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ESTIMATES ON SCHIPPERS' HIGHER-ORDER SCHWARZIAN DERIVATIVES OF A SUBCLASS OF UNIVALENT ANALYTIC FUNCTIONS

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Abstract. In the theory of univalent analytic functions, the properties and characteristics of convex functions and starlike functions are widely studied, such as the estimates on Schippers' higher-order Schwarzian derivatives at zero. In this paper, we consider the bounds for Schippers' higher-order Schwarzian derivatives of $f(z)$ at $z = 0$ when the function f belongs to the class of a kind of combination of convex functions and starlike functions, which consists of analytic functions f in the open unit disk $\mathbb{D}$ with the normalized conditions given by

$$f(0) = f'(0) - 1 = 0$$

and satisfying the following inequality:

$$\Re\left(\frac{1}{2}\frac{zf'(z)}{f(z)} + \frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0 \qquad (z \in \mathbb{D}).$$

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1. INTRODUCTION AND THE MAIN RESULT

We denote by $\mathbb{D}$ the open unit disk given by

$$\mathbb{D} = \{z\colon z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{A}$ be the class of all analytic functions f in $\mathbb{D}$ with

$$f(0) = f'(0) - 1 = 0.$$

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Hence, $f \in \mathcal{A}$ can be expanded in the following form (see, for details, [1] and [4])

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.1}$$

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$.

The Schwarzian derivative of a locally univalent function f is defined by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

It is well known that S_f plays an important role in the study of univalent functions. Nehari [16] proved that, if $f \in \mathcal{S}$, then

$$\|S_f\| = \sup_{z\in\mathbb{D}} |S_f(z)|(1-|z|^2)^2 \leq 6.$$

For some subclasses of $\mathcal{S}$, the upper bounds for $\|S_f\|$ have been widely studied (see, for example, [8] and [17]).

Owing to the importance of the Schwarzian derivatives, several different higher-order Schwarzian derivatives are given (see, for details, [5, 6, 17, 20]). Among these different higher-order Schwarzian derivatives, the following kind defined by Schippers was also widely studied.

Let f be locally univalent analytic in $\mathbb{D}$. Schippers [17] introduced a higher-order Schwarzian derivative $\sigma_n(f)$ as follows:

$$\sigma_{n+1}(f) = \sigma_n'(f) - (n-1)\sigma_n(f)\frac{f''}{f'} \qquad (n \geqq 3),$$

where $\sigma_3(f) = S_f$. The norm $\|\sigma_{n+1}(f)\|$ of $\sigma_{n+1}(f)$ is defined as follows:

$$\|\sigma_{n+1}(f)\| = \sup_{z\in\mathbb{D}} |\sigma_{n+1}(f)(z)|(1-|z|^2)^n \qquad (n \geqq 2).$$

The sharp upper bound for $\|\sigma_n(f)\|$ of $f \in \mathcal{S}$ was given by Schippers [17]. Analogously, for subclasses of $\mathcal{S}$, the upper bounds for $|\sigma_n(f)(0)|$ $(n = 3,4,5)$ have also been investigated. For instance, Dorff and Szynal [3] derived the bounds of $\sigma_n(f)(0)$ for convex functions. Cho *et al.* [2] investigated the bounds of $\sigma_n(f)(0)$ $(n = 3,4,5)$ in general forms of these classes consisting of the Janowski classes satisfying the following subordination conditions (see also [2] and [19]):

$$\left\{ f\colon f \in \mathcal{S} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$\left\{ f\colon f \in \mathcal{S} \quad \text{and} \quad 1+\frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\},$$

where $-1 \leq B < A \leq 1$, which generalized the results in [3]. Recently, Kumar et al. [10] studied the bounds on the first three consecutive higher-order Schwarzian derivatives for the class given by

$$\left\{ f\colon f \in \mathcal{S} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \mathrm{e}^{\mathrm{e}^{z}-1} \right\}.$$

More recently, for some subclasses of close-to-convex functions, the upper bounds for $|\sigma_4(f)(0)|$ were considered by Hu et al. (see [7]).

