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NEW ASYMPTOTICALLY ISOMETRIC PROPERTIES THAT IMPLY THE FAILURE OF THE FIXED POINT PROPERTY IN COPIES OF ℓ^1

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Abstract. In this study, we introduce three new notions which may occur for some Banach spaces. We call these new properties AAI1, AAI2 and AAI3 where AAI stands for "alternative asymptotically isometric". We prove that if a Banach space has any of them, then it fails to have the fixed point property for nonexpansive mappings. We provide alternative ways of detecting if a Banach space has any of these properties. We show that AAI1 is an equivalent property for a Banach space to have an asymptotically isometric copy of ℓ^1. That is, a Banach space contains an asymptotically isometric copy of ℓ^1 if and only if it has the property AAI1. In fact, we obtain generalized version of property AAI1, which we call property AAI3, obtained as a conclusion of property AAI2. We prove that all properties AAI1, AAI2 and AAI3 are equivalent. We also support our ideas with some examples and remarks for our equivalent properties to contain asymptotically isometric copy of ℓ^1.

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1. PRELIMINARIES AND INTRODUCTION

A Banach space is said to have the fixed point property for nonexpansive mappings [fpp(ne)] if every self-map of any closed, bounded and convex domain in that space satisfying the condition of nonexpansiveness has a fixed point (see for example page 37 in [8]). Here, by nonexpansive map we mean a map $T : C \to C$ such that $\|Tx - Ty\| \leq \|x-y\|$ for every $x, y \in C$.

It is a fact that either c_0 or ℓ^1 is almost isometrically embedded in any nonreflexive Banach spaces with an unconditional basis (see e.g. [9]). Thus, all of the classical nonreflexive spaces fail fpp(ne); that is, there exists a closed, bounded and convex subset and a nonexpansive self-map T defined on that set such that T is fixed point free. This result depends on well-known facts (Theorems 1.c.12 in [12] and 1.c.5 in

[13]) stated by the following: a Banach lattice or a Banach space with an unconditional basis is reflexive if and only if it contains no isomorphic copies of c_0 or ℓ^1. Hence, if it can be shown that neither c_0 nor ℓ^1 can be renormed to have fpp(ne), it would follow that fpp(ne) in either a Banach lattice or in a Banach space with an unconditional basis would imply reflexivity.

In this matter, for many years, researchers have asked the question whether or not either ℓ^1 or c_0 can be equivalently renormed so that the renormed space has fpp(ne). For ℓ^1, there is a fact suggesting the contrary by Lin [11].

In [9], James showed that if a Banach space contains an isomorphic copy of ℓ^1 (respectively, c_0), then it contains almost isometric copies of ℓ^1 (respectively, c_0) and then he provided a tool that helped researches investigate the question of whether ℓ^1 or c_0 can be renormed to have fpp(ne). Using this tool and strengthening that, Dowling, Lennard and Turett, in several articles, have inquired relations between spaces containing nice copies of c_0 or ℓ^1 and the failure of fpp(ne). After strengthening James Distortion Theorems, they investigated these structures and then worked on the notion of a Banach space containing an asymptotically isometric (ai) copy of ℓ^1 or c_0 which are used as important tools in identifying failure of fpp(ne) for Banach spaces. It can be said that at first, their aim of introducing the notion of Banach spaces containing ai copies of ℓ^1 was to prove the fact that nonreflexive subspaces of $L^1[0,1]$ fail fpp(ne). This result can be seen in [3] by Dowling and Lennard.

Later, Dowling, Lennard and Turett [6] showed that if a Banach space contains an ai c_0 copy, then it fails fpp(ne). They also showed in [5] that if a Banach space contains an ai copy of c_0 then its isometric dual contains an ai copy of ℓ^1. Furthermore, in [6], they prove that there exists a Banach space failing fpp(ne) whereas it does not contain any ai copy of c_0. In fact, in 2010 [7], they show that any infinite dimensional subspace of their example of asymptotically isometric c_0 free space fails fpp(ne). Hence, one can inquire of existence of a property which may strengthen the notion of ai copy of c_0 so that it can help see failure of fpp(ne) in the case of lack of ai c_0 copy.

