+1}\right)$-solution of (1.1). By the other side, by the investigation of $P_\omega(\lambda_0)$-solutions of Equation (1.1) it will be clear, that (1.1) has the $P_\omega\left(\frac{\gamma-n+2}{\gamma-n+1}\right)$-solutions, where $\gamma \in R\setminus\{0,1,...,n-1\}$, only if the function p is regularly varying as $t\uparrow\omega$ of index $\gamma-n+1-\sum\limits_{j=0}^{n-1}(\gamma-j)\sigma_j$.

It follows from the definition of $P_\omega(\lambda_0)$-solution, that every $P_\omega(1)$-solution of Equation (1.1) is rapidly varying as $t\uparrow\omega$. By the other side, by the investigation of $P_\omega(\lambda_0)$-solutions of Equation (1.1) it will be clear, that (1.1) has the $P_\omega(1)$-solutions only if the function p is rapidly varying as $t\uparrow\omega$.

The first results about the asymptotic properties and the existence of $P_\omega(\lambda^0_{n-1})$-solutions of Equation (1.4) are found in [4]. All $P_\omega(\lambda^0_{n-1})$-solutions of Equation (1.4) were investigated in [5,6]. In case $n=2$ for all $P_\omega(\lambda^0_{n-1})$-solutions of Equation (1.1) the necessary and sufficient conditions of existence and asymptotic representations as $t\uparrow\omega$ were found later in [1–3,8].

The aim of the work is to establish the necessary and sufficient conditions of the existence $P_\omega(\lambda^0_{n-1})$-solutions of Equation (1.1) in general case $n\ge 2$ for $\lambda^0_{n-1}\in\mathbb{R}\setminus\left\{0,\frac{1}{2},\frac{2}{3},...,\frac{n-2}{n-1}\right\}$.

2. Results

Let us introduce subsidiary notations

$$\gamma_0=1-\sum_{j=0}^{n-1}\sigma_j,\quad \mu_n=\sum_{j=0}^{n-1}(n-j-1)\sigma_j,\quad \pi_\omega(t)=\begin{cases} t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases}$$

$$\theta_i(z)=\varphi_i(z)|z|^{-\sigma_i},\quad a_{0i}=(n-i)\lambda^0_{n-1}-(n-i-1)\quad (i=1,...,n),$$

$$C=\alpha_0|\lambda^0_{n-1}-1|^{\mu_n}\prod_{k=0}^{n-2}\left|\prod_{j=k+1}^{n-1}a_{0j}\right|^{-\sigma_k} signy^0_{n-1},$$

$$I_0(t)=\int_{A^0_\omega}^{t}Cp(\tau)|\pi_\omega(\tau)|^{\mu_n}\,d\tau,\quad I_1(t)=\int_{A^1_\omega}^{t}\alpha_0 p(\tau)\,d\tau,$$

$$A^0_\omega=\begin{cases} a, & \text{if } \int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0}\,d\tau=+\infty,\\ \omega, & \text{if } \int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0}\,d\tau<+\infty,\end{cases}\quad A^1_\omega=\begin{cases} a, & \text{if } \int_a^\omega p(\tau)\,d\tau=+\infty,\\ \omega, & \text{if } \int_a^\omega p(\tau)\,d\tau<+\infty,\end{cases}$$

$$J(t)=\int_{B_\omega}^{t}|\gamma_0 I_1(\tau)|^{\frac{1}{\gamma_0}}\,d\tau,\quad B_\omega=\begin{cases} a, & \text{if } \int_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}}\,d\tau=+\infty,\\ \omega, & \text{if } \int_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}}\,d\tau<+\infty.\end{cases}$$

The following conclusions take place for Equation (1.1).

Theorem 1. *The next conditions are necessary for the existence of $P_\omega(\lambda^0_{n-1})$-solutions ($\lambda^0_{n-1} \in \mathbb{R}\setminus\{0, 1, \frac{1}{2}, \frac{2}{3}, ..., \frac{n-2}{n-1}\}$) of Equation (1.1)*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) I_0'(t)}{I_0(t)} = \frac{\gamma_0}{\lambda^0_{n-1} - 1}, \quad \lim_{t\uparrow\omega} y_i^0 |\pi_\omega(t)|^{\dfrac{a_{0i+1}}{\lambda^0_{n-1} - 1}} = Y_i, \tag{2.1}$$

$$y_i^0 y_{i+1}^0 a_{0i+1} (\lambda^0_{n-1} - 1) \pi_\omega(t) > 0 \quad as\ t \in [a, \omega[, \tag{2.2}$$

where $y_n^0 = \alpha_0$, $i = 0, ..., n-1$.

If Equation

$$\sum_{k=0}^{n-1} \sigma_k \prod_{i=k+1}^{n-1} a_{0i} \prod_{i=1}^{k} (a_{0i} + \lambda) = (1 + \lambda) \prod_{i=1}^{n-1} (a_{0i} + \lambda) \tag{2.3}$$

has no roots with zero real part, then Conditions (2.1) and (2.2) are sufficient for the existence of $P_\omega(\lambda^0_{n-1})$-solutions of Equation (1.1). For any such solution the next asymptotic representations as $t \uparrow \omega$

$$\begin{gathered} \frac{|y^{(n-1)}(t)|^{\gamma_0}}{\prod_{j=0}^{n-1} \theta_j(y^{(j)}(t))} = \gamma_0 I_0(t)[1 + o(1)], \\ \frac{y^{(i)}(t)}{y^{(n-1)}(t)} = \frac{\left[(\lambda^0_{n-1} - 1)\pi_\omega(t)\right]^{n-i-1}}{\prod_{j=i+1}^{n-1} a_{0j}} [1 + o(1)], \end{gathered} \tag{2.4}$$

where $i = 0, \ldots, n-2$, take place.

Theorem 2. *The next conditions are necessary for the existence of $P_\omega(1)$-solutions of Equation (1.1)*

$$\lim_{t\uparrow\omega} \frac{I_1'(t) J(t)}{I_1(t) J'(t)} = \gamma_0, \quad \lim_{t\uparrow\omega} y_i^0 |I_1(t)|^{\frac{1}{\gamma_0}} = Y_i, \tag{2.5}$$

$$\alpha_0 y^0_{n-2} > 0, \quad y_i^0 y_{i+1}^0 J(t) > 0 \ as\ t \in [a, \omega[, \tag{2.6}$$

where $i = 0, ..., n-1$.

If Equation

$$\sum_{k=0}^{n-1} \sigma_k (1 + \lambda)^k = (1 + \lambda)^n \tag{2.7}$$

has no roots with zero real part, then Conditions (2.5) and (2.6) are sufficient for the existence of $P_\omega(1)$-solutions of Equation (1.1). For any such solution the asymptotic *representations as $t \uparrow \omega$*

$$\frac{|y^{(n-1)}(t)|^{\gamma_0}}{\prod_{j=0}^{n-1} \theta_j(y^{(j)}(t))} = \gamma_0 I_1(t) \left| \frac{J(t)}{J'(t)} \right|^{\mu_n} signy^0_{n-1}[1+o(1)], \tag{2.8}$$

$$\frac{y^{(i)}(t)}{y^{(n-1)}(t)} = \left(\frac{J(t)}{J'(t)} \right)^{n-i-1} [1+o(1)],$$

where $i = 0, \dots, n-2$, take place.