In order to introduce the aims of this paper, we first recall the definitions of starlike and convex functions as follows. A function $f \in \mathcal{S}$ is said to be starlike if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{D}). \tag{1.2}$$

A function $f \in \mathcal{S}$ is said to be convex if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{D}). \tag{1.3}$$

Let $\mathcal{S}^*$ and $\mathcal{C}$ be the classes of starlike functions and convex functions, respectively. A well-known generalization of these classes is the class $\mathcal{M}_\alpha$ of α-convex functions, which was given by Mocanu [15]. It is shown in Mocanu [15] that $\mathcal{M}_\alpha \subset \mathcal{S}^*$ for all real α. Moreover, if $\alpha \geqq 1$, then $\mathcal{M}_\alpha \subset \mathcal{C}$ (see, for details, [15]). A great deal of attention has been given to problems concerning functions in the class $\mathcal{M}_\alpha$, particularly relating to the coefficient problems (see [9, 13, 14, 18, 21]).

Let $f \in \mathcal{A}$ be given by (1.1) and let $f \in \mathcal{M}_{\frac{1}{2}}$ if

$$\Re\left(\frac{1}{2}\frac{zf'(z)}{f(z)} + \frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0 \qquad (z \in \mathbb{D}).$$

We see from [15] that $\mathcal{M}_{\frac{1}{2}} \subset S^*$. Motivated by the above-cited papers [2, 3, 7, 10, 17], the following question arises naturally.

Question. *Can we give the upper bounds on $|\sigma_3(f)(0)|$ and $|\sigma_4(f)(0)|$ if $f \in \mathcal{M}_{\frac{1}{2}}$?*

In this paper, we investigate the bounds for $|\sigma_3(f)(0)|$ and $|\sigma_4(f)(0)|$ of analytic functions f in $\mathbb{D}$ at zero if $f \in \mathcal{M}_{\frac{1}{2}}$. We will also give a positive answer to the above question.

We now state our result as follows.

Theorem 1. *If $f \in \mathcal{M}_{\frac{1}{2}}$, then*

$$|\sigma_3(f)(0)| \leq 3 \tag{1.4}$$

and

$$|\sigma_4(f)(0)| \leq \frac{32}{5}\sqrt{\frac{18}{11}}. \tag{1.5}$$

The inequality (1.4) is sharp.

The paper is organized as follows. In Section 2, we give the preliminaries and the proof of Theorem 1. In the concluding section (Section 3), we present our concluding remarks and observations.

2. Proof of Theorem 1

In order to prove Theorem 1, we need each of the following lemmas.

Lemma 1 ([4]). *Let $\mathcal{P}$ be the class of analytic functions p in $\mathbb{D}$ with positive real part and given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \tag{2.1}$$

If $p(z) \in \mathcal{P}$, then the following sharp inequality holds true:

$$|c_k| \leq 2 \qquad (k \geqq 1).$$

Lemma 2 ([11, 12]). *If $p \in \mathcal{P}$ with $c_1 > 0$, then*

$$2c_2 = c_1^2 + \zeta(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)\zeta - c_1(4 - c_1^2)\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta$$

for some ζ and η such that

$$\max\{|\zeta|, |\eta|\} \leq 1.$$

We next prove our main result (Theorem 1 above).

