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NO COMPRESSION OF SWITCHING MOMENTS AND STABILITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES

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Abstract. In this paper, we investigate the switching moments have no compression point and stability of the zero solution for fractional differential equations with variable structure and impulses. In order to study properties of the solution, we seek some sufficient conditions, under which the switching moments do not have a compression point. Then several sufficient conditions for stability, uniform stability, uniform asymptotically stability of the zero solution for the systems investigated by Lyapunov functions. Finally, two examples are given to illustrate our results.

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Keywords: fractional differential equations, variable structure, impulses, compression point, stability, Lyapunov functions

1. INTRODUCTION

Fractional calculus extend the classical integer order calculus to arbitrary order case, playing a crucial part in several research areas, such as engineering, physics, finance and signal analysis. There are many important monographs on fractional differential equations and their applications, we recommend the reader to read [11–13, 17, 22, 26, 27, 30] and reference therein.

Fractional differential equations with fixed moments of impulses have been regarded as an interesting and challenging subfield of impulsive differential equations, because fractional order derivatives display a global character and highlight the memory accumulated by the long-term effects throughout the entire evolution process, including some fixed impulsive moments. This is much different from the same problem arises from the classical integer order differential equations with fixed moments of impulses, which has local character and can be solved by dividing the whole time span into several subintervals related to fixed moments of impulses, using a step-by-step methodology.

The main trends to offer fractional differential equations with fixed moments of impulses by defining the lower limit for the fractional order derivatives. Consider the following initial problem for fractional order differential equations with fixed moments of impulses on the initial condition and impulsive moments:

$$\begin{cases} {}^{c}D^{\alpha}_{t_i,t}x(t) = g(t),\ t \neq t_i,\ \alpha \in (0,1), \\ x(t_i+0) = x(t_i) + I_i(x(t_i)),\ \ i = 1,2,\cdots, \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\alpha}_{t_i,t}$ denotes *the classical Caputo fractional derivative of the order* α *(see Definition 3) via changing the lower limit associated with fixed moments of impulses* t_i, $0 \leq t_0 < t_1 < t_2 < \cdots$ and $t_i \to \infty$ as $i \to \infty$, $g : R^+ \to R^n$ is a given function.

For this case, Agarwal et al. [3,5] introduced the concepts of solution and derived the new representation of piecewise continuous solution of (1.1) as follows:

$$\begin{aligned} x(t;t_0,x_0) = x_0 &+ \sum_{t_0<t_i<t} I_i(x(t_i)) \\ &+ \frac{1}{\Gamma(\alpha)}\left(\sum_{t_0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)ds + \int_{t_i}^{t}(t-s)^{\alpha-1}g(s)ds\right) \end{aligned} \tag{1.2}$$

for $t \in (t_i, t_{i+1}]$, $i = 1,2,\cdots$. The main idea to derive the representation of (1.2) is motivated by the framework of seeking the formula of solution to integer order impulsive differential equations. There are many works on the existence, stability of solution and controllability, see[7, 8, 14–16, 28, 29]. One of the main properties in the qualitative theory of differential equations is stability of solution. Stability enables us to compare the behavior of solution starting at different points, and Lyapunov approach is one of the methods for studying stability of fractional order systems [2, 4, 24, 25]. Stability for the zero solution of fractional nonlinear equations is studied in [6, 10], which requires differentiability of the applied Lyapunov functions.

In the real world life there are many processes and phenomena that are characterized by rapid changes in their state, so called impulses. The study of processes with a short-term rapid change is a subject of several sciences, and classical instantaneous impulses are suitable to show some evolution processes in engineering, physics, biology, medical science, etc, see [1, 18, 23]. Moreover, in the study of discontinuous processes using mathematical modeling in the form of differential equations with impulsive effects, as a rule, is required.

The differential equations with variable structure without impulsive were originated in the works by T.Vogel [19], in these systems there is a change in the right hand sides when the trajectory reaches give set defined, primarily applied in the control theory.