Remark 1. If

$$\sum_{i=0}^{n-2} |\sigma_i| < |1-\sigma_{n-1}|,$$

Conditions (2.1) and (2.2) are necessary and sufficient for the existence of $P_\omega(\lambda^0_{n-1})$-solutions ($\lambda^0_{n-1} \in \mathbb{R} \setminus \{0, 1, \frac{1}{2}, \frac{2}{3}, ..., \frac{n-2}{n-1}\}$) of Equation (1.1) and Conditions (2.5) and (2.6) are necessary and sufficient for the existence of $P_\omega(1)$-solutions of (1.1).

By additional conditions on the functions φ_0, φ_1, ..., φ_{n-1} the asymptotic representations as $t \uparrow \omega$ of $P_\omega(\lambda^0_{n-1})$-solutions and their derivatives from the first to $(n-1)$-th order are found in the explicit form.

In order to formulate our following results we present the next definition.

Definition 1. We call the slowly varying function θ as $z \to Y$ $(z \in \Delta)$ satisfies the condition S if for every continuously differentiable function $L : \Delta \to]0; +\infty[$ such that

$$\lim_{\substack{z \to Y \\ z \in \Delta}} \frac{zL'(z)}{L(z)} = 0,$$

the representation

$$\theta(zL(z)) = \theta(z)[1+o(1)] \quad \text{as } z \to Y \quad (z \in \Delta)$$

takes place.

The next conclusions follow from Theorems 1 and 2.

Corollary 1. *Let the functions θ_0, ..., θ_{n-1} satisfy the condition S. Then for any $P_\omega(\lambda^0_{n-1})$-solution ($\lambda^0_{n-1} \in \mathbb{R} \setminus \{0, 1, \frac{1}{2}, \frac{2}{3}, ..., \frac{n-2}{n-1}\}$) of Equation (1.1) the next asymptotic representations as $t \uparrow \omega$*

$$y^{(n-1)}(t) = \left| \gamma_0 I_0(t) \prod_{j=0}^{n-1} \theta_j \left(y^0_j |\pi_\omega(t)|^{\frac{a_{0j+1}}{\lambda^0_{n-1}-1}} \right) \right|^{\frac{1}{\gamma_0}} signy^0_{n-1}[1+o(1)],$$

$$y^{(i)}(t)=y^{(n-1)}(t)\frac{\left[(\lambda_{n-1}^{0}-1)\pi_\omega(t)\right]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1}a_{0j}}[1+o(1)],\quad i=0,\dots,n-2$$

take place.

Corollary 2. *Let the functions θ_0, ..., θ_{n-1} satisfy the condition S. Then for any $P_\omega(1)$-solution of Equation (1.1) the next asymptotic representations as $t\uparrow\omega$*

$$y^{(n-1)}(t)=\left|\gamma_0 I_1(t)\left|\frac{J'(t)}{J(t)}\right|^{\mu_n}\prod_{j=0}^{n-1}\theta_j\left(y_j^0|J(t)|\right)\right|^{\frac{1}{\gamma_0}} sign y_{n-1}^0[1+o(1)],$$

$$y^{(i)}(t)=y^{(n-1)}(t)\left(\frac{J(t)}{J'(t)}\right)^{n-i-1}[1+o(1)],\quad i=0,\dots,n-2$$

take place.

2.1. *Preliminary considerations*

The following lemma is true.

Lemma 1. *Let $y:[t_0,\omega[\to\Delta_{Y_0}$ be a $P_\omega(\lambda_{n-1}^0)$-solution of Equation (1.1), where $\lambda_{n-1}^0\in\mathbb{R}\setminus\left\{0,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\right\}$. Then for every function $y^{(i)}$ ($i\in\{0,\dots,n-1\}$), the inverse function $\left(y^{(i)}\right)^{-1}$ exists. Moreover, the function $y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)$ is regularly varying as $z\to Y_i$ ($z\in\Delta_{Y_i}$) of index $\frac{a_{0j+1}}{a_{0i+1}}$ for every $j\in\{0,\dots,n-1\}$.*

Proof. Let $y:[t_0,\omega[\to\Delta_{Y_0}$ be a $P_\omega(\lambda_{n-1}^0)$-solution of Equation (1.1), where $\lambda_{n-1}^0\in\mathbb{R}\setminus\left\{0,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\right\}$. By (1.3) and (1.1) the function $y^{(i)}(t)$ is strongly monotone on $[t_0,\omega[$ for every $i\in\{0,\dots,n-1\}$, because its derivative $y^{(i+1)}$ has a fixed sign on $[t_0,\omega[$. Therefore $y^{(i)}$ has an inverse function $\left(y^{(i)}\right)^{-1}$. Due to (1.5) and Lemmas 10.1 and 10.2 from [7] we would have the asymptotic representations as $t\uparrow\omega$

$$\frac{\pi_\omega(t)y^{(k+1)}(t)}{y^{(k)}(t)}=\frac{a_{0k+1}}{(\lambda_{n-1}^0-1)}[1+o(1)],\quad (k=0,..,n-1) \tag{2.9}$$

in case $\lambda_{n-1}^0\in\mathbb{R}\setminus\left\{0,1,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\right\}$. Then, for every $j\in\{0,\dots,n-1\}$, according to the equality

$$\frac{z\left(y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)\right)'}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}=\frac{y^{(j+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(i+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}\cdot\frac{y^{(i)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} \tag{2.10}$$

and (2.9), we have

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{z\left(y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)\right)'}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} =$$

$$= \lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{\pi_\omega\left(\left(y^{(i)}\right)^{-1}(z)\right) y^{(j+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} \times$$

$$\times \frac{y^{(i)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{\pi_\omega\left(\left(y^{(i)}\right)^{-1}(z)\right) y^{(i+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} = \frac{a_{0j+1}}{a_{0i+1}}.$$

Therefore by M_3 the function $y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)$ is regularly varying as $z\to Y_i$ $(z\in \Delta_{Y_i})$ of index $\frac{a_{0j+1}}{a_{0i+1}}$ in case $\lambda^0_{n-1}\in\mathbb{R}\setminus\left\{0,1,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\right\}$.

By Lemma 10.1 from [7] and (1.5) the asymptotic representations as $t\uparrow\omega$

$$\frac{y^{(k+1)}(t)}{y^{(k)}(t)} = \frac{y^{(l+1)}(t)}{y^{(l)}(t)}[1+o(1)] \quad (l,k=0,..,n-1). \tag{2.11}$$

take place in case $\lambda^0_{n-1}=1$. Then, according to (2.10), in this case we have

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{z\left(y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)\right)'}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} = \lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{y^{(j+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}$$

$$\times \frac{y^{(i)}\left(\left(y^{(i)}\right)^{-1}(z)\right)}{y^{(i+1)}\left(\left(y^{(i)}\right)^{-1}(z)\right)} = 1 = \frac{(n-j-1)-(n-j-2)}{(n-i-1)-(n-i-2)} = \frac{a_{0j+1}}{a_{0i+1}}.$$

Therefore by M_3 the function $y^{(j)}\left(\left(y^{(i)}\right)^{-1}(z)\right)$ is regularly varying as $z\to Y_i$ $(z\in \Delta_{Y_i})$ of index $\frac{a_{0j+1}}{a_{0i+1}}$ in case $\lambda^0_{n-1}=1$. □

