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AMPLY ESSENTIAL SUPPLEMENTED LATTICES

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Abstract. In this work, amply essential supplemented (briefly, amply e-supplemented) lattices are defined and some properties of these lattices are investigated. All lattices are complete modular lattices with the greatest element 1 and the smallest element 0 in this work. Let L be a lattice. If $b/0$ is essential supplemented for every $b \in L$, then L is amply essential supplemented. Let L be an amply e-supplemented lattice, b be a supplement of a in L and $b \trianglelefteq L$. Then $b/0$ is amply e-supplemented. Here also $x/0$ is amply supplemented for every e-supplement element x in L. Let L be a lattice. Then L is amply essential supplemented if and only if $a \wedge b$ has a supplement in $b/0$ for every $a \trianglelefteq L$ and $b \in L$ with $1 = a \vee b$. Let L be a lattice. If $a \wedge b$ lies above an e-supplement element in L for every $a, b \in L$ with $a \trianglelefteq L$ and $1 = a \vee b$, then L is amply e-supplemented.

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1. INTRODUCTION

In this paper, every lattice is a complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x, y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \leq a \leq y\}$ is called a *quotient* sublattice of L and denoted by y/x. An element a of a lattice L is called a *complement* of b in L if $a \vee b = 1$ and $a \wedge b = 0$, in this case we denote $1 = a \oplus b$ (here we also call a and b are *direct summands* of L). L is said to be *complemented* if each element of L has at least one complement in L. An element x of L is said to be *small* or *superfluous* in L and denoted by $x \ll L$ if $y = 1$ for every $y \in L$ such that $x \vee y = 1$. Let L be a lattice and $k \in L$. If $t = 0$ for every $t \in L$ with $k \wedge t = 0$, then k is called an *essential* element of L and denoted by $k \trianglelefteq L$. The meet of all maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. If $a \ll L$, then $a \leq r(L)$ holds. An element a of L is called a *supplement* of b in L if it is minimal for $1 = b \vee a$. a is a supplement of b in a lattice L if and only if $1 = b \vee a$ and $b \wedge a \ll a/0$. A lattice L is said to be *supplemented* if every element of L has a supplement in L. If every essential element of L has a supplement in L, then L

is called an *essential supplemented* (briefly, *e-supplemented*) lattice. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$. An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. A lattice L is said to be *weakly supplemented*, if every element of L has a weak supplement in L. Let L be a lattice. L is said to be *hollow* if every element distinct from 1 is superfluous in L. If every essential element of L with distinct from 1 is small in L or L have no essential elements with distinct from 1, then L is called an *e-hollow* lattice. L is said to be *local* if L has the greatest element $(\neq 1)$. We say an element $x \in L$ has *ample supplements* in L if for every $y \in L$ with $x \vee y = 1$, x has a supplement z in L with $z \leq y$. L is said to be *amply supplemented* if every element of L has ample supplements in L. Let L be a lattice. It is defined β_* relation on the elements of L by $a\beta_*b$ with $a,b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

More information about (amply) supplemented lattices are in [1,2,5,8]. More results about (amply) supplemented modules are in [7,13–15]. More information about essential supplemented lattices are in [12]. More information about (amply) essential supplemented modules are in [10, 11]. The definition of β_* relation on lattices and some properties of this relation are in [9]. This relation is a generalization of β^* relation on modules. The definition of β^* relation on modules and some properties of this relation are in [4].

