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A CHARACTERIZATION OF PERMUTABILITY OF 2-UNIFORM TOLERANCES ON POSETS

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Abstract. Tolerance relations were investigated by several authors in various algebraic structures, see e.g. the monograph [1]. Recently G. Czédli [5] proved that two 2-uniform tolerances on a lattice containing no infinite chain permute if and only if they are amicable. We extend this result to tolerances on posets. Since in posets we have no lattice operations, we modify the notion of amicability in such a way that in case of lattices it coincides with the original definition. With this new definition we can prove that two 2-uniform tolerances on a poset containing no infinite chain permute if and only if they are amicable.

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1. Introduction

By a *tolerance* on an algebra $\mathbf{A} = (A, F)$ is meant a reflexive and symmetric binary relation on A having the Substitution Property with respect to all operations of F. For the theory of tolerances see e.g. the monograph [1]. Important are in particular tolerances on lattices since G. Czédli showed in [4] that every lattice $\mathbf{L} = (L, \vee, \wedge)$ can be factorized in a natural way by any tolerance T on $\mathbf{L}$, i.e. the set L/T of all blocks of T forms a lattice again. Such a situation is rather exceptional and does not hold for other types of algebras in general, but many varieties whose members have this property were described in [2].

Recall that a *block* of a tolerance T on an algebra (A, F) is a maximal subset B of A satisfying $B^2 \subseteq T$.

Recently G. Czédli proved in [5] that so-called 2-uniform tolerances on a lattice containing no infinite chain permute if and only if they are amicable.

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The concept of a tolerance on a lattice was generalized to posets by the present authors [3]. Despite the fact that we do not have explicit operations on a poset, the concept was defined in a way that the blocks are convex again and that posets can be factorized by these tolerances. For other interesting properties of tolerances on posets the reader is referred to [3]. Hence the natural question arises if also the result of G. Czédli concerning the permutability of 2-uniform tolerances can be extended to posets. This is the topic of the present paper.

2. Basic concepts

At first we recall several concepts introduced in [3] and [5].

Let $\mathbf{P} = (P, \leq)$ be a poset. A *tolerance* on $\mathbf{P}$ is a reflexive and symmetric binary relation T on P satisfying the following conditions:

(1) If $(x,y),(z,u) \in T$ and $x \vee z$ and $y \vee u$ exist then $(x \vee z, y \vee u) \in T$.
(2) If $(x,y),(z,u) \in T$ and $x \wedge z$ and $y \wedge u$ exist then $(x \wedge z, y \wedge u) \in T$.
(3) If $x,y,z \in P$ and $(x,y),(y,z) \in T \neq P^2$ then there exist $u,v \in P$ with $u \leq x,y,z \leq v$ and $(u,y),(y,v) \in T$.
(4) If $(x,y) \in T \neq P^2$ then there exists some $(z,u) \in T$ with both $z \leq x, y \leq u$ and $(v,z),(v,u) \in T$ for all $v \in P$ with $(v,x),(v,y) \in T$.

Conditions (3) and (4) are quite natural since they are satisfied by every tolerance on a lattice. In condition (3) one can take $u := x \wedge y \wedge z$ and $v := x \vee y \vee z$, and in condition (4) one can take $z := x \wedge y$ and $u := x \vee y$.

A *block* of a tolerance T on $\mathbf{P}$ is a maximal subset B of P satisfying $B^2 \subseteq T$. Let P/T denote the set of all blocks of T. Clearly, $T = \bigcup_{B \in P/T} B^2$. In [3] we proved that every block of T is convex. According to Zorn's Lemma every subset A of P satisfying $A^2 \subseteq T$ is contained in a block of T. Following [5] a tolerance is called 2-*uniform* if every of its blocks consists of exactly two elements. Let $a,b \in P$. We call a a *lower T-neighbor of b* and b an *upper T-neighbor of a* if $a \prec b$ and $(a,b) \in T$.

3. A characterization of permutability of 2-uniform tolerances on posets

It is evident that 2-uniform tolerances on a poset are very specific. Their basic properties are as follows.

Lemma 1. *Let $\mathbf{P} = (P, \leq)$ be a poset, $a,b \in P$ and T be a 2-uniform tolerance on $\mathbf{P}$. Then the following holds:*

(i) The element a has at most one lower T-neighbor and at most one upper T-neighbor.
(ii) If $(a,b) \in T$ then $a = b$ or $a \prec b$ or $b \prec a$.

Proof. Without loss of generality, assume $|P| > 2$. Then $T \neq P^2$.