2.2. *Proof of the main results.*

Proof of Theorems 1 and 2. Necessity. Let $y:[t_0,\omega[\to\Delta_{Y_0}$ be a $P_\omega(\lambda^0_{n-1})$-solution of Equation (1.1), where $\lambda^0_{n-1}\in\mathbb{R}\setminus\left\{0,\frac{1}{2},\frac{2}{3},\dots,\frac{n-2}{n-1}\right\}$. Then by Lemmas 10.1 and 10.2 from [7], according to (1.3) and (1.5), the asymptotic representations (2.9) take place as $\lambda^0_{n-1}\neq 1$. It follows from (2.9), that (2.2) and the second of conditions (2.1) take place in this case. We get (2.11) if $\lambda^0_{n-1}=1$ like in the proof of Lemma 1. Putting in (2.11) $k=n-1$, $l=n-2$, we get the first of inequalities (2.6). By virtue of (1.5), corollary 10.1 and note 10.1 from [7] we have as $t\uparrow\omega$ for $i=0,..,n-2$

$$y^{(i)}(t)=\begin{cases}\dfrac{\left[(\lambda_{n-1}^0-1)\pi_\omega(t)\right]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1}a_{0j}}\,y^{(n-1)}(t)[1+o(1)], \text{ if } \lambda_{n-1}^0\neq 1,\\[2ex] y^{(n-1)}(t)\left(\dfrac{y^{(n-1)}(t)}{y^{(n)}(t)}\right)^{n-i-1}[1+o(1)], \text{ if } \lambda_{n-1}^0=1.\end{cases}\tag{2.12}$$

This means that the second of representations (2.4) takes place.

By Lemma 1 the inverse function $\left(y^{(n-1)}\right)^{-1}$ exists for the function $y^{(n-1)}$ on $[t_0,\omega[$. So, we have

$$\theta_i(y^{(i)}(t))=\theta_i\left(y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}\left(y^{(n-1)}(t)\right)\right)\right)\quad \forall i\in\{0,\dots,n-2\}.$$

From M_2 and Lemma 1 it follows, that every function $\theta_i\left(y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)\right)$ $(i\in\{0,\dots,n-2\})$ is slowly varying as $z\to Y_{n-1}$ $(z\in\Delta_{Y_{n-1}})$. In case $\lambda_{n-1}^0=1$ for all $i\in\{1,\dots,n\}$ $a_{0i}=1$. Therefore by Lemma 1 in this case every function $y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)$ $(i\in\{0,\dots,n-2\})$ is regularly varying of index 1 as $z\to Y_{n-1}$ $(z\in\Delta_{Y_{n-1}})$. Then the function

$$\psi(z)=\begin{cases}\dfrac{|z|^{\gamma_0}}{\prod\limits_{i=0}^{n-1}\theta_i\left(y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)\right)} \text{ if } \lambda_{n-1}^0\neq 1,\\[3ex] \dfrac{z\prod\limits_{i=0}^{n-1}\left|y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)\right|^{-\sigma_i}}{\prod\limits_{i=0}^{n-1}\theta_i\left(y^{(i)}\left(\left(y^{(n-1)}\right)^{-1}(z)\right)\right)} \text{ if } \lambda_{n-1}^0=1,\end{cases}$$

is regularly varying as $z\to Y_{n-1}$ $(z\in\Delta_{Y_{n-1}})$ of index $\gamma_0\neq 0$.

For all $i=0,\dots,n-2$ using (2.12) and (1.1), we get as $t\uparrow\omega$

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)}\psi\left(y^{(n-1)}(t)\right)=I_k'(t)[1+o(1)],\quad k=\begin{cases}0, \text{ if } \lambda_{n-1}^0\neq 1,\\ 1, \text{ if } \lambda_{n-1}^0=1.\end{cases}\tag{2.13}$$

By (2.13), Theorem 2.1 from [11] and the definition of the functions $I_k(t)$, we have as $t \uparrow \omega$

$$\psi\left(y^{(n-1)}(t)\right) = \gamma_0 I_k(t)[1+o(1)], \quad k = \begin{cases} 0, \text{ if } \lambda^0_{n-1} \neq 1, \\ 1, \text{ if } \lambda^0_{n-1} = 1. \end{cases} \tag{2.14}$$

If $\lambda_0 \neq 1$ ($k = 0$) the first of representations (2.4) follows from this.

Using (2.14) and (2.13) we have as $t \uparrow \omega$

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{I'_j(t)}{\gamma_0 I_j(t)}[1+o(1)], \quad j = \begin{cases} 0, \text{ if } \lambda^0_{n-1} \neq 1, \\ 1, \text{ if } \lambda^0_{n-1} = 1. \end{cases} \tag{2.15}$$

Therefore the first of conditions (2.1) follows from (2.15), where $j = 0$ and (2.9), where $k = n-1$. The second of conditions (2.5) follows from (2.15), (2.11) and (1.3) in case $\lambda^0_{n-1} = 1$.

In view of M_2 and Lemma 1 every function $\theta_i\left(y^{(i)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right)\right)$ ($i \in \{0, \dots, n-1\}$) is slowly varying as $z \to Y_{n-1}$ ($z \in \Delta_{Y_{n-1}}$). Then by Lemma 1 the function

$$\psi_1(z) = \frac{z}{y^{(n-1)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right)}$$

$$\times \left| y^{(n-1)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right) \prod_{i=0}^{n-1} \frac{\left| y^{(i)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right)\right|^{-\sigma_i}}{\theta_i\left(y^{(i)}\left(\left(y^{(n-2)}\right)^{-1}(z)\right)\right)} \right|^{\frac{1}{\gamma_0}}$$

is regularly varying as $z \to Y_{n-1}$ ($z \in \Delta_{Y_{n-1}}$) of index 1 in case $\lambda^0_{n-1} = 1$. By rewriting (2.14), where $k = 1$ as

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\psi_1\left(y^{(n-2)}(t)\right) = J'(t)[1+o(1)] \quad \text{as } t \uparrow \omega, \tag{2.16}$$

in case $\lambda^0_{n-1} = 1$ due to the Theorem 2.1 from [11] we get

$$\psi_1\left(y^{(n-2)}(t)\right) = J(t)[1+o(1)] \quad \text{as } t \uparrow \omega. \tag{2.17}$$

By (2.11) this representation leads to the second of inequalities (2.6). From (2.17) and (2.16) we have

$$\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = \frac{J'(t)}{J(t)}[1+o(1)] \quad \text{as } t \uparrow \omega. \tag{2.18}$$

Using (2.11), we get from this representation the second of representations (2.8). By putting the second of representations (2.8) in (2.14), where $k = 1$, we get the first of representations (2.8). From (2.11), (2.15) and (2.18) the first of conditions (2.5) follows.

Sufficiency. Let $\lambda^0_{n-1} \in \mathbb{R}\setminus\{0, \frac{1}{2}, \frac{2}{3}, ..., \frac{n-2}{n-1}\}$. Let also Conditions (6 – 8) take place as $\lambda^0_{n-1} \neq 1$ and Conditions (10 – 12) take place as $\lambda^0_{n-1} = 1$.