Lemma 1. *Let L be a lattice and $a,b,c \in L$ with $a \leq b$. If c is a supplement of b in L, then $a \vee c$ is a supplement of b in $1/a$.*

Proof. See [3, Lemma 1]. □

Lemma 2. *Let a be a supplement of b in L and $x,y \in a/0$. Then y is a supplement of x in $a/0$ if and only if y is a supplement of $b \vee x$ in L.*

Proof. See [8, Lemma 11]. □

Lemma 3. *Let L be a lattice and $a,b,c \in L$. If $1 = a \vee b$ and $c \leq b$, then c is a supplement of a in L if and only if c is a supplement of $a \wedge b$ in $b/0$.*

Proof. ($\Longrightarrow$) Since c is a supplement of a in L, $1 = a \vee c$ and $a \wedge c \ll c/0$. Since $1 = a \vee c$ and $c \leq b$, by modularity, $b = b \wedge 1 = b \wedge (a \vee c) = (a \wedge b) \vee c$. On the other hand, $a \wedge b \wedge c = a \wedge c \ll c/0$. Hence c is a supplement of $a \wedge b$ in $b/0$.

($\Longleftarrow$) Since c is a supplement of $a \wedge b$ in $b/0$, $b = (a \wedge b) \vee c$ and $a \wedge c = a \wedge b \wedge c \ll c/0$. Since $1 = a \vee b$, $1 = a \vee b = a \vee (a \wedge b) \vee c = a \vee c$. On the other hand, $a \wedge c \ll c/0$. Hence c is a supplement of a in L. □

Corollary 1. *Let L be a lattice and $a \in L$. Then a has ample supplements in L if and only if $a \wedge b$ has a supplement in $b/0$ for every $b \in L$ with $1 = a \vee b$.*

Proof. ($\Longrightarrow$) Let $1 = a \vee b$ with $b \in L$. Since a has ample supplements in L, then a has a supplement c in L with $c \leq b$. By Lemma 3, c is a supplement of $a \wedge b$ in $b/0$. Hence $a \wedge b$ has a supplement in $b/0$.

($\Longleftarrow$) Let $1 = a \vee b$ with $b \in L$. By hypothesis, $a \wedge b$ has a supplement c in $b/0$. By Lemma 3, c is a supplement of a in L. Moreover, $c \leq b$. Hence a has ample supplements in L. □

Corollary 2. *Let L be a lattice and $a \in L$. If $b/0$ is supplemented for every $b \in L$ with $1 = a \vee b$, then a has ample supplements in L.*

Proof. Clear from Corollary 1. □

Proposition 1. *Let L be a lattice. Then L is amply supplemented if and only if $a \wedge b$ has a supplement in $b/0$ for every $a, b \in L$ with $1 = a \vee b$.*

Proof. Clear from Corollary 1. □

Corollary 3. *Let L be a lattice. If $b/0$ is supplemented for every $b \in L$, then L is amply supplemented.*

Proof. Clear from Proposition 1. □

Corollary 4. *Let L be a lattice. If $a \wedge b$ is a direct summand of $b/0$ for every $a, b \in L$ with $1 = a \vee b$, then L is amply supplemented.*

Proof. Clear from Proposition 1. □

Lemma 4. *Let b be a supplement of a in L. If $a \vee x$ has ample supplements in L for every $x \in b/0$, then $b/0$ is amply supplemented.*

Proof. Let $x \in b/0$ and $b = x \vee y$ with $y \in b/0$. Then $1 = a \vee b = a \vee x \vee y$. By hypothesis, $a \vee x$ has ample supplements in L. Then $a \vee x$ has a supplement z in L with $z \leq y$. Since b is a supplement of a in L and $x, z \in b/0$, by Lemma 2, z is a supplement of x in $b/0$. Moreover, $z \leq y$. Hence $b/0$ is amply supplemented. □

Corollary 5. *Let b be a supplement of a in L. If $a \vee x$ has ample supplements in L for every $x \in L$, then $b/0$ is amply supplemented.*

Proof. Clear from Lemma 4. □

Corollary 6. *Let b be a supplement of a in L. If x has ample supplements in L for every $x \in 1/a$, then $b/0$ is amply supplemented.*

Proof. Clear from Corollary 5. □

2. Amply essential supplemented lattices

Definition 1. Let L be a lattice. If every essential element of L has ample supplements in L, then L is called an amply essential supplemented (briefly, amply e-supplemented) lattice. (See also [6]).