(i) Assume a to have two distinct lower T-neighbors c and d. Then $(c,a),(a,d) \in T$. According to (3) there exists some $e \in P$ with $e \leq c,a,d$ and $(e,a) \in T$. Let $B \in P/T$ with $e,a \in B$. Since $e \leq c,d \leq a$ and B is convex we conclude $c,d,a \in B$ contradicting $|B| = 2$. This shows that a can have at most one lower T-neighbor. The second assertion follows by duality.

(ii) Assume $(a,b) \in T$. First suppose $a \parallel b$. According to (4) there exist $c,d \in P$ with $(c,d) \in T$ and $c \leq a,b \leq d$. Let $B \in P/T$ with $c,d \in B$. Since B is convex we conclude $c,a,b \in B$. Now we have $a \parallel b$ and $c \leq a,b$ and hence $c \neq a,b$ contradicting $|B| = 2$. This shows $a \leq b$ or $b \leq a$. Let $C \in P/T$ with $a,b \in C$. Since C is convex we obtain $[a,b] \subseteq C$ if $a \leq b$ and $[b,a] \subseteq C$ if $b \leq a$. Now $|C| = 2$ implies $a = b$ or $a \prec b$ or $b \prec a$.

□

Let $\mathbf{P}$ be a poset, $a,b \in P$ and T,S be 2-uniform tolerances on $\mathbf{P}$.

In what follows we adopt some key concepts from [5] for 2-uniform tolerances on posets.

The element a is called a *split* (T,S)*-bottom* if there exists some upper T-neighbor b of a and some upper S-neighbor of a being different from b. Further, a is called an *adherent* (T,S)*-bottom* if there exists some common upper T-neighbor and S-neighbor of a. Finally, a is called a (T,S)*-bottom* if it is either a split (T,S)-bottom or an adherent (T,S)-bottom. The notions *split* (T,S)*-top*, *adherent* (T,S)*-top* and (T,S)*-top* are defined dually. A T*-top* (T*-bottom*) is an upper (a lower) T-neighbor of some element of P.

The tolerances T and S are called *amicable* if the following four conditions hold:

(5) If $a \neq b$ and there exists some lower T-neighbor of a being a lower S-neighbor of b then there exists some upper S-neighbor of a being an upper T-neighbor of b.

(6) If $a \neq b$ and there exists some upper T-neighbor of a being an upper S-neighbor of b then there exists some lower S-neighbor of a being a lower T-neighbor of b.

(7) If a is a (T,S)-top and b is either an upper T-neighbor or an upper S-neighbor of a then b is a (T,S)-top, too.

(8) If a is a (T,S)-bottom and b is either a lower T-neighbor or a lower S-neighbor of a then b is a (T,S)-bottom, too.

Let us note that any two 2-uniform tolerances on a lattice $(L,\vee,\wedge)$ satisfy (5) and (6). Namely, G. Czédli [5] showed that if $a \neq b$ and there exists some lower T-neighbor of a being a lower S-neighbor of b then $a \vee b$ is an upper S-neighbor of a being an upper T-neighbor of b, and if $a \neq b$ and there exists some upper T-neighbor of a being an upper S-neighbor of b then $a \wedge b$ is a lower S-neighbor of a being a lower T-neighbor of b.

Remark 1. Note that (7) and (8) coincide with conditions (A1) and (A2) from [5], respectively. Hence, for lattices the concept of amicable 2-uniform tolerances as defined in [5] coincides with the one defined above.

The proof of our main theorem will broadly use the ideas from [5].

Theorem 1. *Let T and S be 2-uniform tolerances on a poset $(P,\leq)$ containing no infinite chain. Then T and S permute if and only if they are amicable.*

Proof. Let $a,b\in P$. First assume T and S to permute.

(5) Suppose $a\neq b$ and suppose c to be some lower T-neighbor of a being a lower S-neighbor of b. Because of (ii) of Lemma 1 we have $a\parallel b$ and hence $(a,b)\notin S\cup T$ again according to (ii) of Lemma 1. Now $(a,b)\in T\circ S=S\circ T$. Hence there exists some S-neighbor d of a being a T-neighbor of b. Because of $a\parallel b$ we have $d<a,b$ or $d>a,b$. Now $d<a,b$ would imply $(a,b)=(a\vee d,c\vee b)\in T$ according to (1) since $(a,c),(d,b)\in T$. But this contradicts (ii) of Lemma 1. Therefore d is an upper S-neighbor of a being an upper T-neighbor of b proving (5).

(6) follows from (5) by duality.