For example, in order to obtain alternative methods for detecting failure of fpp(ne), in 2011, Lennard and Nezir [10] proved that if a Banach space contains an ai c_0-summing basic sequence, it fails fpp(ne). In fact, they showed that closed convex hull of any ai c_0-summing basic sequence fails fpp(ne). So there was another tool of detecting spaces failing fpp(ne). Moreover, recently, a nice property was developed by Álvaro, Cembranos and Mendoza [1] to detect failure of fpp(ne). They introduced a new property, N1, that implies an isomorphic copy of c_0 fails fpp(ne) such that their property is more general than the property of Banach spaces to contain ai copies of c_0.

Similarly to their works for Banach spaces containing ai copies of c_0, starting in 1996, Dowling, Lennard and Turett explored Banach spaces containing ai copies of ℓ^1 and they obtained important results leading researchers to test failure of fpp(ne) in

Banach spaces they work on. For example, it can be said that Lin was inspired by the norm developed by Dowling, Johnson, Lennard and Turett [2] when they showed ℓ^1 with that norm does not contain any ai copy of ℓ^1 whereas Lin later proved in [11] that ℓ^1 considered with a special case of the norm has fpp(ne). As there are interests to search alternative properties for detecting asymptotically isometric copy of c_0, it must be significant to find better or even equivalent property for a Banach space to contain an ai copy of ℓ^1.

Hence, aiming to find different tools to detect failure of fpp(ne) in Banach spaces, in this study, we introduce new notions which may occur for some Banach spaces. We call these new properties AAI1, AAI2 and AAI3 where AAI stands for "alternative asymptotically isometric". We prove that if a Banach space has any of them, then it fails to have the fixed point property for nonexpansive mappings [fpp(ne)]. We provide alternative ways of detecting if a Banach space has any of these properties. We show that AAI1 is an equivalent property for a Banach space to have an ai copy of ℓ^1. That is, a Banach space contains an ai copy of ℓ^1 if and only if it has the property AAI1. In fact, we obtain generalized version of property AAI1, which we call property AAI3, obtained as a conclusion of property AAI2. We prove that all properties AAI1, AAI2 and AAI3 are equivalent. Moreover, we support our ideas with some examples and remarks for our equivalent properties to contain ai copy of ℓ^1. For example, we demonstrate an example of a Banach space satisfying property AAI1 where we show that the sequence allowing us to have the property is also an ai ℓ^1 sequence. Furthermore, as our final example, which is an obvious remark due to our result, we give an example of a Banach space failing AAI1 property.

We need to note that in [14], Nezir introduced another alternative asymptotically isometric property inside copies of ℓ^1 but our properties AAI1, AAI2 and AAI3 are still more general than that.

Now, we give some preliminaries required for our work.

Throughout the paper our scalar field is $\mathbb{R}$, c_0 represents the Banach space of scalar sequences converging to 0, ℓ^1 is for the Banach space of absolutely summable sequences, and c_{00} is for the space of sequences of finitely nonzero terms. Furthermore, in the paper, we will be using the unconditional basis $(e_n)_{n\in\mathbb{N}}$ for both $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$ such that it has 1 in its n^{th} coordinate, and 0 in its other coordinates for each $n \in \mathbb{N}$.

Firstly, we will remind the definition of an ai ℓ^1 sequence and we will give the theorem providing alternative way of detecting if there is any as in [4] by Dowling, Lennard and Turett.

Definition 1. [4] We call a Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sum_{n=1}^{\infty}|a_n| \text{ , for all } (a_n)_n \in \ell^1.$$

Theorem 1. *[4] A Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1 if and only if there is a sequence $(x_n)_n$ in X and there are constants $0 < m < M < \infty$ such that for all $(t_n)_n \in \ell^1$,*

$$m \sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n|$$

and $\lim_{n \to \infty} \|x_n\| = m$.