Let us consider the function

$$F(s_0, s_1, ..., s_{n-1}) = \begin{pmatrix} \prod\limits_{i=0}^{n-1} \Phi_i(s_i) \\ \frac{s_1}{s_0} \\ \frac{s_2}{s_1} \\ ... \\ \\ \frac{s_{n-1}}{s_{n-2}} \end{pmatrix},$$

where

$$\Phi_i(z) = \int\limits_{Y_i^*}^{z} \frac{d\tau}{\varphi_i(\tau)|\tau|^{1-c_i-\sigma_i}}, \tag{2.19}$$

$$c_i = \begin{cases} \frac{\gamma_0(1+n-m)}{m}, & \text{if } (n-i-1)\lambda^0_{n-1} > n-i-2, \\ -\gamma_0, & \text{if } (n-i-1)\lambda^0_{n-1} < n-i-2, \end{cases} \tag{2.20}$$

m is the quantity of numbers $i \in \{0, ..., n-1\}$, for which $(n-i-1)\lambda^0_{n-1} > n-i-2$,

$$Y_i^* = \begin{cases} y_i^0, & \text{if } \left| \int\limits_{y_i^0}^{Y_i} \frac{dz}{\varphi_i(z)|z|^{1-c_i-\sigma_i}} \right| = +\infty, \\ Y_i, & \text{if } \left| \int\limits_{y_i^0}^{Y_i} \frac{dz}{\varphi_i(z)|z|^{1-c_i-\sigma_i}} \right| < +\infty \end{cases} \qquad (i = 0, ..., n-1)$$

on the set $\Delta = \Delta^1_{Y_0} \times ... \times \Delta^1_{Y_{n-1}}$. For every $i \in \{0, ..., n-1\}$ the number $y_i^1 \in \Delta_{Y_i}$ is chosen in such a way, that for $z_i \in \Delta^1_{Y_i}$, where

$$\Delta^1_{Y_i} = \begin{cases} [y_i^1, Y_i[, & \text{if } \Delta_{Y_i} = [y_i^0, Y_i[, \\ ]Y_i, y_i^1], & \text{if } \Delta_{Y_i} =]Y_i, y_i^0], \end{cases}$$

the next inequality holds

$$\left|\frac{z_i\Phi_i'(z_i)}{\Phi_i(z_i)}-c_i\right| < \frac{\left|1-\sum\limits_{i=0}^{n-1}\sigma_i\right|}{8mn}. \tag{2.21}$$

It is easy to see that the function Φ_i is monotone on Δ_{Y_i} for every $i\in\{0,\dots,n-1\}$. In view of the second of conditions (2.1) and (2.20) the following equality holds

$$\prod_{i=0}^{n-1}\lim_{\substack{z_i\to Y_i\\ z\in\Delta_{Y_i}}}\Phi_i(z_i)=\Phi_{0n}, \tag{2.22}$$

where

$$\Phi_{0n}=\begin{cases}\infty, & \text{if } \pi_\omega(t)(\lambda_{n-1}^0-1)\gamma_0>0, \text{ or } \alpha_0 I_1(t)>0 \text{ if } \lambda_{n-1}^0=1,\\ 0, & \text{if } \pi_\omega(t)(\lambda_{n-1}^0-1)\gamma_0<0, \text{ or } \alpha_0 I_1(t)<0 \text{ if } \lambda_{n-1}^0=1.\end{cases} \tag{2.23}$$

We have also

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}}\frac{z_i\Phi_i'(z_i)}{\Phi_i(z_i)}=c_i \quad (i=0,\dots,n-1), \quad \sum_{i=0}^{n-1}c_i=\gamma_0. \tag{2.24}$$

We will show that F is the one to one correspondence between the set Δ and the set

$$F(\Delta)=\begin{cases}\left[\prod\limits_{i=0}^{n-1}\Phi_i(y_i^1);\Phi_{0n}\right)\times\Delta_0\times\dots\times\Delta_{n-2}, & \text{if } \prod\limits_{i=0}^{n-1}\Phi_i(y_i^1)<\Phi_{0n},\\ \left(\Phi_{0n};\prod\limits_{i=0}^{n-1}\Phi_i(y_i^1)\right]\times\Delta_0\times\dots\times\Delta_{n-2}, & \text{if } \prod\limits_{i=0}^{n-1}\Phi_i(y_i^1)>\Phi_{0n},\end{cases}$$

where

$$\Delta_i=\begin{cases}(0;+\infty), & \text{if } a_{0i+2}a_{0i+1}>0,\ y_i^0y_{i+1}^0>0,\\ (-\infty;0), & \text{if } a_{0i+2}a_{0i+1}>0,\ y_i^0y_{i+1}^0<0,\\ \left[\frac{y_{i+1}^1}{y_i^1};Y_i^0\right), & \text{if } a_{0i+2}a_{0i+1}<0,\ \frac{y_{i+1}^1}{y_i^1}<Y_i^0,\\ \left(Y_i^0;\frac{y_{i+1}^1}{y_i^1}\right], & \text{if } a_{0i+2}a_{0i+1}<0,\ \frac{y_{i+1}^1}{y_i^1}>Y_i^0,\end{cases}$$

$$Y_i^0=\begin{cases}Y_i, & \text{if } Y_{i+1}=0,\\ -\infty, & \text{if } Y_{i+1}=\infty,\end{cases} \qquad i=0,\dots,n-2.$$

Let us suppose that F is not a one to one mapping. Then

$$\exists\,(p_0,...,p_{n-1}),(q_0,...,q_{n-1})\in\Delta,\ (p_0,...,p_{n-1})\neq(q_0,...,q_{n-1})$$

such that

$$F(p_0,...,p_{n-1})=F(q_0,...,q_{n-1}).$$

By definition of the set Δ the last equality means that

$$\prod_{i=0}^{n-1}\Phi_i(p_i)=\prod_{i=0}^{n-1}\Phi_i(q_i),\quad \frac{p_{j+1}}{p_j}=\frac{q_{j+1}}{q_j}=k_j\in R\setminus\{0\},\quad j=1,...,n-1. \tag{2.25}$$

Then the points $(p_0,...,p_{n-1})$ and $(q_0,...,q_{n-1})$ lie on the one line

$$\frac{s_0}{1}=\frac{s_1}{k_1}=\frac{s_2}{k_1k_2}=...=\frac{s_{n-1}}{k_1...k_{n-1}}.$$

On this line $\prod_{i=0}^{n-1}\Phi_i(s_i)=\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)$. We obtain also

$$\left(\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)\right)'_{s_0}=$$

$$=\frac{\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)}{s_0}\left(\frac{s_0\Phi_0'(s_0)}{\Phi_0(s_0)}+\sum_{i=1}^{n-1}\frac{k_1...k_is_0\Phi_i'(k_1...k_is_0)}{\Phi_i(k_1...k_is_0)}\right).$$

By (2.22), (2.21) and the definition of the set Δ this means, that

$$sign\left(\left(\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)\right)'_{s_0}\right)=sign\left(y_0^0\Phi^0\left(1-\sum_{i=0}^{n-1}\sigma_i\right)\right),$$

where

$$\Phi^0=\prod_{i=0}^{n-1}sign\Phi_i(z_i),\quad (z_i\in\Delta^1_{Y_i}). \tag{2.26}$$

