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A NOVEL METHOD FOR MILNE-TYPE INEQUALITIES USING PROPORTIONAL CAPUTO-HYBRID OPERATOR

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Abstract. In mathematics and the applied sciences, fractional calculus is a significant generalization and a very useful tool as it overcomes many limitations of classical analysis. More importantly, it is better to use the new hybrid fractional operator, which merges the proportional and Caputo operator. In this study, because of its numerous applications, we concentrate on the proportional Caputo-hybrid operator. First, we propose a new integral identity with the help of twice-differentiable mappings for the proportional Caputo-hybrid operator. Then, with the help of this newly derived identity, we establish several integral inequalities related to the Milne-type integral inequalities for proportional Caputo-hybrid operator. Also, we obtain various Milne-type inequalities for bounded mappings and mappings of bounded variation. Finally, we point out that the obtained results enhance and generalize some of the previous findings in the field of integral inequalities. These results are the first kind of such results in this direction.

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1. INTRODUCTION

In advanced analysis, the theory of convexity holds an important and alluring position because of its many applications. With the help of this theory, we can effectively tackle a wide range of issues that come up in both pure and applied mathematics. Additionally, the use of integral inequalities and its applications has grown quickly, and it has influenced the various modern mathematical fields, including measure theory, approximation theory, and information theory, along with a number of fields within science and engineering. Also, the integral inequalities are an extremely useful tool for figuring out the error bounds of numerical integration formulas for a variety of mappings such as mappings with bounded variation, Lipschitz mappings, n-times differentiable mappings and bounded mappings. Some well-known inequalities like Simpson-type, Ostrowski-type, Hermite–Hadamard-type and Grüss-type analyze the

remainder term and establish error bounds for quadrature rules. Due to the fact that the concept of convexity is a crucial property for improving these integral inequalities, the development of the theory of inequalities shares a strong connection with the theory of convexity. Because of the fundamental properties of these theories, many researchers have been quite interested in combining them, which helps to improve and generalize the integral inequalities.

One of the most significant and most commonly needed inequalities is Simpson's inequality, which is as follows:

$$\left|\frac{1}{3}\left[\frac{\psi(\varsigma)+\psi(\vartheta)}{2}+2\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{(\vartheta-\varsigma)^4}{2880}\left\|\psi^{(4)}\right\|_{\infty},$$

where $\psi\colon[\varsigma,\vartheta]\to\mathbb{R}$ is four times continuously differantiable mapping on (ς,ϑ) and $\left\|\psi^{(4)}\right\|_{\infty}=\sup_{\varkappa\in(\varsigma,\vartheta)}\left|\psi^{(4)}(\varkappa)\right|<\infty$. This inequality establishes an upper limit on the error that occurs when estimating a definite integral using Simpson's rule. It is widely recognized that when the mapping ψ is not four times differentiable or when its fourth derivative $\psi^{(4)}$ is unbounded on the interval (ς,ϑ), the classical Simpson quadrature formula cannot be employed. Simpson-type inequalities provide boundaries and approximations, which aid in the solution of many geometric problems. Moreover, numerical integration, which is crucial in applied mathematics, engineering, and physics, can be estimated using Simpson-type inequalities. Therefore, in recent years, several authors have highlighted Simpson-type inequalities for various classes of mappings due to its abundant geometric importance and applications. For instance, in [14], Dragomir et al. demonstrated some recent advancements in Simpson's inequality, where the remaining part is expressed in terms of derivatives lower than the fourth order. In [2], Alomari introduced Simpson's type inequalities for s-convex functions. Sarikaya et al. gave some Simpson's type inequalities via twice differentiable functions in [27]. For the other results, one can refer to [8, 10, 17, 18, 20, 28].

According to the similar conditions as the Simpson inequality, the Milne inequality is the one that gives estimates of the error boundaries for the Milne formula:

$$\left|\frac{1}{3}\left[2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right]-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{7(\vartheta-\varsigma)^4}{23040}\left\|\psi^{(4)}\right\|_{\infty},$$

where $\psi\colon[\varsigma,\vartheta]\to\mathbb{R}$ is a four times differentiable mapping on (ς,ϑ) and $\left\|\psi^{(4)}\right\|_{\infty}=\sup_{\varkappa\in(\varsigma,\vartheta)}\left|\psi^{(4)}(\varkappa)\right|<\infty$. In mathematical analysis, Milne inequality is essential, particularly when studying differential equations and dynamical systems. With the help of this inequality, researchers can derive bounds on solutions and establish significant results regarding the stability, convergence, and system control. Besides, the use of

the inequality can be seen in its applications to the proof of existence and uniqueness theorems for differential and integral equation solutions, which form the basis of numerous theoretical and applied fields. In the recent times, researchers' attention to the Milne inequality has been considerable. Alomari and Liu [3] established error estimations for the Milne's rule for mappings of bounded variation and for absolutely continuous mappings. Román-Flores et al. [22] proved some Milne type inequalities for interval-valued functions. Budak et al. [11] investigated Milne-type inequalities for bounded functions, Lipschitz functions and functions of bounded variation. Ali et al. [1] gave the fractional version of Milne's formula-type inequalities for differentiable convex functions and Riemann–Liouville fractional integrals. Many recent articles have been published on this subject, as in [7, 9, 19].

On the other hand, fractional calculus is a branch of mathematics that deals with derivatives and integrals of non-integer order. So, it plays a vital role in the generalization of classical calculus, modeling complex systems, solving fractional differential equations, analyzing fractal geometry and various scientific and engineering applications. Furthermore, it provides a framework for analyzing and understanding systems with fractional dynamics, allowing for a more comprehensive mathematical description of complex phenomena. Therefore, due to the new fractional integral and derivative such as Caputo-Fabrizio [12], Atangana-Baleanu [5] and tempered [23], this calculus has gained more importance and has found applications in various fields of science and engineering.