Then, the following theorem is given as a result in [3].

Theorem 2. *If a Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1, then X fails fpp(ne).*

2. Alternative Asymptotically Isometric Properties

In this section, we will introduce our AAI properties that will provide alternatives for well-known notions of ai copy of ℓ^1.

Definition 2. We will say that a Banach space $(X, \|\cdot\|)$ has property AAI1 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that

$$\sqrt{\left(\sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n| \right)^2 + \left(\sum_{n=1}^{\infty} (1-\varepsilon_n) \frac{|t_n|}{2^n} \right)^2} \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sqrt{\left(\sum_{n=1}^{\infty} |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \right)^2},$$

for all $(t_n)_n \in \ell^1$.

Definition 3. We will say that a Banach space $(X, \|\cdot\|)$ has property AAI2 if there exist a sequence $(x_n)_n$ in X, a null sequence $(\varepsilon_n)_n$ in $(0,1)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that

$$\sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n| + \rho\left(((1-\varepsilon_n)t_n)_n\right) \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right),$$

for all $(t_n)_n \in \ell^1$.

Definition 4. We will say that a Banach space $(X, \|\cdot\|)$ has property AAI3 if there exist a sequence $(x_n)_n$ in X, a null sequence $(\varepsilon_n)_n$ in $(0,1)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in$

c_0 so that

$$\sqrt{\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|\right)^2+[\rho\left(((1-\varepsilon_n)t_n)_n\right)]^2} \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq \sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+[\rho\left((t_n)_n\right)]^2}\ ,$$

for all $(t_n)_n \in \ell^1$.

In this section, we explore alternative properties to the notion of ai ℓ^1 copy. We will see that our properties will imply the failure of fpp(ne) in copies of ℓ^1. We prove that each of AAI1, AAI2 and AAI3 is equivalent property to have ai copy of ℓ^1. Then, in the final subsection, we give an example of a Banach space containing an ai ℓ^1 sequence such that the same sequence satisfies the condition of property AAI1. Furthermore, we show an example of a Banach space without AAI1 property or any ai ℓ^1 sequence by a quick remark since we know that a Banach space's failure of having property AAI1 is equivalent to failure of containing an ai copy of ℓ^1.

2.1. *Property AAI1*

In this section, we explore property AAI1 and show that property AAI1 implies having ai copy of ℓ^1 and the converse is also true: that is, both properties imply each other.

First, we give an alternative way of detecting our property AAI1 which will help us prove that a Banach space contains an ai copy of ℓ^1 if and only if it has property AAI1.

Theorem 3. *A Banach space $(X, \|\cdot\|)$ has property AAI1 if and only if there is a sequence $(x_n)_n$ in X such that there exists $R \in [1,\infty)$ so that for any $(t_n)_n \in \ell^1$,*

$$\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2} \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq R\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2} \tag{2.1}$$

and

$$\lim_{n\to\infty}\|x_n\| = 1. \tag{2.2}$$

Proof. Suppose that $(X,\|\cdot\|)$ has property AAI1. Then, there exist a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$\sqrt{\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|\right)^2+\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)\frac{|t_n|}{2^n}\right)^2} \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq \sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}, \tag{2.3}$$

for all $(t_n)_n \in \ell^1$.