Therefore the function $\Phi_0(s_0)\prod_{i=1}^{n-1}\Phi_i(k_1...k_is_0)$ is strongly monotone on this line. But then (2.25) is impossible. So, there exists the inverse function $F^{-1}:\ F(\Delta)\to\Delta$. Looking on the function F, we obtain

$$F^{-1}(w_0,...,w_{n-1})=$$

$$= \begin{pmatrix} F_0^{-1}(w_0,...,w_{n-1}) \\ F_1^{-1}(w_0,...,w_{n-1}) \\ ... \\ F_{n-1}^{-1}(w_0,...,w_{n-1}) \end{pmatrix} = \begin{pmatrix} F_0^{-1}(w_0,...,w_{n-1}) \\ w_1 F_0^{-1}(w_0,...,w_{n-1}) \\ ... \\ w_1...w_{n-1} F_0^{-1}(w_0,...,w_{n-1}) \end{pmatrix}.$$

Let us show that we can choose the number $t_0 \in [a,\omega[$ in such a way, that

$$(H_0(t),...,H_{n-1}(t)) \in F(\Delta) \text{ if } t \in [t_0,\omega[, \tag{2.27}$$

where

$$H_0(t) = \gamma_0 B T(t) \left| \frac{L(t)}{L'(t)} \right|^{\sum\limits_{i=0}^{n-1}(n-i-1)c_i},$$

$$H_i(t) = \begin{cases} \dfrac{a_{0i}}{(\lambda_{n-1}^0 - 1)\pi_\omega(t)} \text{ if } \lambda_{n-1}^0 \neq 1, \\ \dfrac{J'(t)}{J(t)} \text{ if } \lambda_{n-1}^0 = 1, \end{cases} \quad i = 1,...,n-1,$$

$$T(t) = \begin{cases} I_0(t) \text{ if } \lambda_{n-1}^0 \neq 1, \\ I_1(t) \left| \dfrac{J(t)}{J'(t)} \right|^{\mu_n} \text{ if } \lambda_{n-1}^0 = 1, \end{cases} \quad L(t) = \begin{cases} \pi_\omega(t) \text{ if } \lambda_{n-1}^0 \neq 1, \\ J(t) \text{ if } \lambda_{n-1}^0 = 1, \end{cases}$$

$$B = \begin{cases} \prod\limits_{i=0}^{n-1} \left(\dfrac{signy_i^0}{c_i} |\lambda_{n-1}^0 - 1|^{(n-i-1)c_i} \prod\limits_{j=i+1}^{n-1} |a_{0j}|^{-c_i} \right) \text{ if } \lambda_{n-1}^0 \neq 1, \\ \prod\limits_{i=0}^{n-1} \dfrac{signy_i^0}{c_i} signy_{n-1}^0 \text{ if } \lambda_{n-1}^0 = 1. \end{cases}$$

By the type of the set $F(\Delta)$it is sufficient to prove that

$$signH_0(t) = \Phi^0 \text{ if } t \in [a,\omega[, \tag{2.28}$$

for all $i = 1,...,n-1$

$$signH_i(t) = signy_i^0 y_{i-1}^0 \text{ if } t \in [a,\omega[, \tag{2.29}$$

$$\lim_{t\uparrow\omega} H_0(t) = \Phi_{0n}, \tag{2.30}$$

and for all $i = 1,...,n-1$ if $a_{0i+1}a_{0i} < 0$

$$\lim_{t\uparrow\omega} H_i(t) = Y_{i-1}^0. \tag{2.31}$$

By (2.24)

$$\Phi_i(z) = \frac{|z|^{c_i}}{c_i\theta_i(z)} sign z[1+o(1)] \text{ as } z \to Y_i \ \left(z \in \Delta^1_{Y_i}\right). \tag{2.32}$$

Therefore $sign \Phi_i(z) = sign(y_i^0 c_i)$. Then by (2.26)

$$\Phi^0 = sign \prod_{i=0}^{n-1} y_i^0 c_i \tag{2.33}$$

Using the first of conditions (2.1), we obtain in case $\lambda^0_{n-1} \neq 1$

$$\pi_\omega(t)(\lambda^0_{n-1}-1)y^0_{n-1} > 0 \text{ if } t \in [a,\omega[.$$

From this inequality and the condition (2.2), where $i=n-1$, it follows that $\gamma_0 I_0(t) > 0$ for $t \in [a,\omega[$ in case $\lambda^0_{n-1} \neq 1$. Therefore, in this case from (2.33) we have (2.28).

Using the first of conditions (2.5), in case $\lambda^0_{n-1} = 1$ we obtain

$$\gamma_0\alpha_0 I_1(t)J(t) > 0 \text{ if } t \in [a,\omega[.$$

From this inequality and the condition (2.6), where $i=n-1$, it follows that $\alpha_0 y^0_{n-1}J(t) > 0$ for $t \in [a,\omega[$ in case $\lambda^0_{n-1} = 1$. Therefore, $\gamma_0 y^0_{n-1} I_1(t) > 0$ for $t \in [a,\omega[$ and from (2.33) we have (2.28) in this case.

The equality (2.29) follows from (2.2) in case $\lambda^0_{n-1} \neq 1$. If $\lambda^0_{n-1} = 1$ we get (2.29) due to the second of conditions (2.6).

It is easy to see, that if $i \in \{1,\dots,n-1\}$ and

$$\frac{a_{0i}}{a_{0i+1}} = \frac{(n-i)\lambda^0_{n-1}-(n-i-1)}{(n-i-1)\lambda^0_{n-1}-(n-i-2)} = 1 + \frac{\lambda^0_{n-1}-1}{(n-i-1)\lambda^0_{n-1}-(n-i-2)} < 0.$$

the next inequality

$$\frac{\lambda^0_{n-1}-1}{(n-i-1)\lambda^0_{n-1}-(n-i-2)} < 0 \tag{2.34}$$

takes place. By (2.34), the second of conditions (2.1), the definition of the function $\pi_\omega(t)$ and the definition of Y_1^0 we have (2.31).

Let us consider the equality

$$\frac{L(t)H_0'(t)}{L'(t)H_0(t)} = \frac{L(t)T'(t)}{L'(t)T(t)} + \left(\sum_{i=0}^{n-1}(n-i-1)c_i\right)\left(1-\frac{L(t)L''(t)}{(L'(t))^2}\right).$$

Here

$$\frac{L(t)T'(t)}{L'(t)T(t)} = \begin{cases} \dfrac{\pi_\omega(t)I_0'(t)}{I_0(t)} & \text{if } \lambda^0_{n-1} \neq 1, \\[2ex] \dfrac{J(t)I_1'(t)}{J'(t)I_1(t)} + \mu_n\left(1-\dfrac{J(t)J''(t)}{(J'(t))^2}\right) & \text{if } \lambda^0_{n-1} = 1, \end{cases}$$

$$\frac{L(t)L''(t)}{(L'(t))^2} = \begin{cases} 0 \ \text{ if } \lambda_{n-1}^0 \neq 1, \\ \frac{J(t)J''(t)}{(J'(t))^2} \ \text{ if } \lambda_{n-1}^0 = 1. \end{cases}$$

In case $\lambda_{n-1}^0 = 1$

$$\frac{J(t)J''(t)}{(J'(t))^2} = \frac{J(t)I_1'(t)}{\gamma_0 J'(t)I_1(t)}. \tag{2.35}$$

So, using the first of conditions (2.1) and the first of conditions (2.5), we have as $t \uparrow \omega$

$$\frac{L(t)H_0'(t)}{L'(t)H_0(t)} = \begin{cases} \sum\limits_{i=0}^{n-1} \frac{a_{0i+1}c_i}{\lambda_{n-1}^0 - 1}[1+o(1)] \text{ if } \lambda_{n-1}^0 \neq 1, \\ \gamma_0[1+o(1)] \text{ if } \lambda_{n-1}^0 = 1. \end{cases} \tag{2.36}$$

The next representation

$$\lim_{t\uparrow\omega} H_0(t) = \begin{cases} \infty, \quad \text{if } \pi_\omega(t)(\lambda_{n-1}^0 - 1)\left(\sum\limits_{i=0}^{n-1} a_{0i+1}c_i\right) > 0 \text{ or } \gamma_0 J(t) > 0, \\ 0, \quad \text{if } \pi_\omega(t)(\lambda_{n-1}^0 - 1)\left(\sum\limits_{i=0}^{n-1} a_{0i+1}c_i\right) < 0 \text{ or } \gamma_0 J(t) < 0. \end{cases} \tag{2.37}$$

follows from (2.36) by the type of functions $L(t)$ and $H_0(t)$.