One of the significant definitions in fractional analysis is the following:

Definition 1 ([24])**.** Let $\rho > 0$ and $\rho \notin \{1,2,\dots\}$, $n = [\rho]+1$, $\psi \in AC^n[\varsigma,\vartheta]$, the space of functions having n-th derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order ρ are defined as follows:

$$ {}^{C}D^{\rho}_{\varsigma^+}\,\psi(\varkappa) = \frac{1}{\Gamma(n-\rho)}\int_{\varsigma}^{\varkappa}(\varkappa-\mathfrak{s})^{n-\rho-1}\psi^{(n)}(\mathfrak{s})\mathrm{d}\mathfrak{s}, \quad \varkappa > \varsigma, $$

and

$$ {}^{C}D^{\rho}_{\vartheta^-}\psi(\varkappa) = \frac{1}{\Gamma(n-\rho)}\int_{\varkappa}^{\vartheta}(\mathfrak{s}-\varkappa)^{n-\rho-1}\psi^{(n)}(\mathfrak{s})\mathrm{d}\mathfrak{s}, \quad \varkappa < \vartheta. $$

If $\rho = n \in \{1,2,3,\dots\}$ and usual derivative $\psi^{(n)}(\varkappa)$ of order n exists, then Caputo fractional derivative ${}^{C}D^{\rho}_{\varsigma^+}\,\psi(\varkappa)$ coincides with $\psi^{(n)}(\varkappa)$ whereas ${}^{C}D^{\rho}_{\vartheta^-}\,\psi(\varkappa)$ with exactness to a constant multiplier $(-1)^n$. For $n=1$ and $\rho=0$, we have ${}^{C}D^{\rho}_{\varsigma^+}\,\psi(\varkappa)$ $= {}^{C}D^{\rho}_{\vartheta^-}\,\psi(\varkappa) = \psi(\varkappa)$.

The Caputo derivative is defined as the application of a fractional integral to a standard derivative of the function whereas the Riemann–Liouville fractional derivative is obtained by differentiating the fractional integral of a function with respect to its independent variable of order n. The Caputo fractional derivative necessitates

more suitable initial conditions in contrast to the conventional Riemann–Liouville fractional derivative considering fractional differential equations [13]. Accordingly, when evaluating other fractional derivatives, the Caputo derivative is advantageous since it yields solutions that are more meaningful in a physical sense for the specific problems. Besides, the operator of proportional derivative denoted as ${}^{P}D_{\rho}\psi(\varkappa)$ is given by the equation [4] :

$$ {}^{P}D_{\rho}\psi(\varkappa) = K_1(\rho,\mathfrak{s})\psi(\mathfrak{s}) + K_0(\rho,\mathfrak{s})\psi'(\mathfrak{s}), $$

where K_1 and K_0 are the functions with respect to $\rho \in [0,1]$ and $\mathfrak{s} \in \mathbb{R}$ subject to certain conditions and also, the function ψ is differentiable with respect to $\mathfrak{s} \in \mathbb{R}$. This mathematical operator is commonly used in control systems and robotics. In recent years, there has been a notable increase in the importance of research conducted on both the Caputo derivative and the proportional derivative [15, 16, 21].

In [6], Baleanu et al. gave the following definition which they merge the concepts of Caputo derivative and proportional derivative in a novel manner, resulting in a hybrid fractional operator that can be represented as a linear combination of Caputo fractional derivative and Riemann–Liouville fractional integral.

Definition 2. Let $\psi\colon I \subset \mathbb{R}^{+} \to \mathbb{R}$ be a differentiable function on I° and ψ, ψ' are locally $L(I)$. Then, the proportional Caputo-hybrid operator may be defined as follows:

$$ {}^{C}D^{\rho}_{\varsigma^{+}}\,\psi(\mathfrak{s}) = \frac{1}{\Gamma(1-\rho)} \int_{0}^{\mathfrak{s}} \left[K_1(\rho,\tau)\psi(\tau) + K_0(\rho,\tau)\psi'(\tau)\right](\mathfrak{s}-\tau)^{-\rho}\mathrm{d}\tau, $$

where $\rho \in [0,1]$ and K_1 and K_0 are functions which satisfy the following conditions:

$$ \lim_{\rho\to 0^{+}} K_0(\rho,\tau) = 0; \quad \lim_{\rho\to 1} K_0(\rho,\tau) = 1; \;\; K_0(\rho,\tau) \neq 0, \;\; \rho \in (0,1]; $$
$$ \lim_{\rho\to 0} K_1(\rho,\tau) = 0; \quad \lim_{\rho\to 1^{-}} K_1(\rho,\tau) = 1; \;\; K_1(\rho,\tau) \neq 0, \;\; \rho \in [0,1). $$

Afterwards, Sarıkaya [25] presented a novel definition by employing distinct K_1 and K_0 functions based on Definition 2. Furthermore, Sarıkaya [25] derived the Hermite–Hadamard inequality utilizing his own new definition as presented below:

Definition 3. Let $\psi\colon I \subset \mathbb{R}^{+} \to \mathbb{R}$ be a differentiable function on I° and ψ, $\psi' \in L(I)$, where $L(I)$ denote the collection of all Riemann integrable functions on I. The left-sided and right-sided proportional Caputo-hybrid operator of order ρ are defined respectively as follows:

$$ {}^{PC}_{\varsigma^{+}}D^{\rho}_{\vartheta}\,\psi(\vartheta) = \frac{1}{\Gamma(1-\rho)} \int_{\varsigma}^{\vartheta} \left[K_1(\rho,\vartheta-\tau)\psi(\tau) + K_0(\rho,\vartheta-\tau)\psi'(\tau)\right](\vartheta-\tau)^{-\rho}\mathrm{d}\tau, $$

and

$$ {}^{PC}_{\vartheta^-}D^{\rho}_{\varsigma}\,\psi(\varsigma)=\frac{1}{\Gamma(1-\rho)}\int_{\varsigma}^{\vartheta}\left[K_1(\rho,\tau-\varsigma)\psi(\tau)+K_0(\rho,\tau-\varsigma)\psi'(\tau)\right](\tau-\varsigma)^{-\rho}\mathrm{d}\tau, $$

where $\rho\in[0,1]$ and $K_0(\rho,\tau)=(1-\rho)^2\tau^{1-\rho}$ and $K_1(\rho,\tau)=\rho^2\tau^{\rho}$.

Theorem 1. *Let $\psi\colon I\subset\mathbb{R}^+\to\mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $\varsigma,\vartheta\in I^\circ$ with $\varsigma<\vartheta$ and let ψ,ψ' be convex functions on I. Then, the following inequalities hold:*

$$ \begin{aligned} &\rho^2(\vartheta-\varsigma)^{\rho}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+\frac{1}{2}(1-\rho)(\vartheta-\varsigma)^{1-\rho}\psi'\left(\frac{\varsigma+\vartheta}{2}\right)\\ &\leq\ \frac{\Gamma(1-\rho)}{2(\vartheta-\varsigma)^{1-\rho}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\rho}_{\varsigma}\psi(\varsigma)\right]\\ &\leq\ \rho^2(\vartheta-\varsigma)^{\rho}\left[\frac{\psi(\varsigma)+\psi(\vartheta)}{2}\right]+(1-\rho))(\vartheta-\varsigma)^{1-\rho}\left[\frac{\psi'(\varsigma)+\psi'(\vartheta)}{4}\right]. \end{aligned} $$