We may assume $(\varepsilon_n)_{n\in\mathbb{N}}$ to be a decreasing sequence since we may replace that with $\zeta_n := \max\limits_{j\geq n} \varepsilon_j$, for all $n \in \mathbb{N}$. Let $y_n = (1-\varepsilon_n)^{-1}x_n$ for each $n \in \mathbb{N}$. Then, for all $(t_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2} \leq \left\|\sum_{n=1}^{\infty} t_n y_n\right\| \leq \sqrt{\left(\sum_{n=1}^{\infty}\frac{|t_n|}{1-\varepsilon_n}\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{(1-\varepsilon_n)2^n}\right)^2}.$$

Let $R = \frac{1}{1-\varepsilon_1}$. Then, condition (2.1) is satisfied for the sequence $(y_n)_n$ in X. Also, it is easy to check (2.2) for the sequence $(y_n)_n$ too since in inequality (2.3), taking $(t_n)_n$ as the unit basis $(e_n)_n$ of c_0 we obtain that $\lim\limits_{n\to\infty}\|x_n\| = 1$ and so $\lim\limits_{n\to\infty}\|y_n\| = 1$.

Conversely, assume that there exist a sequence $(x_n)_n$ in X and $R \in [1,\infty)$ so that for all $(t_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2} \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq R\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2} \tag{2.4}$$

and $\lim\limits_{n\to\infty}\|x_n\| = 1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0,1)$. Since $\lim\limits_{n\to\infty}\|x_n\| = 1$, and $\|x_n\| \geq 1$ for all $n\in\mathbb{N}$, by passing to subsequences, if necessary, we can assume that $1 \leq \|x_n\| \leq 1+\varepsilon_n$ for all $n \in \mathbb{N}$. Define $y_n = \frac{x_n}{1+\varepsilon_n}$ for all $n \in \mathbb{N}$. Then, since $\|y_n\| \leq 1$, we have

$$\left\|\sum_{n=1}^{\infty} t_n y_n\right\| \leq \sum_{n=1}^{\infty}|t_n| \leq \sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}, \text{ for all } (t_n)_n \in \ell^1.$$

Also, from the left hand side inequality of (2.4), we have

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} t_n y_n\right\| &= \left\|\sum_{n=1}^{\infty} t_n \frac{x_n}{(1+\varepsilon_n)}\right\| \geq \sqrt{\left(\sum_{n=1}^{\infty} \frac{|t_n|}{1+\varepsilon_n}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{(1+\varepsilon_n)2^n}\right)^2} \\ &\geq \sqrt{\left(\sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n|\right)^2 + \left(\sum_{n=1}^{\infty} (1-\varepsilon_n)\frac{|t_n|}{2^n}\right)^2}.\end{aligned}$$

□

Now, we show property AAI1 is equivalent for a Banach space to contain an ai copy of ℓ^1.

Theorem 4. *Let $(X, \|.\|)$ be a Banach space. Then, X contains an ai copy of ℓ^1 if and only if X has property AAI1.*

Proof. Assume that a Banach space X contains an ai copy of ℓ^1. Then, by Theorem 1, there exist a sequence $(x_n)_n$ in X with $\lim_{n\to\infty} \|x_n\| = 1$ and $M \in [1, \infty)$ such that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq M \sum_{n=1}^{\infty} |t_n|.$$

Then, letting $y_n = \left(1 + \frac{1}{2^n}\right) x_n$ for each $n \in \mathbb{N}$ and $R = \frac{3M}{2}$, we have

$$\begin{aligned}\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} &\leq \sum_{n=1}^{\infty} |t_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \\ &\leq \left\|\sum_{n=1}^{\infty} t_n y_n\right\| \leq M \left(\sum_{n=1}^{\infty} |t_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} &\leq \left\|\sum_{n=1}^{\infty} t_n y_n\right\| \leq R \sum_{n=1}^{\infty} |t_n| \\ &\leq R \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2}\end{aligned}$$

and $\lim_n \|y_n\| = 1$.

Thus, by the previous theorem, X has property AAI1.

Conversely assume that X has property AAI1. Then, there exist a sequence $(x_n)_n$ in X satisfying $\lim_{n\to\infty} \|x_n\| = 1$ and a constant $R \in [1,\infty)$ so that for any $(t_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le R\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2}$$

and so

$$\sum_{n=1}^{\infty} |t_n| \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le \frac{3R}{2}\sum_{n=1}^{\infty} |t_n| \,.$$

Hence, we are done by Theorem 1. □

2.2. *A more generalized property, property AAI2*

Firstly, we give an alternative way of detecting our property AAI2 which will help us prove that a Banach space contains an ai copy of ℓ^1 if and only if it has property AAI2.