In this paper, we discuss the following initial problem for fractional order differential equations with variable structure and impulses:

$$\begin{cases} {}^{c}D^{\alpha}_{t_{i-1},t}x(t) = f_i(t,x),\ \varphi_i(x(t)) \neq 0,\ t_{i-1} < t < t_i,\ \alpha \in (0,1),\ i = 1,2,\cdots, \\ x(t_i+0) = x(t_i) + I_i(x(t_i)),\ \varphi_i(x(t_i)) = 0,\ \ i = 1,2,\cdots, \\ x(t_0) = x_0, \end{cases} \tag{1.3}$$

where ${}^{c}D^{\alpha}_{t_{i-1},t}$ denotes the classical Caputo fractional derivative of the order α via changing the lower limit associated with fixed moments of impulses t_{i-1}. The functions $f_i : R^+ \times D \to R^n$, D is a nonempty domain in R^n, $t_i \in R^+$, $0 \leq t_0 < t_1 < t_2 < \cdots$, $\varphi_i : D \to R$, $I_i : D \to R^n$, $Id + I_i : D \to D$ and $x_0 \in D$. The identity in R^n is denoted by Id. $x(t_i+0) = \lim_{\varepsilon \to 0+} x(t_i+\varepsilon)$ represent the right limit of $x(t)$ at $t = t_i$.

At the moments $t_1, t_2, \cdots$, the right hand side of the system is changing and the impulsive effect takes place, these moments are called switching moments. We set $\Phi_i = \{x \in D; \varphi_i(x) = 0\},\ i = 1,2,\cdots$ and $\Phi = \bigcup_{i=1,2,\cdots} \Phi_i$. If the switching moments have a compression point, then the solution not continuable at the right hand side of this point. In this case, a number of important properties (stability, monotony, periodicity equivalence, etc.) of the solution can not be studied. Therefore, we need find the switching moments do not have a compression point in sufficient conditions, lay the foundations for studying the properties of solution to the equations.

Since the concepts of solution and existence results have been reported in [21] on finite time interval, one can adopt the same methods [20] to obtain the existence of solution of (1.3) on infinite time interval $[t_0, \infty)$ by using contracting mapping principle. Thus, we follows the concepts in [9] and focus on the unlimited moments of switching of (1.3) in the present paper. In order to discuss the stability, uniform stability, uniform asymptotically stability of the zero solution for the system investigated by Lyapunov functions.

The representation of the solution to the problem (1.3) is denoted by $x(\cdot; x_0) \in PC([t_0, \infty), R^n)$ as follows:

$$\begin{aligned} x(t;x_0) &= x(t;x_0, f_1, \varphi_1, I_1, f_2, \varphi_2, I_2, \cdots) \\ &= \begin{cases} x(t;x_0,f_1), & t_0 < t < t_1, \\ x(t;x_0,f_1,\varphi_1,I_1,f_2), & t_1 < t < t_2, \\ \vdots & \\ x(t;x_0,f_1,\varphi_1,I_1,f_2,\varphi_2,I_2,\cdots,f_i,\varphi_i,I_i,f_{i+1}), & t_i < t < t_{i+1}, \\ \vdots & \end{cases} \end{aligned}$$

The differential equations with variable structures and impulses have been explored about integer order, the breadth of research into fractional differential equations with similar attributes is notably less extensive.

Note that the properties for various aspects of the dynamic impulsive mathematical models from the epidemiology and pharmacokinetics are connected with unlimited moments of switching for fractional differential equations with variable structure and

impulses. In [9], Dishlieva and Dishliev initially study the properties in respect of integer order differential equations with variable structure and impulses. To fix this gap, we discuss the characteristics in respect of fractional differential equations with variable structure and impulses, which is an extension for integer order system.

The rest of this paper is organized as follows. In Section 2, we collect some concepts of fractional calculus and Lyapunov functions, give the definitions of stability. In Section 3, we present main results for our original fractional problem with variable structure and impulses. Some examples are given in final section to demonstrate the application of our main results.

2. PRELIMINARIES

In this section, we introduce the notations, definitions and preliminary facts which are used throughout this paper.

We define $PC(R^+,R^n) := \{x : R^+ \to R^n : x \in C((t_k,t_{k+1}],R^n),\ k = 0,1,\cdots$ and there exist $x(t_k^-)$ and $x(t_k^+), k = 1,2,\cdots$, with $x(t_k) = x(t_k^-)\}$ with the norm $\|x\|_{PC} := \sup\{\|x(t)\| : t \in R^+\}$.

The Euclidean distance between non-empty sets $X,Y \subset R^n$ is

$$\rho(X,Y) = \inf\{\|x-y\|; x \in X, y \in Y\}.$$

Define $\Delta = \bigcap_{i=1,2,\cdots}\{x \in D : V_i(t,x) \le \omega(\sigma)\}$, for $t \ge t_0$, where the $\sigma > 0$ is a constant such that $B_\sigma(0) = \{x \in R^n, \|x\| < \sigma\}$.

Let us recall the following known definitions of fractional calculus. For more details see [4, 12].