(7) Suppose b to be either an upper T-neighbor or S-neighbor of a. Without loss of generality assume $(a,b)\in T$. First suppose a to be a split (T,S)-top. Let c and d denote the lower T-neighbor and S-neighbor of a, respectively. Then $c\parallel d$. Since $(d,b)\in S\circ T=T\circ S$ there exists some $e\in P$ with $(d,e)\in T$ and $(e,b)\in S$. Now $b\leq e$ would imply $d<a<b\leq e$ which together with $(d,e)\in T$ would contradict (ii) of Lemma 1. Again according to Lemma 1 we have $e\prec b$. Now a and e are a lower T-neighbor and S-neighbor of b, respectively, showing b to be a (T,S)-top. Next assume a to be an adherent (T,S)-top. Let c denote the common lower T-neighbor and S-neighbor of a. Since $(c,b)\in S\circ T=T\circ S$ there exists some $d\in P$ with $(c,d)\in T$ and $(d,b)\in S$. Now $d\leq c$ would imply $d\leq c<a<b$ which together with $(d,b)\in S$ would contradict (ii) of Lemma 1, and $b\leq d$ would imply $c<a<b\leq d$ which together with $(c,d)\in T$ would again contradict (ii) of Lemma 1. Thus a further application of (ii) of Lemma 1 yields $c\prec d\prec b$. Now a and d are a lower T-neighbor and S-neighbor of b, respectively, showing b to be a (T,S)-top.

(8) follows from (7) by duality.

Hence T and S are amicable.

Conversely, assume T and S to be amicable. Suppose $(a,b)\in T\circ S$. Then there exists some $c\in P$ with $(a,c)\in T$ and $(c,b)\in S$. We want to show $(a,b)\in S\circ T$. If $a=b$ then $(a,b)=(a,a)\in S\circ T$. If $c=a$ then $(a,b)=(c,b)\in S$ and $(b,b)\in T$ and hence $(a,b)\in S\circ T$. If $c=b$ then $(a,a)\in S$ and $(a,b)=(a,c)\in T$ and hence $(a,b)\in S\circ T$. So we can assume that a,b,c are mutually distinct. According to (ii) of Lemma 1 we consider the following four cases.

$\underline{c \prec a}$ **and** $\underline{c \prec b}$**:** Because of (5) there exists a common upper S-neighbor d of a and T-neighbor of b and hence $(a,d) \in S$ and $(d,b) \in T$ showing $(a,b) \in S \circ T$.

$\underline{a \prec c}$ **and** $\underline{b \prec c}$ **:** This case is dual to the previous one.

$\underline{a \prec c \prec b}$ **:** Put $a_0 := a$, $a_1 := c$ and $a_2 := b$. For $i \geq 3$ define a_i as follows: If i is odd and a_{i-1} is a T-bottom then define a_i to be the unique upper T-neighbor of a_{i-1}. If i is even and a_{i-1} is an S-bottom then define a_i to be the unique upper S-neighbor of a_{i-1}. Note that a_0 is a T-bottom, a_1 is the unique upper T-neighbor of a_0, a_1 is an S-bottom and a_2 is the unique upper S-neighbor of of a_1. Since $a_2 \prec a_3 \prec a_4 \prec \cdots$, but **P** has no infinite chain, there exists some $n \geq 2$ such that $a_2, \ldots, a_n$ are defined, but a_{n+1} is not. First assume n to be even. Then a_n is not a T-bottom. Since every element of P belongs to at least one 2-element block of T, a_n is a T-top. But it is also an S-top and therefore a (T,S)-top. If n is odd then a similar reasoning shows that a_n is a (T,S)-top. So a_n is a (T,S)-top in any case. Now assume a_n to be an adherent (T,S)-top. Then, according to (i) of Lemma 1, a_{n-1} is an adherent (T,S)-bottom. Now (8) shows that a_{n-2} is a (T,S)-bottom, too. Next assume a_n to be a split (T,S)-top. Suppose n is even. Then a_{n-1} is the unique lower S-neighbor of a_n, a_n has a unique lower T-neighbor d and $a_{n-1} \neq d$. According to (6) there exists some lower S-neighbor e of d being a lower T-neighbor of a_{n-1}. Since a_{n-2} is a lower T-neighbor of a_{n-1}, too, (i) of Lemma 1 yields $e = a_{n-2}$. Therefore a_{n-2} is a (T,S)-bottom. With roles of T and S interchanged, one can prove that a_{n-2} is a (T,S)-bottom also in the case when n is odd. Hence in any case a_{n-2} is a (T,S)-bottom. Applying (8) finitely many times yields that $a_0 = a$ is a (T,S)-bottom, too. First assume a to be a split (T,S)-bottom. Then there exists some upper S-neighbor f of a being distinct from c. According to (5) there exists some upper S-neighbor g of c being an upper T-neighbor of f. Since b is an upper S-neighbor of c, too, (i) of Lemma 1 yields $g = b$. Hence $(a,f) \in S$ and $(f,b) = (f,g) \in T$ showing $(a,b) \in S \circ T$. Now suppose a to be an adherent (T,S)-bottom. Then c is a (T,S)-top. According to (7), b is a (T,S)-top, too. Hence b has a unique lower T-neighbor h. Assume $c \neq h$. According to (6) there exists some lower S-neighbor i of h being a lower T-neighbor of c. Since a is a lower T-neighbor of c, too, (i) of Lemma 1 yields $i = a$. On the other hand, $i = a$ is a lower S-neighbor of h. Hence c and h are two distinct upper S-neighbors of a contradicting (i) of Lemma 1. This shows $c = h$. Now $(a,c) \in S$ and $(c,b) = (h,b) \in T$ proving $(a,b) \in S \circ T$.