Proof of Theorem 1. Let the function $f \in \mathcal{M}_{\frac{1}{2}}$ be of the form (1.1). Then, by applying (2.1), we can write

$$\frac{1}{2}\frac{zf'(z)}{f(z)} + \frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z), \tag{2.2}$$

where $p \in \mathcal{P}$. Equating the coefficients in (2.2), we have

$$a_2 = \frac{2c_1}{3}, \tag{2.3}$$

$$a_3 = \frac{10c_1^2 + 9c_2}{36} \tag{2.4}$$

and

$$a_4 = \frac{33c_1^3 + \frac{189}{2}c_1c_2 + 54c_3}{405}. \tag{2.5}$$

We first estimate the upper bound for $|\sigma_3(f)(0)|$ if $f \in \mathcal{M}_{\frac{1}{2}}$. Since $\mathcal{M}_\alpha$ and $\sigma_3(f)(0)$ are rotationally invariant, by Lemma 1, we assume that $c_1 \in [0, 2]$. Moreover, by Lemma 2, and the equations (2.3) and (2.4), we have

$$\sigma_3(f)(0) = 6(a_3 - a_2^2) = -c_1^2 + \frac{3}{2}c_2 = \frac{1}{4}\left[-c_1^2 + 3\left(4 - c_1^2\right)\zeta\right], \tag{2.6}$$

where $|\zeta| \leq 1$.

Let $x = |\zeta|$. Then, by (2.6), we have

$$|\sigma_3(f)(0)| \leq \frac{1}{4}\left[-c_1^2 + 3(4-c_1^2)\right] \leq 3 \tag{2.7}$$

for $f \in \mathcal{M}_{\frac{1}{2}}$. The equality in (2.7) holds true for

$$c_1 = 0, \quad c_2 = 2 \quad \text{and} \quad \zeta = 1.$$

Therefore, the equality in (2.7) holds true for

$$\frac{1}{2}\frac{zf'(z)}{f(z)} + \frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{1+z^2}{1-z^2}.$$

We next estimate the upper bound for $|\sigma_4(f)(0)|$ if $f \in \mathcal{M}_{\frac{1}{2}}$. Indeed, noting that $\mathcal{M}_{\frac{1}{2}}$ and $\sigma_4(f)(0)$ are rotationally invariant, by Lemma 1, we assume that

$$c_1 =: t \in [0,2].$$

Furthermore, by Lemma 2, and the equations (2.3), (2.4) and (2.5), we have

$$\begin{aligned}
\sigma_4(f)(0) &= 24(a_4 - 3a_3a_2 + 2a_2^3) \\
&= \frac{64}{45}\left(2c_1^3 - \frac{9}{2}c_1c_2 + \frac{9}{4}c_3\right) \\
&= \frac{16}{45}\left(\frac{5}{4}c_1^3 - \frac{9}{2}(4-c_1^2)c_1\zeta - \frac{9}{4}(4-c_1^2)c_1\zeta^2 + \frac{9}{2}(4-c_1^2)(1-|\zeta|^2)\eta\right),
\end{aligned} \tag{2.8}$$

where

$$\max\{|\zeta|, |\eta|\} \leq 1.$$

Let $x = |\zeta|$. Then, by (2.8), we have

$$\begin{aligned}
|\sigma_4(f)(0)| &\leq \frac{16}{45}\left(\frac{5}{4}c_1^3 + \frac{9}{2}(4-c_1^2)c_1x + \frac{9}{4}(4-c_1^2)c_1x^2 + \frac{9}{2}(4-c_1^2)(1-x^2)\right) \\
&= \frac{16}{45}T(t,x),
\end{aligned}$$

where

$$(t,x) \in \Delta := [0,2] \times [0,1]$$

and

$$T(t,x) = \frac{5}{4}t^3 + \frac{9}{2}(4-t^2)tx + \frac{9}{4}(4-t^2)tx^2 + \frac{9}{2}(4-t^2)(1-x^2). \tag{2.9}$$

In order to complete the proof of Theorem 1, we consider five cases as follows.