In [26], Sarıkaya also gave the following Simpson's type inequality using his own definition of the proportional Caputo operator:

Lemma 1. *Let $\psi\colon I\subset\mathbb{R}^+\to\mathbb{R}$ be differantiable function on I°, the interior of the interval I, where $\varsigma,\vartheta\in I^\circ$ with $\varsigma<\vartheta$, and $\psi',\psi''\in L[\varsigma,\vartheta]$. Then, the following identity holds:*

$$ \begin{aligned} S(\varsigma,\vartheta;\rho)=&\frac{\rho^2(\vartheta-\varsigma)^{\rho}}{6}\left[\psi(\varsigma)+4\psi\left(\frac{\varsigma+\vartheta}{2}\right)+\psi(\vartheta)\right]\\ &+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{12}\left[\psi'(\varsigma)+4\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+\psi'(\vartheta)\right]\\ &-\frac{\Gamma(1-\rho)}{2(\vartheta-\varsigma)^{1-\rho}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\rho}_{\varsigma}\psi(\varsigma)\right], \end{aligned} $$

where

$$ \begin{aligned} S(\varsigma,\vartheta;\rho)=&\frac{\rho^2(\vartheta-\varsigma)^{1+\rho}}{2}\int_0^1 P(\mathfrak{s})[\psi'(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)+\psi'(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)]\,\mathrm{d}\mathfrak{s}\\ &+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{4}\int_0^1 Q(\mathfrak{s})[\psi''(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)+\psi''(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)]\,\mathrm{d}\mathfrak{s}, \end{aligned} $$

$$ P(\mathfrak{s})=\begin{cases}\frac{1}{6}-\mathfrak{s}, & 0\leq\mathfrak{s}<\frac{1}{2},\\ \frac{5}{6}-\mathfrak{s}, & \frac{1}{2}\leq\mathfrak{s}\leq 1,\end{cases} $$

and

$$Q(\mathfrak{s})=\begin{cases}\frac{1}{6}-\mathfrak{s}^{2-2\rho}, & 0\leq\mathfrak{s}<\frac{1}{2},\\ \\ \frac{5}{6}-\mathfrak{s}^{2-2\rho}, & \frac{1}{2}\leq\mathfrak{s}\leq 1.\end{cases}$$

The motivation of this paper is to investigate the analogous forms of the Milne-type inequalities with regard to Riemann integrals by using the proportional Caputo-hybrid operator. In line with this purpose, firstly, we present an identity with the help of the newly defined proportional Caputo-hybrid operator. Then, we establish many important Milne-type inequalities by utilizing convexity, the Hölder inequality, and the power mean inequality. Moreover, we investigate Milne-type inequalities for bounded mappings and mappings of bounded variation. Through the consideration of appropriate assumptions of ρ, these results improve and generalize the inequalities derived in earlier works.

2. Main results

Our theorems rely primarily on the following lemma. So, we will first provide the proof of this lemma.

Lemma 2. *Let $\psi\colon I\subset\mathbb{R}^{+}\to\mathbb{R}$ be a twice differentiable function on I^{o}, the interior of the interval I, where $\varsigma,\vartheta\in I^{o}$ satisfying $\varsigma<\vartheta$ and let $\psi,\psi',\psi''\in L[\varsigma,\vartheta]$. Then, the following identity is satisfied:*

$$\begin{aligned}&\rho^{2}(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\int_{0}^{1}\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\\ &\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\\ &\quad\times\int_{0}^{1}\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\\ &=\frac{\rho^{2}(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\\ &\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\ &\quad-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^{+}}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^{-}}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right].\end{aligned}\tag{2.1}$$

Proof. Through the process of integration by parts, and using the change of the variable we get

$$\int_{0}^{1}\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}$$

$$
\begin{aligned}
&= \frac{4}{3(\vartheta-\varsigma)}\psi(\vartheta) - \frac{2}{6(\vartheta-\varsigma)}\psi\left(\frac{\varsigma+\vartheta}{2}\right) - \frac{1}{\vartheta-\varsigma}\int_0^1 \psi\left(\frac{1-\mathfrak{s}}{2}\varsigma + \frac{1+\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= \frac{4}{3(\vartheta-\varsigma)}\psi(\vartheta) - \frac{2}{6(\vartheta-\varsigma)}\psi\left(\frac{\varsigma+\vartheta}{2}\right) - \frac{2}{(\vartheta-\varsigma)^2}\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta} \psi(\tau)\,\mathrm{d}\tau, \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \left(\frac{\mathfrak{s}}{2} + \frac{1}{6}\right)\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma + \frac{1-\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= -\frac{4}{3(\vartheta-\varsigma)}\psi(\varsigma) + \frac{2}{6(\vartheta-\varsigma)}\psi\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{1}{\vartheta-\varsigma}\int_0^1 \psi\left(\frac{1+\mathfrak{s}}{2}\varsigma + \frac{1-\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= -\frac{4}{3(\vartheta-\varsigma)}\psi(\varsigma) + \frac{2}{6(\vartheta-\varsigma)}\psi\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{2}{(\vartheta-\varsigma)^2}\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}} \psi(\tau)\,\mathrm{d}\tau, \\
&\int_0^1 \left(\frac{\mathfrak{s}^{2-2\rho}}{2} + \frac{1}{6}\right)\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma + \frac{1+\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= \frac{4}{3(\vartheta-\varsigma)}\psi'(\vartheta) - \frac{2}{6(\vartheta-\varsigma)}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) - \frac{2-2\rho}{\vartheta-\varsigma}\int_0^1 \mathfrak{s}^{1-2\rho}\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma + \frac{1+\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= \frac{4}{3(\vartheta-\varsigma)}\psi'(\vartheta) - \frac{2}{6(\vartheta-\varsigma)}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) - \frac{2^{3-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{3-2\rho}}\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left(\tau - \frac{\varsigma+\vartheta}{2}\right)^{1-2\rho}\psi'(\tau)\,\mathrm{d}\tau
\end{aligned} \qquad (2.3)
$$

and

$$
\begin{aligned}
&\int_0^1 \left(\frac{\mathfrak{s}^{2-2\rho}}{2} + \frac{1}{6}\right)\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma + \frac{1-\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= -\frac{4}{3(\vartheta-\varsigma)}\psi'(\varsigma) + \frac{2}{6(\vartheta-\varsigma)}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{2-2\rho}{\vartheta-\varsigma}\int_0^1 \mathfrak{s}^{1-2\rho}\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma + \frac{1-\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= -\frac{4}{3(\vartheta-\varsigma)}\psi'(\varsigma) + \frac{2}{6(\vartheta-\varsigma)}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{2^{3-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{3-2\rho}}\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\left(\frac{\varsigma+\vartheta}{2} - \tau\right)^{1-2\rho}\psi'(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Multiplying (2.2) by $2^{1-2\rho}\rho^2(\vartheta-\varsigma)^{1+\rho}$ and (2.3) by $\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}$, and combining them side by side, we obtain the following result:

$$\begin{aligned}
&2^{1-2\rho}\rho^2(\vartheta-\varsigma)^{1+\rho}\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\
&=\frac{2^{3-2\rho}\rho^2(\vartheta-\varsigma)^{\rho}}{3}\psi(\vartheta)-\frac{2^{2-2\rho}\rho^2(\vartheta-\varsigma)^{\rho}}{6}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\\
&\quad+\frac{2(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\psi'(\vartheta)-\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{6}\psi'\left(\frac{\varsigma+\vartheta}{2}\right)\\
&\quad-\frac{2^{2-2\rho}}{(\vartheta-\varsigma)^{1-\rho}}\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left[\rho^2\left(\tau-\frac{\varsigma+\vartheta}{2}\right)^{\rho}\psi(\tau)\right.\\
&\quad\left.+(1-\rho)^2\left(\tau-\frac{\varsigma+\vartheta}{2}\right)^{1-\rho}\psi'(\tau)\right]\left(\tau-\frac{\varsigma+\vartheta}{2}\right)^{-\rho}d\tau. \qquad (2.4)
\end{aligned}$$

By employing similar steps, the following result emerges:

$$\begin{aligned}
&2^{1-2\rho}\rho^2(\vartheta-\varsigma)^{1+\rho}\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\
&=-\frac{2^{3-2\rho}\rho^2(\vartheta-\varsigma)^{\rho}}{3}\psi(\varsigma)+\frac{2^{2-2\rho}\rho^2(\vartheta-\varsigma)^{\rho}}{6}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\\
&\quad-\frac{2(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\psi'(\varsigma)+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{6}\psi'\left(\frac{\varsigma+\vartheta}{2}\right)\\
&\quad+\frac{2^{2-2\rho}}{(\vartheta-\varsigma)^{1-\rho}}\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\left[\rho^2\left(\frac{\varsigma+\vartheta}{2}-\tau\right)^{\rho}\psi(\tau)\right.\\
&\quad\left.+(1-\rho)^2\left(\frac{\varsigma+\vartheta}{2}-\tau\right)^{1-\rho}\psi'(\tau)\right]\left(\frac{\varsigma+\vartheta}{2}-\tau\right)^{-\rho}d\tau. \qquad (2.5)
\end{aligned}$$

The required equality (2.1) is obtained by applying equations (2.4) and (2.5). □

Remark 1. Letting the limit as $\rho \to 1$ in Lemma 2, it follows that

$$\frac{(\vartheta-\varsigma)}{2}\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}$$
$$=\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_\varsigma^\vartheta\psi(\varkappa)d\varkappa,$$

which is the same with Lemma 1 provided by Budak et al. [11] under the condition $\rho=1$.

Corollary 1. *In the limiting case* $\rho=0$ *in Lemma 2, we obtain*

$$\frac{(\vartheta-\varsigma)}{2}\int_0^1\left(\frac{\mathfrak{s}^2}{2}+\frac{1}{6}\right)\left[\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}$$
$$=\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2\left(\psi(\vartheta)-\psi(\varsigma)\right)}{\vartheta-\varsigma}$$
$$-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_\varsigma^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)d\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^\vartheta\psi(\varkappa)d\varkappa\right).$$

Now, we present some important Milne-type inequalities by taking into account the convexity, power mean inequality and the Hölder inequality.

Theorem 2. *Let* $\psi\colon I\subset\mathbb{R}^+\to\mathbb{R}$ *be a twice differentiable function on* I^o*, the interior of the interval I, where* $\varsigma,\vartheta\in I^o$ *satisfying* $\varsigma<\vartheta$ *and let* $\psi,\psi',\psi''\in L[\varsigma,\vartheta]$*. If* $|\psi'|^q$ *and* $|\psi''|^q$ *are convex on* $[\varsigma,\vartheta]$ *for* $q\geq 1$*, then the following inequality holds:*

$$\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.$$
$$+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)$$
$$\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|$$
$$\leq\ \rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\left(\frac{5}{12}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{12}\left|\psi'(\varsigma)\right|^q+\frac{1}{3}\left|\psi'(\vartheta)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{3}\left|\psi'(\varsigma)\right|^q+\frac{1}{12}\left|\psi'(\vartheta)\right|^q\right)^{\frac{1}{q}}\right]+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{6}+\frac{1}{2(3-2\rho)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(A(\rho)|\psi''(\varsigma)|^q+B(\rho)|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}+\left(B(\rho)|\psi''(\varsigma)|^q+A(\rho)|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}\right],\quad(2.6)$$

where

$$A(\rho)=\frac{1}{24}+\frac{1}{4(3-2\rho)}-\frac{1}{4(4-2\rho)},$$

and

$$B(\rho)=\frac{1}{8}+\frac{1}{4(3-2\rho)}+\frac{1}{4(4-2\rho)}.$$

Proof. First, let us examine the case in which $q=1$. Using the convexity of $|\psi'|$ and $|\psi''|$, we deduce from Lemma 2 that

$$\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\leq\ \rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\left(\frac{1-\mathfrak{s}}{2}\left|\psi'(\varsigma)\right|+\frac{1+\mathfrak{s}}{2}\left|\psi'(\vartheta)\right|\right)\right.\\
&\quad\left.+\left(\frac{1+\mathfrak{s}}{2}\left|\psi'(\varsigma)\right|+\frac{1-\mathfrak{s}}{2}\left|\psi'(\vartheta)\right|\right)\right]d\mathfrak{s}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\left(\frac{1-\mathfrak{s}}{2}\left|\psi''(\varsigma)\right|+\frac{1+\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|\right)\right.\\
&\quad\left.+\left(\frac{1+\mathfrak{s}}{2}\left|\psi''(\varsigma)\right|+\frac{1-\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|\right)\right]d\mathfrak{s}. \qquad (2.7)
\end{aligned}$$

Hence, from the fact that

$$\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\frac{1-\mathfrak{s}}{2}d\mathfrak{s}=\frac{1}{12},\qquad\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\frac{1+\mathfrak{s}}{2}d\mathfrak{s}=\frac{1}{3},$$

$$\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\frac{1-\mathfrak{s}}{2}d\mathfrak{s}=\frac{1}{24}+\frac{1}{4(3-2\rho)}-\frac{1}{4(4-2\rho)},$$

$$\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\frac{1+\mathfrak{s}}{2}d\mathfrak{s}=\frac{1}{8}+\frac{1}{4(3-2\rho)}+\frac{1}{4(4-2\rho)},$$

it follows that the expression on the right-hand side of inequality (2.7) is

$$\frac{5\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{12}\left(\left|\psi'(\varsigma)\right|+\left|\psi'(\vartheta)\right|\right)$$

$$+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{6}+\frac{1}{2(3-2\rho)}\right)\left(\left|\psi''(\varsigma)\right|+\left|\psi''(\vartheta)\right|\right).$$