$\underline{b \prec c \prec a}$ **:** This case is dual to the previous one.

Hence in any case we have $(a,b) \in S \circ T$. This shows $T \circ S \subseteq S \circ T$. By duality, we obtain $S \circ T \subseteq T \circ S$ and hence $T \circ S = S \circ T$, i.e. T and S permute. □

4. EXAMPLES

Example 1. Consider the poset **P** depicted in Figure 1

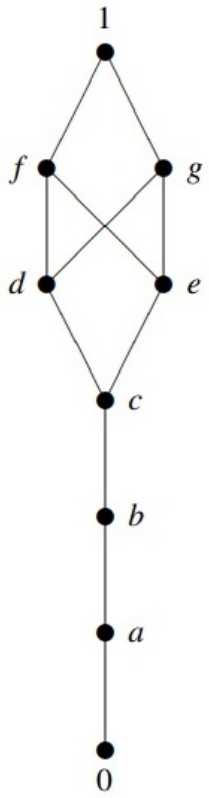

FIGURE 1. Poset

and put

$$T := \{0,a\}^2 \cup \{a,b\}^2 \cup \{c,e\}^2 \cup \{d,g\}^2 \cup \{f,1\}^2,$$
$$S := \{0,a\}^2 \cup \{a,b\}^2 \cup \{c,e\}^2 \cup \{d,f\}^2 \cup \{g,1\}^2.$$

Then T and S are permuting 2-uniform tolerances on **P** since

$$T \circ S = T \cup S \cup \{(0,b),(b,0),(d,1),(f,g),(g,f),(1,d)\} = S \circ T$$

and hence they are amicable according to Theorem 1.

Example 2. Consider the poset **P** visualized in Figure 2

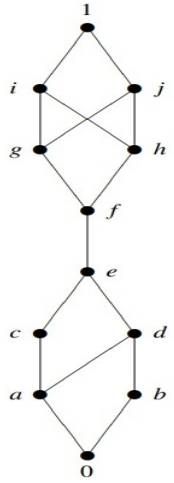

FIGURE 2. Poset

and put

$$T := \{0,a\}^2 \cup \{a,c\}^2 \cup \{b,d\}^2 \cup \{d,e\}^2 \cup \{f,h\}^2 \cup \{g,j\}^2 \cup \{i,1\}^2,$$
$$S := \{0,b\}^2 \cup \{a,d\}^2 \cup \{c,e\}^2 \cup \{f,g\}^2 \cup \{h,j\}^2 \cup \{i,1\}^2.$$

Then T and S are permuting 2-uniform tolerances on **P** since

$$T \circ S = T \cup S \cup \{(0,d),(a,b),(a,e),(b,a),(c,d),(d,0),$$
$$(d,c),(e,a),(f,j),(g,h),(h,g),(j,f)\} = S \circ T$$

and hence they are amicable according to Theorem 1.

Example 3. Consider the poset **P** depicted in Figure 3

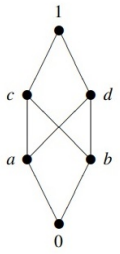

FIGURE 3. Poset

and put

$$T := \{0,a\}^2 \cup \{b,d\}^2 \cup \{c,1\}^2,$$
$$S := \{0,b\}^2 \cup \{a,c\}^2 \cup \{d,1\}^2.$$

Then T and S do not permute since $(a,b) \in (T \circ S) \setminus (S \circ T)$, and T and S are not amicable since 0 is a lower T-neighbor of a being a lower S-neighbor of b, but there exists no upper S-neighbor of a being an upper T-neighbor of b.

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