Theorem 5. *A Banach space $(X, \|\cdot\|)$ has property AAI2 if and only if there exist a sequence $(x_n)_n$ in X, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ such that there exists $R \in [1,\infty)$ so that for any $(t_n)_n \in \ell^1$,*

$$\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le R\left[\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right)\right] \tag{2.5}$$

and

$$\lim_{n\to\infty} \|x_n\| = 1. \tag{2.6}$$

Proof. Suppose that $(X, \|\cdot\|)$ has property AAI2. Then, there exist a sequence $(x_n)_n$ in X, a decreasing null sequence $(\varepsilon_n)_n$ in $(0,1)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that

$$\sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n| + \rho\left(((1-\varepsilon_n)t_n)_n\right) \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le \sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right), \tag{2.7}$$

for all $(t_n)_n \in \ell^1$.

Let $y_n = (1-\varepsilon_n)^{-1} x_n$ for each $n \in \mathbb{N}$. Then, for all $(t_n)_n \in \ell^1$, taking ρ being a lattice norm into consideration,

$$\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \le \left\|\sum_{n=1}^{\infty} t_n y_n\right\| \le \sum_{n=1}^{\infty} \frac{|t_n|}{1-\varepsilon_n} + \rho\left(\left(\frac{|t_n|}{1-\varepsilon_n}\right)_n\right) .$$

Let $R = \frac{1}{1-\varepsilon_1}$. Then, condition (2.5) is satisfied for the sequence $(y_n)_n$ in X. Also, it is easy to check (2.6) for the sequence $(y_n)_n$ too since in inequality (2.7), taking $(t_n)_n$ as the unit basis $(e_n)_n$ of c_0 and recalling ρ is equivalent to $|||.|||$, we obtain that $\lim_{n\to\infty} \|x_n\| = 1$ and so $\lim_{n\to\infty} \|y_n\| = 1$.

Conversely, assume that there exist $R \in [1,\infty)$, a sequence $(x_n)_n$ in X, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \le \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \le R\left[\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right)\right] \tag{2.8}$$

and $\lim_{n\to\infty} \|x_n\| = 1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0,1)$. Since $\lim_{n\to\infty} \|x_n\| = 1$, and $\|x_n\| \ge 1$ for all $n \in \mathbb{N}$, by passing to subsequences, if necessary, we can assume that $1 \le \|x_n\| \le 1+\varepsilon_n$ for all $n \in \mathbb{N}$. Define $y_n = \frac{x_n}{1+\varepsilon_n}$ for all $n \in \mathbb{N}$. Then, since $\|y_n\| \le 1$, we have

$$\left\| \sum_{n=1}^{\infty} t_n y_n \right\| \le \sum_{n=1}^{\infty} |t_n| \le \sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \quad \text{for all } (t_n)_n \in \ell^1 .$$

Also from the left hand side inequality of (2.8) and recalling ρ is a lattice norm, we have

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} t_n y_n \right\| &= \left\| \sum_{n=1}^{\infty} t_n \frac{x_n}{1+\varepsilon_n} \right\| \ge \sum_{n=1}^{\infty} \frac{|t_n|}{1+\varepsilon_n} + \rho\left(\left(\frac{|t_n|}{1+\varepsilon_n}\right)_n\right) \\ &\ge \sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n| + \rho\left(((1-\varepsilon_n)t_n)_n\right). \end{aligned}$$

$\square$

Theorem 6. *Let $(X, \|.\|)$ be a Banach space. Then, X has property AAI2 if and only if X contains an ai copy of ℓ^1*