For $q\geq 1$, we can now achieve the following by using Lemma 2, the power mean inequality, and accounting for the convexity of $|\psi'|^q$ and $|\psi''|^q$:

$$\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\leq \rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\\
&\quad\times\left\{\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\frac{1-\mathfrak{s}}{2}|\psi'(\varsigma)|^q+\frac{1+\mathfrak{s}}{2}|\psi'(\vartheta)|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\frac{1+\mathfrak{s}}{2}|\psi'(\varsigma)|^q+\frac{1-\mathfrak{s}}{2}|\psi'(\vartheta)|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right\}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left\{\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)d\mathfrak{s}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\frac{1-\mathfrak{s}}{2}|\psi''(\varsigma)|^q+\frac{1+\mathfrak{s}}{2}|\psi''(\vartheta)|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\frac{1+\mathfrak{s}}{2}|\psi''(\varsigma)|^q+\frac{1-\mathfrak{s}}{2}|\psi''(\vartheta)|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Consequently, it follows that inequality (2.6) is true since

$$\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)d\mathfrak{s}=\frac{5}{12},\quad\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)d\mathfrak{s}=\frac{1}{6}+\frac{1}{2(3-2\rho)}.$$

□

Remark 2. Letting the limit as $\rho\to 1$ and putting $q=1$ in Theorem 2, it follows that

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)d\varkappa\right|\leq\frac{5(\vartheta-\varsigma)}{24}\left(|\psi'(\varsigma)|+|\psi'(\vartheta)|\right),$$

which was proved by Budak et al. in [11]. Also, when ρ converges to 1 and with $q \geq 1$, the inequality described in Theorem 2 is given by

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|$$
$$\leq \frac{5(\vartheta-\varsigma)}{24}\left[\left(\frac{|\psi'(\varsigma)|^q+4|\psi'(\vartheta)|^q}{5}\right)^{1/q}+\left(\frac{4|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{5}\right)^{1/q}\right],$$

which was proved by Budak et al. in [11].

Corollary 2. *As* ρ *approaches* 0 *and for* $q \geq 1$ *in Theorem 2, we obtain*

$$\left|\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2(\psi(\vartheta)-\psi(\varsigma))}{\vartheta-\varsigma}\right.$$
$$\left.-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)\mathrm{d}\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right)\right|$$
$$\leq \frac{(\vartheta-\varsigma)}{6}\left[\left(\frac{9}{48}|\psi''(\varsigma)|^q+\frac{39}{48}|\psi''(\vartheta)|^q\right)^{1/q}+\left(\frac{39}{48}|\psi''(\varsigma)|^q+\frac{9}{48}|\psi''(\vartheta)|^q\right)^{1/q}\right].$$

We present a concrete example to verify the validity of our theorem.

Example 1. Consider the function $\psi(\varkappa)=\varkappa^3$ defined on the interval $[0,2]$. Then, we can determine the right-hand side of inequality (2.6) for $q=1$ as follows:

$$5\rho^2 2^{2-\rho}+6(1-\rho)2^{2-\rho}\left(\frac{1}{6}+\frac{1}{2(3-2\rho)}\right):=\Psi_1.$$

Furthermore, it can be observed

$$\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.$$
$$+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)$$
$$\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|$$
$$=2^{1-\rho}\left(6\rho^2+7-7\rho-\frac{3(1-\rho)(3-2\rho)}{\rho-2}\right):=\Psi_2.$$

Therefore, for all $0<\rho<1$ and $q=1$, the left side of the inequality (2.6) is constantly located below the right side of this inequality, as shown in Figure 1.

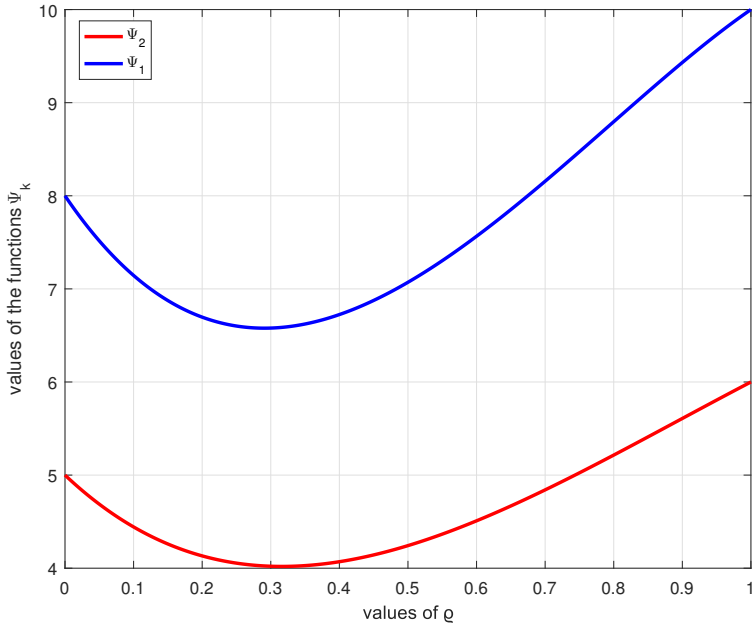

FIGURE 1. The graph of both sides of the inequality (2.6) depending on $0<\varrho<1$ and $q=1$, which is computed and drawn in MATLAB program.