By definition of numbers c_i, the inequality $\gamma_0 a_{0i+1} c_i > 0$ takes place for all $i \in \{0,\dots,n-1\}$. Therefore, if $\lambda_{n-1}^0 \neq 1$ we obtain (2.30) using (2.37) and (2.2). The existence of such $b_0 \in [a,\omega[$ that for any $t \in [b_0,\omega[$ the inequality $\alpha_0 I(t)\gamma_0 J(t) > 0$ takes place, follows from the first of representations (2.5) in case $\lambda_{n-1}^0 = 1$. This leads to (2.30) by (2.23).

So, we have (2.27).

Let us introduce the next notations for all $i \in \{0,\dots,n-1\}$

$$Y^{[i]}(t) = F_i^{-1}\left(H_0(t),\dots,H_{n-1}(t)\right).$$

From the properties of the function F and the functions H_0,...,H_{n-1} it is easy to see, that for all $i \in \{0,\dots,n-1\}$ $Y^{[i]}$ is continuously differentiable on $[t_0,\omega[$.

Using (2.1), (2.2), (2.5), (2.6) and the transformation of Equation (1.1)

$$y^{(i)}(t) = Y^{[i]}(t)[1+z_i(x)], \tag{2.38}$$

where

$$x = \beta \ln|L(t)|, \quad \beta = \begin{cases} 1 & \text{if} \quad \lim\limits_{t\uparrow\omega} L(t) = \infty, \\ -1 & \text{if} \quad \lim\limits_{t\uparrow\omega} L(t) = 0, \end{cases} \tag{2.39}$$

we get the system of differential equations

$$\begin{cases} z_i' = \dfrac{\beta L(t(x))}{L'(t(x))Y^{[i]}(t(x))}\Big[-(Y^{[i]})'(t(x))[1+z_i]+ \\ \qquad\qquad +Y^{[i+1]}(t(x))[1+z_{i+1}]\Big],\ (i=0,...,n-2) \\ z_{n-1}' = \dfrac{\beta L(t(x))}{L'(t(x))}\Bigg[G(t(x))\prod\limits_{j=0}^{n-1}|1+z_j|^{\sigma_j}Q_j(x,z_j)- \\ \qquad\qquad -\dfrac{\left(Y^{[n-1]}\right)'(t(x))}{Y^{[n-1]}(t(x))}[1+z_{n-1}]\Bigg], \end{cases} \tag{2.40}$$

where

$$G(t) = \frac{\alpha_0 p(t)\prod\limits_{j=0}^{n-1}\varphi_j\left(Y^{[j]}(t)\right)}{Y^{[n-1]}(t)}, \quad Q_j(x,z_j) = \frac{\theta_j\left(Y^{[j]}(t(x))[1+z_j]\right)}{\theta_j\left(Y^{[j]}(t(x))\right)},$$

$j=0,...,n-1$, $t(x)$ is the inverse function for the function $x=\beta\ln|L(t)|$.

Then we consider the system (2.40) on the set

$$\Omega = [x_0,+\infty[\times D, \qquad x_0=\beta\ln|\pi_\omega(t_0)|,$$
$$D=\{(z_1,z_2): |z_i|\le\varepsilon,\ i=1,2\}.$$

The system (2.40) can be rewritten as the system

$$z_i' = \sum_{j=0}^{n-1}A_{ij}z_j + R_1^{[i]}(x,z_0,...,z_{n-1}) + R_2^{[i]}(x,z_0,...,z_{n-1}), \quad i=0,...,n-1, \tag{2.41}$$

where

$$A_{ij} = \begin{cases} -\dfrac{\beta a_{0i+1}}{\lambda_{n-1}^0-1} & \text{as } j=i,\ \lambda_{n-1}^0\ne 1, \\ \dfrac{\beta a_{0i+1}}{\lambda_{n-1}^0-1} & \text{as } j=i+1,\ \lambda_{n-1}^0\ne 1, \\ -\beta & \text{as } j=i,\ \lambda_{n-1}^0=1, \\ \beta & \text{as } j=i+1,\ \lambda_{n-1}^0=1, \\ 0 & \text{as } j\notin\{i,i+1\}, \end{cases}$$
$$if \quad i=0,...,n-2,\ j=0,...,n-1,$$

$$A_{n-1\ j} = \begin{cases} \dfrac{\beta\sigma_j}{\lambda^0_{n-1}-1} & \text{as } j \neq n-1,\ \lambda^0_{n-1} \neq 1, \\ \dfrac{\beta(\sigma_{n-1}-1)}{\lambda^0_{n-1}-1} & \text{as} j = n-1,\ \lambda^0_{n-1} \neq 1, \\ \beta\sigma_j & \text{as } j \neq n-1,\ \lambda^0_{n-1} = 1, \\ \beta(\sigma_{n-1}-1) & \text{as } j = n-1,\ \lambda^0_{n-1} = 1, \end{cases}$$

$$if \quad j = 0,...,n-1,$$

$$R_1^{[i]}(x,z_0,...,z_{n-1}) =$$

$$= \begin{cases} 0 & \text{as } i = 0,...,n-2, \\ \dfrac{\prod\limits_{j=0}^{n-1}|1+z_j|^{\sigma_j} - \sum\limits_{j=0}^{n-1}\sigma_j z_j}{\beta(\lambda^0_{n-1}-1)} & \text{as } i = n-1,\ \lambda^0_{n-1} \neq 1, \\ \beta\left(\prod\limits_{j=0}^{n-1}|1+z_j|^{\sigma_j} - \sum\limits_{j=0}^{n-1}\sigma_j z_j\right) & \text{as } i = n-1,\ \lambda^0_{n-1} = 1, \end{cases}$$

$$R_2^{[i]}(x,z_0,...,z_{n-1}) = \beta[1+z_i]\left(G_0 a_{0i+1} - \frac{L(t(x))(Y^{[i]})'(t(x))}{L'(t(x))Y^{[i]}(t(x))}\right) +$$

$$+\beta[1+z_{i+1}]\left(\frac{L(t(x))Y^{[i+1]}(t(x))}{L'(t(x))Y^{[i]}(t(x))} - G_0 a_{0i+1}\right) \text{ as } i = 0,...,n-2,$$

$$R_2^{[n-1]}(x,z_0,...,z_{n-1}) =$$

$$= \beta\prod_{j=0}^{n-1}|1+z_j|^{\sigma_j}\left(\frac{L(t(x))}{L'(t(x))}G(t(x))\left(\prod_{j=0}^{n-1}Q_j(x,z_j)-1\right)+\right.$$

$$\left.+\frac{L(t(x))}{L'(t(x))}G(t(x)) - G_0\right) + \left(\frac{L(t(x))(Y^{[n-1]})'(t(x))}{L'(t(x))Y^{[n-1]}(t(x))} - G_0\right)(1+z_{n-1})$$

$$G_0 = \begin{cases} \dfrac{1}{\lambda^0_{n-1}-1} \text{ as} \lambda^0_{n-1} \neq 1, \\ 1 \text{ if } \lambda^0_{n-1} = 1. \end{cases}$$

It is clear that

$$\lim_{|z_0|+...+|z_{n-1}|\to 0} \frac{R_1^{[i]}(x,z_0,...,z_{n-1})}{|z_0|+...+|z_{n-1}|} = 0 \quad (i = 0,...,n-1)$$

uniformly on $x \in [x_0, +\infty[$.