Proposition 2. *Every amply e-supplemented lattice is e-supplemented.*

Proof. Let L be an amply e-supplemented lattice and $a \trianglelefteq L$. Since L is amply e-supplemented, a has ample supplements in L. Since $1 = a \vee 1$, a has a supplement x in L with $x \leq 1$. Hence L is e-supplemented. $\square$

Corollary 7. *Let L be an amply essential supplemented lattice. Then $1/r(L)$ does not have any essential element with distinct from 1.*

Proof. Since L is amply essential supplemented, by Proposition 2, L is essential supplemented. Then by [12, Proposition 4], $1/r(L)$ does not have any essential element with distinct from 1. $\square$

Proposition 3. *Let L be an amply e-supplemented lattice. If every element of L with distinct from 0 is essential in L, then L is amply supplemented.*

Proof. Clear from definitions. $\square$

Proposition 4. *Every e-hollow lattice is amply e-supplemented.*

Proof. Let L be an e-hollow lattice, $a \trianglelefteq L$ and $1 = a \vee b$ with $b \in L$. If $a = 1$, then 0 is a supplement of a in L. Moreover, $0 \leq b$. Let $a \neq 1$. Since L is e-hollow, $a \ll L$ and $b = 1$. Here 1 is a supplement of a in L with $1 \leq b$. Hence a has ample supplements in L and L is amply e-supplemented. $\square$

Proposition 5. *Let L be a finite lattice. Then L is amply e-supplemented.*

Proof. Clear from definitions. $\square$

Proposition 6. *Let L be a lattice. If $b/0$ is essential supplemented for every $b \in L$, then L is amply essential supplemented.*

Proof. Let $a \trianglelefteq L$ and $1 = a \vee b$ with $b \in L$. Since $a \trianglelefteq L$, $a \wedge b \trianglelefteq b/0$. Since $b/0$ is essential supplemented, $a \wedge b$ has a supplement c in $b/0$. By Lemma 3, c is a supplement of a in L. Moreover, $c \leq b$. Hence L is amply essential supplemented. $\square$

Proposition 7. *Let L be an amply e-supplemented lattice, b be a supplement of a in L and $b \trianglelefteq L$. Then $b/0$ is amply e-supplemented.*

Proof. Let $x \trianglelefteq b/0$ and $b = x \vee y$ with $y \in b/0$. Then $1 = a \vee b = a \vee x \vee y$. Since $x \trianglelefteq b/0$ and $b \trianglelefteq L$, $x \trianglelefteq L$. Then $a \vee x \trianglelefteq L$ and since L is amply e-supplemented, $a \vee x$ has a supplement z in L with $z \leq y$. By Lemma 2, z is a supplement of x in $b/0$. Hence $b/0$ is amply e-supplemented. $\square$

Lemma 5. *Let L be an amply e-supplemented lattice and $x \in L$. Then $1/x$ is also amply e-supplemented.*

Proof. Let a be any essential element of $1/x$. Then $a \trianglelefteq L$ and since L is amply e-supplemented, a has ample supplements in L. Let $1 = a \vee b$ with $b \in 1/x$. Since $1 = a \vee b$, a has a supplement y in L with $y \leq b$. By Lemma 1, $x \vee y$ is a supplement of a in $1/x$. Moreover, $x \vee y \leq b$. Hence a has ample supplements in $1/a$ and $1/a$ is amply e-supplemented. $\square$

Corollary 8. *Let L be an amply e-supplemented lattice. Then $x/0$ is amply e-supplemented for every direct summand x of L.*

Proof. Since x is a direct summand of L, there exists $y \in L$ such that $1 = x \oplus y$. By Lemma 5, $1/y$ is amply e-supplemented. Then by $\frac{1}{y} = \frac{x \vee y}{y} \cong \frac{x}{x \wedge y} = \frac{x}{0}$, $x/0$ is also amply e-supplemented. □