Definition 1. The fractional integral of order $\alpha \in (0,1)$ with the lower limit t_0 for an integrable function $f : [t_0,\infty) \to R^n$ is defined as

$$I^\alpha_{t_0,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,\ t > t_0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. The Riemann-Liouville derivative of order $\alpha \in (0,1)$ with the lower limit t_0 for an integrable function $f : [t_0,\infty) \to R^n$ can be written as

$${}^L D^\alpha_{t_0,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^t \frac{f(s)}{(t-s)^\alpha} ds,\ t > t_0.$$

Definition 3. The classical Caputo derivative of order $\alpha \in (0,1)$ with the lower limit t_0 for a differentiable function $f : [t_0,\infty) \to R^n$ be defined by

$${}^C D^\alpha_{t_0,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} f'(s) ds,\ t > t_0.$$

Definition 4. The Grünwald-Letnikov fractional derivative of order $\alpha \in (0,1)$ with the lower limit t_0 for a function $f : [t_0, \infty) \to R^n$ is given by

$$^{GL}D^{\alpha}_{t_0,t} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} f(t - rh),\ t > t_0,$$

and the Grünwald-Letnikov fractional Dini derivative by

$$^{GL}_{t_0,t}D^{\alpha}_{+} f(t) = \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} f(t - rh),\ t > t_0,$$

where $\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$ and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

Definition 5. The generalized Caputo derivative of order $\alpha \in (0,1)$ with the lower limit t_0 for an integrable function $f : [t_0, \infty) \to R^n$ be defined by

$$^{c}D^{\alpha}_{t_0,t} f(t) = {}^{L}D^{\alpha}_{t_0,t}\big[f(t) - f(t_0)\big],\ t > t_0.$$

If the function f is sufficiently smooth, then $^{L}D^{\alpha}_{t_0,t} f(t) = {}^{GL}D^{\alpha}_{t_0,t} f(t)$ for $t > t_0$. Also, according to Definition 5, we have

$$^{c}D^{\alpha}_{t_0,t} f(t) = {}^{L}D^{\alpha}_{t_0,t}\big[f(t) - f(t_0)\big] = {}^{GL}D^{\alpha}_{t_0,t}\big[f(t) - f(t_0)\big],\ t > t_0.$$

From the relation between Caputo fractional derivative and Grünwald-Letnikov fractional derivative, using the Grünwald-Letnikov fractional Dini derivative, we define the Caputo fractional Dini derivative of a function as

$$\begin{aligned} {}^{c}_{t_0,t}D^{\alpha}_{+} f(t) &= {}^{GL}_{t_0,t}D^{\alpha}_{+}\big[f(t) - f(t_0)\big] \\ &= \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} [f(t) - f(t_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{\alpha}{r} (f(t - rh) - f(t_0))]. \end{aligned} \quad (2.1)$$

Remark 1. If $f \in C^{\alpha}([t_0, T], R^n)$, then ${}^{c}_{t_0,t}D^{\alpha}_{+} f(t) = {}^{c}D^{\alpha}_{t_0,t} f(t)$.

Definition 6. The zero solution of the problem (1.3) is said to be

- stable if $\forall t_0 \in R^+, \forall \varepsilon > 0, \exists \delta = \delta(t_0, \varepsilon) > 0$, such that for $\forall x_0 \in D$, $\|x_0\| < \delta$, then $\|x(t; x_0)\| < \varepsilon$ for $t \geq t_0$.
- uniformly stable if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$, such that for $t_0 \in R^+, x_0 \in D$ with $\|x_0\| < \delta$, then $\|x(t; x_0)\| < \varepsilon$ for $t \geq t_0$.
- uniformly attractive if for $\lambda > 0$, $\forall \varepsilon > 0$, $\exists T = T(\varepsilon) > 0$ such that for $t_0 \in R^+, x_0 \in D$ with $\|x_0\| < \lambda$, then $\|x(t; x_0)\| < \varepsilon$ for $t \geq t_0 + T$.
- uniformly asymptotically stable if the zero solution is uniformly stable and uniformly attractive.

Definition 7. The piecewise continuous Lyapunov functions is given:

$$\{V_i; V_i : R^+ \times D \to R^+,\ i = 1, 2, \cdots\},$$

corresponding to the problem (1.3), if $V_i \in C[R^+ \times D\backslash\Phi_i, R^+]$, $i = 1,2,\cdots$, and for every point $(t,x_{\Phi_i}) \in R^+ \times \Phi_i$, $i = 1,2,\cdots$, there exist limits:

$$\lim_{x\to x_{\Phi_i},\Phi_i(x)<0} V_i(t,x) = V_i(t,x_{\Phi_i}-0) = V_i(t,x_{\Phi_i}),\ for\ (t,x) \in R^+ \times D,$$

and

$$\lim_{x\to x_{\Phi_i},\Phi_i(x)>0} V_i(t,x) = V_i(t,x_{\Phi_i}+0),\ for\ (t,x) \in R^+ \times D,$$

implies

$$V_i(t,x_{\Phi_i}-0) = V_i(t,x_{\Phi_i}), for (t,x_{\Phi_i}) \in R^+ \times \Phi_i,\ i = 1,2,\cdots.$$