Case 1. On the vertices of Δ, we have

$$T(0,0) = 18, \quad T(0,1) = 0 \quad \text{and} \quad T(2,0) = T(2,1) = 10.$$

Case 2. On the side $x=0$, the function $T(t,x)$ becomes

$$T(t,0)=\frac{5}{4}t^3+\frac{9}{2}(4-t^2),$$

which, upon differentiating with respect to t, yields

$$T'(t,0)=\frac{15}{4}t^2-9t,$$

so that $t_0=\frac{12}{5}>2$ is the root of $T'(t,0)=0$. Consequently, we have

$$T'(t,0)<0 \qquad \big(t\in(0,2)\big).$$

It follows that

$$T(t,0)\leq T(0,0)=18.$$

Case 3. On the side $x=1$, the function $T(t,x)$ becomes

$$T(t,1)=\frac{5}{4}t^3+\frac{9}{2}(4-t^2)t+\frac{9}{4}(4-t^2)t.$$

Therefore, we have

$$T'(t,1)=\frac{-33}{2}t^2+27.$$

A short calculation shows that the root of $T'(t,1)=0$ is given by

$$t_0=\sqrt{\frac{18}{11}}<2.$$

It follows that

$$T(t,1)\leq T(t_0,1)=-\frac{11}{2}t_0^3+27t_0=18\sqrt{\frac{18}{11}}.$$

Case 4. It is easy to show that

$$T(2,x)=10 \qquad \text{and} \qquad T(0,x)\leq 18.$$

Case 5. It now remains to consider the interior of Δ. We first solve the following equations:

$$T_t(t,x)=\frac{15}{4}t^2-2t\left(\frac{9}{2}tx+\frac{9}{4}\left[(t-2)x^2+2\right]\right)+(4-t^2)\left(\frac{9}{2}x+\frac{9}{4}x^2\right)=0 \quad (2.10)$$

and

$$T_x(t,x)=\frac{9}{2}(4-t^2)(t+tx-2x)=0. \quad (2.11)$$

Secondly, we will prove that that the equations (2.10) and (2.11) have no critical points in the interior of Δ. We thus find from (2.11) that

$$x=\frac{t}{2-t}. \quad (2.12)$$

So, putting (2.12) into (2.10), we get

$$\rho(t) = \frac{15}{4}t^2 - 2t\left(\frac{9}{2}t\frac{t}{2-t} + \frac{9}{4}\left[(t-2)\left(\frac{t}{2-t}\right)^2 + 2\right]\right)$$
$$+ (4-t^2)\left(\frac{9}{2}\frac{t}{2-t} + \frac{9}{4}\left(\frac{t}{2-t}\right)^2\right). \tag{2.13}$$

Calculations show that $\rho(t) = 0$ has no root in $t \in (0,2)$. Consequently, the equations (2.10) and (2.11) have no critical points in the interior of Δ.

Upon summarizing all of cases which we have considered above, we have the following inequality:

$$|\sigma_4(f)(0)| \leq \frac{32}{5}\sqrt{\frac{18}{11}}.$$

The proof of Theorem 1 is thus completed. □

3. Concluding remarks and observations

The present investigation is motivated essentially by the fact that, in the theory of analytic and univalent functions, the properties and characteristics of various subclasses of such analytic functions as (for example) convex functions, starlike functions and close-to-convex functions are widely- and extensively-studied. Specifically, there is a great deal of literature on the bounds for the Taylor-Maclaurin coefficients and the estimates on Schippers' higher-order Schwarzian derivatives at zero. Here, in this article, we have successfully derived the bounds for the Schippers' higher-order Schwarzian derivatives of $f(z)$ at $z = 0$ when the function f belongs to the class of a kind of combination of convex functions and starlike functions, which consists of analytic functions f in the open unit disk $\mathbb{D}$ with the normalized conditions given by

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and satisfying the following inequality:

$$\Re\left(\frac{1}{2}\frac{zf'(z)}{f(z)} + \frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0 \qquad (z \in \mathbb{D}).$$

We have also discussed relevant connections of the results, which we have presented here, with those available in earlier works on the subject.

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