Proposition 8. *Let L be a lattice. Then L is amply essential supplemented if and only if $a \wedge b$ has a supplement in $b/0$ for every $a \trianglelefteq L$ and $b \in L$ with $1 = a \vee b$.*

Proof. Clear from Corollary 1. □

Proposition 9. *Let b be a supplement of a in L. If $a \vee x$ has ample supplements in L for every $x \trianglelefteq b/0$, then $b/0$ is amply e-supplemented.*

Proof. Similar to proof of Lemma 4. □

Definition 2. Let L be a lattice and $a \in L$. If a is a supplement of an essential element in L, then a is called an e-supplement element in L. (See also [12]).

Lemma 6. *Let L be an amply e-supplemented lattice. Then $x/0$ is amply supplemented for every e-supplement element x in L.*

Proof. Since x is an e-supplement element in L, there exists an element $a \trianglelefteq L$ such that x is a supplement of a in L. Let $y \in x/0$ and $x = y \vee z$ with $z \in x/0$. Then $1 = a \vee x = a \vee y \vee z$. Since $a \trianglelefteq L$, $a \vee y \trianglelefteq L$ and since L is amply e-supplemented, $a \vee y$ has a supplement t in L with $t \leq z$. Since x is a supplement of a in L and $y, t \in x/0$, by Lemma 2, t is a supplement of y in $x/0$. Moreover, $t \leq z$. Hence $x/0$ is amply supplemented. □

Corollary 9. *Let L be an amply e-supplemented lattice. Then $x/0$ is amply e-supplemented for every e-supplement element x in L.*

Proof. Clear from Lemma 6. □

Proposition 10. *Let L be a lattice. If $a \wedge b$ lies above an e-supplement element in L for every $a, b \in L$ with $a \trianglelefteq L$ and $1 = a \vee b$, then L is amply e-supplemented.*

Proof. Let $a \trianglelefteq L$ and $1 = a \vee b$ with $b \in L$. By hypothesis, $a \wedge b$ lies above an e-supplement element x in L. Since x is an e-supplement element in L, there exists an element $y \trianglelefteq L$ such that x is a supplement of y in L. Here $1 = x \vee y$ and since $x \leq a \wedge b \leq b$, $1 = x \vee y = b \vee y$. By hypothesis, $b \wedge y$ lies above an e-supplement element c in L. Since $x \leq a \wedge b$, $1 = x \vee y = (a \wedge b) \vee y$ and by modularity, $b = b \wedge 1 = b \wedge ((a \wedge b) \vee y) = (a \wedge b) \vee (b \wedge y)$ and $1 = a \vee b = a \vee (a \wedge b) \vee (b \wedge y) = a \vee (b \wedge y)$. Since $b \wedge y$ lies above c in L, by [8, Lemma 5], $(b \wedge y)\, \beta_* c$ in L and since $1 = a \vee (b \wedge y)$, $1 = a \vee c$. Since x is a supplement of y in L, y is a weak supplement of x in L and since $a \wedge b$ lies above x in L, by [8, Lemma 15], y is a weak supplement of $a \wedge b$ in L. Here $a \wedge b \wedge y \ll L$ and since $c \leq b \wedge y$, $a \wedge c \ll L$. Since $a \wedge c \ll L$ and c is a supplement element in L, by [8, Lemma 10], $a \wedge c \ll c/0$. Hence c is a supplement of a in L. Moreover, $c \leq b$. Hence L is amply e-supplemented. $\square$

The converse of Proposition 4 is not true in general.

Example 1. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & & \nwarrow & \\ b & & & & c \\ & \nwarrow & & \nearrow & \\ & & a & & \\ & & \uparrow & & \\ & & 0 & & \end{array}$$

Then L is amply e-supplemented but not e-hollow.

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