Proof. Let $(X, \|\cdot\|)$ be a Banach space and assume that X has property AAI2. So there exist a sequence $(x_n)_n$ in X satisfying $\lim_{n\to\infty} \|x_n\| = 1$, a constant $R \in [1,\infty)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that for any $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \le \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \le R\left[\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right)\right] .$$

Then, there exist constants $0 < A \leq B < \infty$ such that for any $x = (\xi_n)_n \in c_0$, $A\rho(x) \leq \sum\limits_{k=1}^{\infty} |\rho_k \xi_k| \leq B\rho(x)$. Let $M = R\left[1 + \sup\limits_n \frac{|\rho_n|}{A}\right]$. Then,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n| \ .$$

So $(x_n)_n$ is an ai ℓ^1 sequence in X by Theorem 1.

Conversely assume that X contains an ai copy of ℓ^1. Then, by Theorem 1, there exist a sequence $(x_n)_n$ in X with $\lim\limits_n \|x_n\| = 1$ and $M \in [1, \infty)$ such that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n| \ .$$

Now, fix a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and fix an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum\limits_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$. For example, we could consider any equivalent norm to $|||.|||$ given by $|||x||| = \sum\limits_{k=1}^{\infty} \frac{|\xi_k|}{2^k}$ for any $x = (\xi_k)_k \in c_0$ but we may work on an arbitrary one. Then, letting $y_n = \left(1 + \frac{|\rho_n|}{A}\right) x_n$ for each $n \in \mathbb{N}$ and $R = \frac{BM}{A}$, we have

$$\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq R\left[\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right)\right] \ .$$

Hence, by the previous theorem, X has property AAI2. □

2.3. *Property AAI3; generalizing AAI1*

Using the ideas of the previous section, we may get a generalized version of property AAI1, which we call property AAI3. Indeed, one can easily obtain the following conclusions.

Corollary 1. *A Banach space $(X, \|\cdot\|)$ has property AAI3 if and only if there exist a sequence $(x_n)_n$ in X, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum\limits_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ such that there exists $R \in [1, \infty)$ so that for any $(t_n)_n \in \ell^1$,*

$$\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left[\rho\left((t_n)_n\right)\right]^2} \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq R \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left[\rho\left((t_n)_n\right)\right]^2}$$

and $\lim\limits_{n \to \infty} \|x_n\| = 1$.

Corollary 2. *Let $(X, \|.\|)$ be a Banach space. Then, X has property AAI3 if and only if X contains an ai copy of ℓ^1*

2.4. *Some Remarks and Examples for Alternatives to ai ℓ^1 Copy*

Example 1. We consider a renorming of ℓ^1 as follows. Consider $(\ell^1, \|.\|)$ with the norm defined by the following way: for every $x = (\xi_n)_n \in \ell^1$,

$$\|x\| = \sup_n \frac{2^n}{1+2^n} \sum_{k=n}^{\infty} |\xi_k| + \sum_{n=1}^{\infty} \frac{|\xi_n|}{2^n} + \frac{1}{3} \sum_{n=1}^{\infty} |\xi_n| \ .$$

Then, we can see that $(\ell^1, \|.\|)$ contains a sequence that satisfies both conditions of property AAI1 and an ai ℓ^1 sequence.

Proof. Fix an increasing sequence $(m_i)_i$ in $\mathbb{N}$ and define for each $k \in \mathbb{N}$, $M_k := \sum_{i=1}^{k} m_i$ with $M_0 := 0$ and $y_k = \frac{1}{1+m_k} \sum_{n=M_{k-1}+1}^{M_k} e_n$. Then, we get for each $k \in \mathbb{N}$,

$$\begin{aligned}
\|y_k\| &\le \frac{1}{m_k+1}(1+\frac{5}{6}(M_k - M_{k-1})) \le \frac{1}{m_k+1}\left(1+\frac{5}{6}m_k\right) \\
&\le \frac{1}{m_k+1}(1+m_k) = 1
\end{aligned}$$

and thus, for all $(t_n)_n \in \ell^1$,

$$\left\|\sum_{n=1}^{\infty} t_n y_n\right\| \le \sum_{n=1}^{\infty} |t_n| \le \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2}.$$