Theorem 3. *Let $\psi: I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $\varsigma,\vartheta\in I^o$ satisfying $\varsigma<\vartheta$ and let $\psi,\psi',\psi''\in L[\varsigma,\vartheta]$. If $|\psi'|^q$ and $|\psi''|^q$ are convex on $[\varsigma,\vartheta]$ for $q>1$, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{\varrho^2(\vartheta-\varsigma)^{\varrho}2^{2-2\varrho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\varrho)(\vartheta-\varsigma)^{1-\varrho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\varrho}\Gamma(1-\varrho)}{(\vartheta-\varsigma)^{-\varrho+1}}\left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\varrho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\leq\frac{\varrho^2(\vartheta-\varsigma)^{\varrho+1}2^{1-2\varrho}}{6}\left(\frac{2^{2p+3}-2}{6p+6}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\psi'(\varsigma)\right|^q+\left|\psi'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{\left|\psi'(\varsigma)\right|^q+3\left|\psi'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}\right]+\frac{(1-\varrho)(\vartheta-\varsigma)^{2-\varrho}}{2}\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\varrho}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left[\left(\frac{3\left|\psi''(\varsigma)\right|^q+\left|\psi''(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\psi''(\varsigma)\right|^q+3\left|\psi''(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{6}\left(\frac{2^{2p+5}-2^3}{6p+6}\right)^{\frac{1}{p}}\left(\left|\psi'(\varsigma)\right|+\left|\psi'(\vartheta)\right|\right)$$

$$+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(4\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}\right)^{\frac{1}{p}}\left(\left|\psi''(\varsigma)\right|+\left|\psi''(\vartheta)\right|\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying the well-known Hölder's inequality and taking into account the convexity of $|\psi'|^q$ and $|\psi''|^q$, based on Lemma 2, we obtain

$$\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\leq\rho^2(\vartheta-\varsigma)^{1+\rho}2^{1-2\rho}\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\int_0^1\left|\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right|^q d\mathfrak{s}\right)^{\frac{1}{q}}+\left(\int_0^1\left|\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|^q d\mathfrak{s}\right)^{\frac{1}{q}}\right]\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\int_0^1\left|\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right|^q d\mathfrak{s}\right)^{\frac{1}{q}}+\left(\int_0^1\left|\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|^q d\mathfrak{s}\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{2.8}$$

When we assess the integrals in the aforementioned inequality, we arrive at

$$\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}=\frac{2^{2p+3}-2}{6^{p+1}(p+1)},$$

$$\int_0^1 \left[\frac{1-\mathfrak{s}}{2}\left|\psi'(\varsigma)\right|^q+\frac{1+\mathfrak{s}}{2}\left|\psi'(\vartheta)\right|^q\right]\mathrm{d}\mathfrak{s}=\frac{|\psi'(\varsigma)|^q+3|\psi'(\vartheta)|^q}{4},$$

$$\int_0^1 \left[\frac{1+\mathfrak{s}}{2}\left|\psi'(\varsigma)\right|^q+\frac{1-\mathfrak{s}}{2}\left|\psi'(\vartheta)\right|^q\right]\mathrm{d}\mathfrak{s}=\frac{3|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{4}.$$

$$\int_0^1 \left[\frac{1-\mathfrak{s}}{2}\left|\psi''(\varsigma)\right|^q+\frac{1+\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right]\mathrm{d}\mathfrak{s}=\frac{|\psi''(\varsigma)|^q+3|\psi''(\vartheta)|^q}{4},$$

$$\int_0^1 \left[\frac{1+\mathfrak{s}}{2}\left|\psi''(\varsigma)\right|^q+\frac{1-\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right]\mathrm{d}\mathfrak{s}=\frac{3|\psi''(\varsigma)|^q+|\psi''(\vartheta)|^q}{4}.$$

Thus, the first inequality of (2.8) can be obtained by replacing the calculated integral results. Also, it is known that we get the property

$$\sum_{k=1}^{n}(\varsigma_k+\vartheta_k)^s\leq\sum_{k=1}^{n}\varsigma_k^s+\sum_{k=1}^{n}\vartheta_k^s,$$

for $0\leq s<1$ and $\varsigma_k,\ \vartheta_k\geq 0$ with $k\in\{1,2,\ldots,n\}$. Therefore, for the proof of the second inequality, if we take $\varsigma_1=|\psi'(\varsigma)|^q$, $\vartheta_1=3\,|\psi'(\vartheta)|^q$, $\varsigma_2=3\,|\psi'(\varsigma)|^q$, $\vartheta_2=|\psi'(\vartheta)|^q$, $\varsigma_1'=|\psi''(\varsigma)|^q$, $\vartheta_1'=3\,|\psi''(\vartheta)|^q$, $\varsigma_2'=3\,|\psi''(\varsigma)|^q$, $\vartheta_2'=|\psi''(\vartheta)|^q$, by $1+3^{\frac{1}{q}}\leq 4$, the desired outcome can be achieved straightforwardly. □

Remark 3. In the particular case when ρ tends to 1 in Theorem 3, we get

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|$$

$$\leq\frac{\vartheta-\varsigma}{12}\left(\frac{2^{2p+2}-1}{3p+3}\right)^{\frac{1}{p}}\left[\left(\frac{3\,|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\psi'(\varsigma)|^q+3\,|\psi'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{\vartheta-\varsigma}{12}\left(\frac{2^{2p+4}-4}{3p+3}\right)^{\frac{1}{p}}\left(\left|\psi'(\varsigma)\right|+\left|\psi'(\vartheta)\right|\right),$$

which was proved by Budak et al. in [11].

Corollary 3. *According to Theorem 3, as* ρ *approaches* 0*, we have the following particular case:*

$$\left|\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2\left(\psi(\vartheta)-\psi(\varsigma)\right)}{\vartheta-\varsigma}\right.$$

$$
\begin{aligned}
&-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)\mathrm{d}\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right)\Bigg| \\
&\leq \frac{(\vartheta-\varsigma)}{2}\left(\int_0^1\left(\frac{\mathfrak{s}^2}{2}+\frac{1}{6}\right)^p\mathrm{d}\mathfrak{s}\right)^{\frac{1}{p}} \\
&\quad\cdot\left[\left(\frac{|\psi''(\varsigma)|^q+3|\psi''(\vartheta)|^q}{4}\right)^{1/q}+\left(\frac{3|\psi''(\varsigma)|^q+|\psi''(\vartheta)|^q}{4}\right)^{1/q}\right] \\
&\leq \frac{(\vartheta-\varsigma)}{2}\left(4\int_0^1\left(\frac{\mathfrak{s}^2}{2}+\frac{1}{6}\right)^p\mathrm{d}\mathfrak{s}\right)^{\frac{1}{p}}\left(\left|\psi''(\varsigma)\right|+\left|\psi''(\vartheta)\right|\right).
\end{aligned}
$$

3. Milne-type inequalities for bounded mappings involving proportional Caputo-hybrid operator

Our task in this section is to obtain some Milne-type inequalities for bounded mappings with the help of a proportional Caputo-hybrid operator.