To prove that

$$\lim_{x\to+\infty} R_2^{[i]}(x, z_0, \dots, z_{n-1}) = 0 \quad (i = 0, \dots, n-1) \tag{2.42}$$

uniformly on $(z_0, \dots, z_{n-1}) \in D$, we have to prove that for all $j \in \{0, \dots, n-2\}$

$$\lim_{t\uparrow\omega} \frac{L(t)(Y^{[j]})'(t)}{L'(t)Y^{[j]}(t)} = G_0 a_{0j+1}, \quad \lim_{t\uparrow\omega} \frac{L(t)Y^{[j+1]}(t)}{L'(t)Y^{[j]}(t)} = G_0 a_{0j+1}, \tag{2.43}$$

$$\lim_{t\uparrow\omega} \frac{L(t)G(t)}{L'(t)} = G_0 \tag{2.44}$$

and for all $i \in \{0, \dots, n-1\}$

$$\lim_{t\uparrow\omega} \frac{\theta_i\left(Y^{[i]}(t)[1+z_i]\right)}{\theta_i\left(Y^{[i]}(t)\right)} = 1, \quad \text{uniformly for} \quad |z_i| < \frac{1}{2}. \tag{2.45}$$

Let us show at first that for all $i \in \{0, \dots, n-1\}$

$$\lim_{t\uparrow\omega} Y^{[i]}(t) = Y_i. \tag{2.46}$$

The inequality

$$\left| \frac{Y^{[i]}(t)\Phi_i'(Y^{[i]}(t))}{\Phi_i(Y^{[i]}(t))} - c_i \right| < \frac{|\gamma_0|}{8mn}$$

for all $t \in [t_0, \omega)$ follows from (2.27) and the definition of the set Δ. By this inequality for $t \in [t_0, \omega[$ we have

$$-\frac{|\gamma_0|}{8mn} + c_i < \frac{Y^{[i]}(t)\Phi_i'(Y^{[i]}(t))}{\Phi_i(Y^{[i]}(t))} < \frac{|\gamma_0|}{8mn} + c_i, \tag{2.47}$$

From (2.47), (2.36), (2.35), the first of conditions (2.2), the first of conditions (2.6) and the equalities

$$\frac{L(t)\left(Y^{[i]}(t)\right)_t'}{L'(t)Y^{[i]}(t)} = \sum_{k=0}^{n-1} \frac{L(t)H_k'(t)}{L'(t)H_k(t)} P_{ik}(t), \tag{2.48}$$

$$\frac{L(t)H_j'(t)}{L'(t)H_j(t)} = \begin{cases} 1 \text{ if } \lambda_{n-1}^0 \neq 1, \\ \dfrac{J''(t)J(t)}{(J'(t))^2} - 1 \text{ if } \lambda_{n-1}^0 = 1, \end{cases} \qquad (j = 1, \dots, n-1)$$

where for all $i \in \{0,...,n-1\}$, $k \in \{1,...,n-1\}$

$$P_{i0}(t) = \frac{1}{\sum\limits_{j=0}^{n-1} \frac{Y^{[j]}(t)\Phi_j'(Y^{[j]}(t))}{\Phi_i(Y^{[j]}(t))}}, \quad P_{ik}(t) = l_k^i + P_{i0}(t)\sum_{j=0}^{k-1} \frac{Y^{[j]}(t)\Phi_j'(Y^{[j]}(t))}{\Phi_i(Y^{[j]}(t))},$$

$$l_k^i = \begin{cases} -1 \text{ if } i \le k-1, \\ 0 \text{ if } i > k-1, \end{cases}$$

$$-\sum_{k=1}^{n-1}\left(l_k^i + \frac{1}{\gamma_0}\sum_{j=0}^{k-1} c_j\right) + \frac{1}{\gamma_0(\lambda_{n-1}^0 - 1)}\sum_{i=0}^{n-1} a_{0i}c_i =$$

$$= -\left(n-i-1+\frac{1}{\gamma_0}\sum_{k=1}^{n-1}\sum_{j=0}^{n-1} l_k^j c_j\right) + \frac{1}{\gamma_0(\lambda_{n-1}^0 - 1)}\sum_{i=0}^{n-1} a_{0i}c_i$$

$$= -\left(n-i-1+\frac{1}{\gamma_0}\sum_{j=0}^{n-1}(n-j-1)c_j\right) + \frac{\sum\limits_{i=0}^{n-1} c_i}{\gamma_0(\lambda_{n-1}^0 - 1)} + \tag{2.49}$$

$$+\frac{1}{\gamma_0}\sum_{i=0}^{n-1}(n-i-1)c_i = \frac{1}{\lambda_{n-1}^0 - 1} - (n-i-1) = \frac{a_{0i+1}}{\lambda_{n-1}^0 - 1},$$

it follows that for every $i \in \{0,1\}$ there exist constants $k_0^i, k_1^i \in \mathbb{R}$ such that

$$sign k_0^i = sign k_1^i = \begin{cases} sign(a_{0i+1}(\lambda_0 - 1)) \text{ if } \lambda_{n-1}^0 \neq 1, \\ 1 \text{ if } \lambda_{n-1}^0 = 1, \end{cases}$$

and the number $t_2 \in [t_1, \omega[$ such that

$$\frac{k_0^i L'(t)}{L(t)} < \frac{\left(Y^{[i]}(t)\right)'}{Y^{[i]}(t)} < \frac{k_1^i L'(t)}{L(t)} \text{ for} t \in [t_2, \omega[.$$

By integrating this inequality on $[t_2, t]$ we will get

$$\ln\frac{|L(t)|^{k_0^i}}{|L(t_2)|^{k_0^i}} + \ln\left|Y^{[i]}(t_2)\right| < \ln\left|Y^{[i]}(t)\right| < \ln\frac{|L(t)|^{k_1^i}}{|L(t_2)|^{k_1^i}} + \ln\left|Y^{[i]}(t_2)\right|.$$

From (2.27) and one to one property of F it is clear, that $sign Y^{[i]}(t) = sign y_i^0$ for every $i \in \{0,...,n-1\}$. So, we have (2.46).

The representation (2.45) follows from (2.46) M_3 and M_4.