On the other hand, we have

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty} t_n y_n\right\| &= \left\| \begin{array}{l} t_1 \frac{1}{m_1+1} \sum_{j=1}^{M_1} e_j + t_2 \frac{1}{m_2+1} \sum_{j=M_1+1}^{M_2} e_j \\ + t_3 \frac{1}{m_3+1} \sum_{j=M_2+1}^{M_3} e_j + \cdots + t_n \frac{1}{m_n+1} \sum_{j=M_{n-1}+1}^{M_n} e_j + \cdots \end{array} \right\| \\
&\ge \frac{2}{3} \sum_{n=1}^{\infty} \frac{|t_n| m_n}{m_n+1} + \sum_{n=1}^{\infty} \frac{m_n |t_n|}{2^n(m_n+1)} + \frac{1}{3} \sum_{n=1}^{\infty} \frac{m_n |t_n|}{m_n+1} \\
&= \sum_{n=1}^{\infty} |t_n| \frac{m_n}{m_n+1} \left(1 + \frac{1}{2^n}\right) \\
&\ge \sqrt{\left(\sum_{n=1}^{\infty} |t_n| \frac{m_n}{m_n+1}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \frac{m_n}{m_n+1}\right)^2}
\end{aligned}$$

Then, letting $m_n \leq 2^n$, $\varepsilon_n = \frac{1}{m_n+1}$ and $\delta_n = \frac{2^n - m_n}{2^n(m_n+1)}$ for each $n \in \mathbb{N}$, we obtain that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n|(1-\delta_n) \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq \sum_{n=1}^{\infty} |t_n|$$

and

$$\sqrt{\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|\right)^2 + \left(\sum_{n=1}^{\infty}(1-\varepsilon_n)\frac{|t_n|}{2^n}\right)^2} \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\|$$
$$\leq \sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2 + \left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}.$$

Thus, $(\ell^1, \|.\|)$ has property AAI1 and it contains ai copy of ℓ^1 such that both properties share a common sequence to exist. $\square$

Remark 1. Dowling, Johnson, Lennard and Turett [2] constructed the following equivalent norm $\||\cdot|\|^{\sim}$ on ℓ^1 and they showed that $\left(\ell^1, \||\cdot|\|^{\sim}\right)$ does not contain an ai copy of ℓ^1 and later Lin [11] showed that ℓ^1 can be renormed to have fpp(ne) with a special version of the norm $\||\cdot|\|^{\sim}$.

Equivalent norm $\||\cdot|\|^{\sim}$ on ℓ^1 is given as follows: for $x = (\xi_k)_k \in \ell^1$, write $\||x|\|^{\sim} := \sup_{k\in\mathbb{N}} \gamma_k \sum_{j=k}^{\infty} \left|\xi_j\right|$ where $\gamma_k \uparrow_k 1$, γ_k is strictly increasing.

Due to our result in previous section, having property AAI1 is equivalent for a Banach space to contain an ai copy of ℓ^1, we have a straight result that $\left(\ell^1, \||\cdot|\|^{\sim}\right)$ does not satisfy AAI1 property.

Remark 2. Note that [5, Theorem 4.1 on page 87] says that if a Banach space contains an ai copy of c_0 then its isometric dual contains an ai copy of ℓ^1. Also note that the converse does not generally hold; that is, a dual Banach space may contain an ai copy of ℓ^1 but this does not generally imply its isometric predual contains an ai copy of c_0; e.g. as a predual, we can consider ℓ^1 with its canonical norm which does not contain an ai copy of c_0 but ℓ^∞ contains an isometric copy of ℓ^1 so it contains an ai copy of ℓ^1. Using the same theorem of Dowling, Lennard and Turett, we can say that if a Banach space fails AAI1 property, then its predual does not contain any ai copy of c_0.

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