Theorem 4. *Suppose that the conditions of Lemma 2 hold. If there exist $m,n,M,N\in\mathbb{R}$ such that $m\leq\psi'(\mathfrak{s})\leq M$ and $n\leq\psi''(\mathfrak{s})\leq N$ for all $\mathfrak{s}\in[\varsigma,\vartheta]$, then we have*

$$
\begin{aligned}
&\Bigg|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right) \\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right) \\
&\quad-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\Bigg| \\
&\leq \frac{5\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{12}(M-m)+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{2(3-2\rho)}+\frac{1}{6}\right)(N-n).
\end{aligned}
$$

Proof. In the identity in Lemma 2, adding and subtracting $\frac{m+M}{2}$ at the point where the ψ' functions appear within the integral, and similarly adding and subtracting $\frac{n+N}{2}$ at the point where the ψ'' functions appear, support us in obtaining

$$
\begin{aligned}
&\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right) \\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right) \\
&\quad-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\left\{\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{m+M}{2}\right]d\mathfrak{s}\right.\\
&\quad\left.+\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\frac{m+M}{2}-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\right\}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left\{\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{n+N}{2}\right]d\mathfrak{s}\right.\\
&\quad\left.+\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\frac{n+N}{2}-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\right\}. \qquad (3.1)
\end{aligned}
$$

After determining absolute value of (3.1), we get

$$
\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\le\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\left\{\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left|\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{m+M}{2}\right|d\mathfrak{s}\right.\\
&\quad\left.+\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left|\frac{m+M}{2}-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|d\mathfrak{s}\right\}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left\{\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left|\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{n+N}{2}\right|d\mathfrak{s}\right.\\
&\quad\left.+\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left|\frac{n+N}{2}-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|d\mathfrak{s}\right\}.
\end{aligned}
$$

Because of $m\le\psi'(\mathfrak{s})\le M$ and $n\le\psi''(\mathfrak{s})\le N$ for all $\mathfrak{s}\in[\varsigma,\vartheta]$, we obtain

$$
\begin{aligned}
\left|\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{m+M}{2}\right|&\le\frac{M-m}{2},\\
\left|\frac{m+M}{2}-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|&\le\frac{M-m}{2},
\end{aligned}
$$

and

$$\left|\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{n+N}{2}\right|\leq\frac{N-n}{2},$$
$$\left|\frac{n+N}{2}-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|\leq\frac{N-n}{2}.$$

Thus, we have

$$\begin{aligned}&\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\&\leq\frac{5\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{12}(M-m)+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{2(3-2\rho)}+\frac{1}{6}\right)(N-n).\end{aligned}$$

□

Corollary 4. *As* ρ *tends to* 0 *in Theorem 4, we obtain the following specific case:*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2\left(\psi(\vartheta)-\psi(\varsigma)\right)}{\vartheta-\varsigma}\right.\\&\quad\left.-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_\varsigma^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)\mathrm{d}\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right)\right|\leq\frac{\vartheta-\varsigma}{6}(N-n).\end{aligned}$$

Corollary 5. *Under the assumptions of Theorem 4, if there exist* $M,N\in\mathbb{R}^+$ *such that* $|\psi'(\mathfrak{s})|\leq M$ *and* $|\psi''(\mathfrak{s})|\leq N$ *for all* $\mathfrak{s}\in[\varsigma,\vartheta]$*, then we have*

$$\begin{aligned}&\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\&\leq\frac{5\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{6}M+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{3-2\rho}+\frac{1}{3}\right)N.\end{aligned}$$

Remark 4. Theorem 4 states that in the specific condition when ρ approaches to 1, we get

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{5(\vartheta-\varsigma)}{24}(M-m),$$

which was shown by Budak et al. in [11].

Remark 5. In the particular case when ρ tends to 1 in Conclusion 5, we have

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{5(\vartheta-\varsigma)}{12}M,$$

which was demonstrated by Alomari and Liu in [3].

4. Milne-type inequalities for mappings of bounded variation involving proportional Caputo-hybrid operator

Here, we present some Milne-type inequalities for proportional Caputo-hybrid operators using mappings of bounded variation.

Theorem 5. *Let $\psi,\psi'\colon[\varsigma,\vartheta]\to\mathbb{R}$ be a mapping of bounded variation on $[\varsigma,\vartheta]$. Then, we get*

$$\begin{aligned}&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\&\leq\frac{2^{3-2\rho}\rho^2}{3(\vartheta-\varsigma)^{-\rho}}\bigvee_{\varsigma}^{\vartheta}(\psi)+\frac{2(1-\rho)}{3(\vartheta-\varsigma)^{-1+\rho}}\bigvee_{\varsigma}^{\vartheta}(\psi').\end{aligned}$$

Here $\bigvee_{\varsigma}^{\vartheta}(\psi)$ and $\bigvee_{\varsigma}^{\vartheta}(\psi')$ denote the total variations of ψ and ψ' on $[\varsigma,\vartheta]$, respectively.

Proof. Consider the mappings $K,L_{\rho}\colon[\varsigma,\vartheta]\to\mathbb{R}$ given by

$$K(\mathfrak{s})=\begin{cases}-\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)-\frac{(\vartheta-\varsigma)}{6}, & \varsigma\leq\mathfrak{s}\leq\frac{\varsigma+\vartheta}{2},\\ \left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)+\frac{(\vartheta-\varsigma)}{6}, & \frac{\varsigma+\vartheta}{2}\leq\mathfrak{s}\leq\vartheta,\end{cases}$$

and

$$L_\rho(\mathfrak{s}) = \begin{cases} -\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{2-2\rho} - \frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}, & \varsigma \le \mathfrak{s} \le \frac{\varsigma+\vartheta}{2}, \\ \left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{2-2\rho} + \frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}, & \frac{\varsigma+\vartheta}{2} \le \mathfrak{s} \le \vartheta. \end{cases}$$

From integration by parts it follows that

$$\begin{aligned} \int_\varsigma^\vartheta K(\mathfrak{s})\mathrm{d}\psi(\mathfrak{s}) &= -\int_\varsigma^{\frac{\varsigma+\vartheta}{2}} \left(\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)+\frac{(\vartheta-\varsigma)}{6}\right)\mathrm{d}\psi(\mathfrak{s}) \\ &\quad + \int_{\frac{\varsigma+\vartheta}{2}}^\vartheta \left(\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)+\frac{(\vartheta-\varsigma)}{6}\right)\mathrm{d}\psi(\mathfrak{s}) \\ &= -\frac{(\vartheta-\varsigma)}{6}\psi\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{2(\vartheta-\varsigma)}{3}\psi(\varsigma) - \int_\varsigma^{\frac{\varsigma+\vartheta}{2}} \psi(\mathfrak{s})\mathrm{d}\mathfrak{s} \\ &\quad + \frac{2(\vartheta-\varsigma)}{3}\psi(\vartheta) - \frac{(\vartheta-\varsigma)}{6}\psi\left(\frac{\varsigma+\vartheta}{2}\right) - \int_{\frac{\varsigma+\vartheta}{2}}^\vartheta \psi(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \int_\varsigma^\vartheta L_\rho(\mathfrak{s})\mathrm{d}\psi'(\mathfrak{s}) &= -\int_\varsigma^{\frac{\varsigma+\vartheta}{2}} \left(\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right)\mathrm{d}\psi'(\mathfrak{s}) \\ &\quad + \int_{\frac{\varsigma+\vartheta}{2}}^\vartheta \left(\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right)\mathrm{d}\psi'(\mathfrak{s}) \\ &= -\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{4(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\psi'(\varsigma) \\ &\quad - (2-2\rho)\int_\varsigma^{\frac{\varsigma+\vartheta}{2}} \left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{1-2\rho}\psi'(\mathfrak{s})\mathrm{d}\mathfrak{s} \\ &\quad + \frac{4(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\psi'(\vartheta) - \frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) \end{aligned}$$

$$-(2-2\rho)\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{1-2\rho}\psi'(\mathfrak{s})\mathrm{d}\mathfrak{s}. \tag{4.2}$$