By the definition of $Y^{[i]}$ $(i=0,...,n-2)$

$$F(Y^{[0]}(t),...,Y^{[n-1]}(t))=(H_0(t),...,H_{n-1}(t)) \quad \forall t\in[t_2,\omega[.$$

Therefore

$$\frac{Y^{[i+1]}(t)}{Y^{[i]}(t)}=H_{i+1}(t) \quad \forall i=0,...,n-2.$$

So

$$\frac{L(t)Y^{[i+1]}(t)}{L'(t)Y^{[i]}(t)}=\begin{cases}\dfrac{a_{0i+1}}{\lambda_{n-1}^0-1} \text{ if } \lambda_{n-1}^0\neq 1,\\[2ex] 1 \text{ if } \lambda_{n-1}^0=1\end{cases}$$

and we have the second of representations (2.43). By (2.46), (2.48) and (2.24), we have

$$\lim_{t\uparrow\omega}\frac{L(t)(Y^{[i]}(t))'}{L'(t)Y^{[i]}(t)}=\begin{cases}-\sum\limits_{k=1}^{n-1}\left(l_k^i+\dfrac{1}{\gamma_0}\sum\limits_{j=0}^{k-1}c_j\right)+\dfrac{\sum\limits_{i=0}^{n-1}a_{0i}c_i}{\gamma_0(\lambda_{n-1}^0-1)} \text{ if } \lambda_{n-1}^0\neq 1,\\[2ex] 1 \text{ if } \lambda_{n-1}^0=1.\end{cases}$$

We get the first of representations (2.43), using the equality (2.49).

By virtue of (2.32), we have the asymptotic representation

$$G(t)=\frac{\alpha_0 p(t)\prod\limits_{i=0}^{n-1}\left|Y^{[i]}(t)\right|^{\sigma_i+c_i}[1+o(1)]}{Y^{[n-1]}(t)\prod\limits_{i=0}^{n-1}\Phi_i\left(Y^{[i]}(t)\right)\prod\limits_{i=0}^{n-1}c_i sign y_i^0} \quad \text{as } t\uparrow\omega.$$

From this representation and the equalities

$$\frac{\alpha_0 p(t)\prod\limits_{i=0}^{n-1}\left|Y^{[i]}(t)\right|^{\sigma_i+c_i}}{Y^{[n-1]}(t)\prod\limits_{i=0}^{n-1}\Phi_i\left(Y^{[i]}(t)\right)\prod\limits_{i=0}^{n-1}c_i sign y_i^0}=\frac{\alpha_0 p(t)\prod\limits_{i=0}^{n-1}\left|\frac{Y^{[i]}(t)}{Y^{[n-1]}(t)}\right|^{\sigma_i+c_i}}{H_0(t)\prod\limits_{i=0}^{n-1}c_i sign y_i^0 sign y_{n-1}^0},$$

$$\frac{Y^{[i]}(t)}{Y^{[n-1]}(t)}=\prod_{k=i}^{n-2}\frac{Y^{[k]}(t)}{Y^{[k+1]}(t)}=\prod_{k=i}^{n-2}\frac{1}{H_{k+1}(t)}$$

it follows that as $t\uparrow\omega$

$$\frac{L(t)G(t)}{L'(t)}=\begin{cases}\dfrac{\pi_\omega(t)I_0'(t)}{\gamma_0 I_0(t)}[1+o(1)] \text{ if } \lambda_{n-1}^0\neq 1,\\[2ex] \dfrac{I_1'(t)J(t)}{\gamma_0 I_1(t)J'(t)}[1+o(1)] \text{ if } \lambda_{n-1}^0=1.\end{cases}$$

This leads to (2.44) by the first of conditions (2.1) and the first of conditions (2.5). So, (2.42) takes place.

Equation (2.3) is the characteristic equation for the matrix of coefficients of linear part of system (2.41) in case $\lambda_{n-1}^0 \neq 1$. Equation (2.7) is the characteristic equation for this matrix in case $\lambda_{n-1}^0 = 1$. By Conditions of Theorems 1 and 2 these equations have no roots with zero real part. Therefore all conditions of Theorem 2.2 from [9] are satisfied for system (2.41). By this Theorem the system (2.40) has at least one solution $\{z_i\}_{i=0}^{n-1} : [x_1, +\infty[\longrightarrow \mathbb{R}^n$ $(x_1 \geq x_0)$, that follows to zero as $x \to +\infty$. So, due to (2.38), (2.39) and the equality

$$F_i\left(Y^{[0]}(t), Y^{[1]}(t), ..., Y^{[n-1]}(t)\right) = H_i(t),$$

we get that Equation (1.1) has at least one solution y that admits the next asymptotic representations as $t \uparrow \omega$

$$\prod_{j=0}^{n-1} \Phi_j(y^{(j)}(t)) = H_0(t)[1+o(1)], \quad \frac{y^{(i)}(t)}{y^{(i-1)}} = H_i(t)[1+o(1)], \quad i = 0, ..., n-2. \tag{2.50}$$

By (2.19) and the second of representations (2.50), the first of representations (2.50) can be rewritten like the first of representations (2.4) in case $\lambda_{n-1}^0 \neq 1$. In case $\lambda_{n-1}^0 = 1$ let us rewrite the first of representations (2.50) like the first of representations (2.8). If $\lambda_{n-1}^0 \neq 1$ we rewrite the second of representations (2.50) like the second of representations (2.8). By these representations and (1.1) it is clear that y is a $P_\omega(\lambda_{n-1}^0)$-solution of (1.1). The Theorems 1 and 2 are proved.

Proof of Corollaries 1, 2. Let $y : [t_0, \omega[\to \Delta_{Y_0}$ be a $P_\omega(\lambda_{n-1}^0)$-solution of Equation (1.1), where $\lambda_{n-1}^0 \in \mathbb{R}\setminus \left\{0, \frac{1}{2}, \frac{2}{3}, ..., \frac{n-2}{n-1}\right\}$. It follows from (2.9), that in case $\lambda_{n-1}^0 \neq 1$ the function $y^{(i)}(t)$ is regularly varying of the index $\dfrac{a_{0i+1}}{\lambda_{n-1}^0 - 1}$. Then, by M_1, using the condition S, from the first of representations (2.4) we get the first of representations in Corollary 1.

In case $\lambda_{n-1}^0 = 1$ it follows from (2.18), that for all $i \in \{0, ...n-1\}$

$$\lim_{t\uparrow\omega} \frac{J(t)\left(\dfrac{y^{(i)}(t)}{J(t)}\right)' \cdot \dfrac{1}{J'(t)}}{\dfrac{y^{(i)}(t)}{J(t)}} = \lim_{t\uparrow\omega} \frac{J(t)}{J'(t)}\left(\frac{y^{(i+1)}(t)}{y^{(i)}(t)} - \frac{J'(t)}{J(t)}\right) =$$

$$= \lim_{t\uparrow\omega} \frac{J(t)y^{(i+1)}(t)}{J'(t)y^{(i)}(t)} - 1 = 0.$$

So, the next representation

$$y^{(i)}(t) = |J(t)| L(|J(t)| y_i^0) sign y_i^0,$$

where the function $L(z)$ is slowly varying as $z \to Y_i$, takes place for all $t \in [t_0, \omega[$. Then, by M_1, using the condition S, we get from the first of representations (2.8) the first of representations in Corollary 2.

The second representations in Corollary 1 are the second of representations (2.4). The second representations in Corollary 2 are the second of representations (2.8).

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