Through the process of multiplying (4.1) by $2^{2-2\rho}\rho^2(\vartheta-\varsigma)^{-1+\rho}$ and (4.2) by $2^{1-2\rho}(1-\rho)(\vartheta-\varsigma)^{-1+\rho}$, and adding them side by side, we achieve the following result:

$$\begin{aligned}
&\frac{2^{2-2\rho}\rho^2}{(\vartheta-\varsigma)^{1-\rho}}\int_{\varsigma}^{\vartheta}K(\mathfrak{s})\mathrm{d}\psi(\mathfrak{s})+\frac{2^{1-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{1-\rho}}\int_{\varsigma}^{\vartheta}L_\rho(\mathfrak{s})\mathrm{d}\psi'(\mathfrak{s})\\
&=\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right].
\end{aligned}$$

It is commonly known that if $g,h\colon[\varsigma,\vartheta]\to\mathbb{R}$ are in the sense that g is continuous on $[\varsigma,\vartheta]$ and h is of bounded variation on $[\varsigma,\vartheta]$, then $\int_{\varsigma}^{\vartheta}g(\mathfrak{s})\mathrm{d}h(\mathfrak{s})$ exists and

$$\left|\int_{\varsigma}^{\vartheta}g(\mathfrak{s})\mathrm{d}h(\mathfrak{s})\right|\leq\sup_{\mathfrak{s}\in[\varsigma,\vartheta]}|g(\mathfrak{s})|\bigvee_{\varsigma}^{\vartheta}(h). \tag{4.3}$$

Thus, making use of (4.3), we find

$$\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\leq\frac{2^{2-2\rho}\rho^2}{(\vartheta-\varsigma)^{1-\rho}}\left[\left|\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\left(\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)+\frac{(\vartheta-\varsigma)}{6}\right)\mathrm{d}\psi(\mathfrak{s})\right|\right.\\
&\quad\left.+\left|\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left(\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)+\frac{(\vartheta-\varsigma)}{6}\right)\mathrm{d}\psi(\mathfrak{s})\right|\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2^{1-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{1-\rho}}\left[\left|\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\left(\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right)\mathrm{d}\psi'(\mathfrak{s})\right|\right.\\
&\left.+\left|\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left(\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right)\mathrm{d}\psi'(\mathfrak{s})\right|\right]\\
&\leq\frac{2^{2-2\rho}\rho^2}{(\vartheta-\varsigma)^{1-\rho}}\left[\sup_{\mathfrak{s}\in[\varsigma,\frac{\varsigma+\vartheta}{2}]}\left|\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)+\frac{(\vartheta-\varsigma)}{6}\right|\bigvee_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}(\psi)\right.\\
&\left.+\sup_{\mathfrak{s}\in[\frac{\varsigma+\vartheta}{2},\vartheta]}\left|\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)+\frac{(\vartheta-\varsigma)}{6}\right|\bigvee_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}(\psi)\right]\\
&+\frac{2^{1-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{1-\rho}}\left[\sup_{\mathfrak{s}\in[\varsigma,\frac{\varsigma+\vartheta}{2}]}\left|\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right|\bigvee_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}(\psi')\right.\\
&\left.+\sup_{\mathfrak{s}\in[\frac{\varsigma+\vartheta}{2},\vartheta]}\left|\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right|\bigvee_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}(\psi')\right]\\
&=\frac{2^{2-2\rho}\rho^2}{(\vartheta-\varsigma)^{1-\rho}}\left[\frac{2(\vartheta-\varsigma)}{3}\bigvee_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}(\psi)+\frac{2(\vartheta-\varsigma)}{3}\bigvee_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}(\psi)\right]\\
&+\frac{2^{1-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{1-\rho}}\left[\frac{4(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\bigvee_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}(\psi')+\frac{4(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\bigvee_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}(\psi')\right]\\
&=\frac{2^{3-2\rho}\rho^2}{3(\vartheta-\varsigma)^{-\rho}}\bigvee_{\varsigma}^{\vartheta}(\psi)+\frac{2(1-\rho)}{3(\vartheta-\varsigma)^{-1+\rho}}\bigvee_{\varsigma}^{\vartheta}(\psi').
\end{aligned}
$$

□

Remark 6. In the specific case when ρ goes to 1 in Theorem 5, we have

$$
\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{2}{3}\bigvee_{\varsigma}^{\vartheta}(\psi),
$$

which is identical to Theorem 6 offered by Budak et al. [11] under the assumption that $\rho=1$.

Corollary 6. *In the particular circumstance where* ρ *approaches* 0 *in Theorem 5, we obtain*

$$\left|\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2\left(\psi(\vartheta)-\psi(\varsigma)\right)}{\vartheta-\varsigma}\right.$$
$$\left.-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)\mathrm{d}\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right)\right|\leq\frac{2}{3}\bigvee_{\varsigma}^{\vartheta}(\psi').$$

5. Conclusion

This study aims to establish novel Milne-type integral inequalities through the utilization of proportional Caputo hybrid operator for twice-differentiable convex mappings. In order to achieve this goal, we begin by proving a new integral identity of the Milne-type associated with proportional Caputo-hybrid operator. Then, we establish a fractional Milne-type inequality for twice-differentiable convex mappings. Moreover, we obtain some Milne-type inequalities for bounded mappings and mappings of bounded variation. The findings of this study are more applicable than in the traditional calculus since previously determined bounds are the special case of our findings for $\rho\to 1$ and our findings hold true across the entire interval given for ρ. With the proportional Caputo-hybrid operator, this is the first paper dedicated to Milne-type inequalities. Therefore, we hope that readers will be motivated to look into this subject further by our approaches and findings. In future work, one can investigate analogous inequalities for different fractional integrals and derive novel Milne-type inequalities by employing various forms of convexity.

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