Definition 8. Let $\{V_i;\ i = 1,2,\cdots\}$ is piecewise continuous Lyapunov functions, then for $\forall (t,x) \in R^+ \times D\backslash\Phi_i$, $i = 1,2,\cdots$, we define the generalized Caputo fractional Dini derivative of Lyapunov functions V_i at the point (t,x) of the problem (1.3):

$$\begin{aligned} {}^{C}_{(1.3)}D^{\alpha}_{+}V_i(t,x) = \limsup_{h\to 0^+} \frac{1}{h^{\alpha}} \Bigg[& V_i(t,x) - V_i(t_0,x_0) \\ & - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{\alpha}{r} (V_i(t-rh, x-h^{\alpha}f(t,x)) - V_i(t_0,x_0)) \Bigg], \end{aligned}$$

where $x, x_0 \in D$, for $\forall t \in R^+, \exists\, h_1 > 0$, such that $t-h \in R^+, x-h^{\alpha}f \in D$, for $0 < h \le h_1$.

3. Main results

3.1. *No Compression of Switching Moments*

The main aim of the section is to find some sufficient conditions which ensure that the switching moments for the problem (1.3) have no compression point.

In the following, we list the assumptions:

- H$_1$ The functions $f_i : R^+ \times D \to R^n, I_i : D \to R^n, i = 1,2,\cdots$ are continuous.
- H$_2$ There exist positive constants L_i such that $\|f_i(t,x)\| \le L_i$, for any $(t,x) \in R^+ \times D$, $i = 1,2,\cdots$.
- H$_3$ For any point $(t_0,x_0) \in R^+ \times D$ and for each $i = 1,2,\cdots$, then the problem (1.3) without impulses has a unique solution in interval $[t_0,\infty)$. That is, the following fractional Cauchy problem

$$\begin{cases} {}^{c}D^{\alpha}_{t_{i-1},t}x(t) = f_i(t,x),\ \alpha \in (0,1), \\ x(t_0) = x_0, \end{cases}$$

  has a unique solution $x_i(t;x_0)$.
- H$_4$ There exist positive constants K_i, $i = 1,2,\cdots$, such that

$$|\varphi_i(x^*) - \varphi_i(x)| \le K_i\|x^* - x\|,\ i = 1,2,\cdots$$

  for each $x^*, x \in D$.

H_5 There exist positive constants C_i such that $|\varphi_{i+1}((Id+I_i)(x))| = |\varphi_{i+1}(x+I_i(x))| \geq C_i$, for any $x \in \Phi_i$, $i = 1,2,\cdots$.

H_6 There exist positive constants K_{φ_i}, $i = 1,2,\cdots$, such that

$$|\varphi_i(x)| \leq K_{\varphi_i}\rho(x,\Phi_i),\ i = 1,2,\cdots$$

for each $x \in D$.

H_7 There exist positive constants C_{N_i} and domains N_i, $\Phi_i \subset N_i \subset D$, such that

$$|\varphi_i(x)| \geq C_{N_i},\ i = 1,2,\cdots$$

for $x \in \partial N_i \cap D$.

H_8 For $x \in \Phi_i$, it is satisfied $x + I_i(x) \in D\backslash N_{i+1}$, $i = 1,2,\cdots$.

Remark 2. The proofs for the existence and uniqueness of the solution to problem (1.3) are trivial. In fact, by repeating the similar procedure in [20, Theorem 4.1], we can verify that the problem (1.3) has a unique solution $x \in PC([t_0,\infty),R^n)$ under the standard assumed conditions.

The solution of the problem (1.3) is denoted by $x(t;x_0)$. For $t_{i-1} < t < t_i$, the solution of the problem (1.3) coincides with the solution $x_i(t;x_{i-1}^+)$ of the problem

$$\begin{cases} {}^cD^{\alpha}_{t_{i-1},t}x(t) = f_i(t,x),\ \alpha \in (0,1), \\ x(t_{i-1}) = x_{i-1}^+, \end{cases}$$

where, we set $\varphi_i(x(t;x_0)) = \varphi_i(x_i(t;x_{i-1}^+)) > 0$, for $t = t_i$, we have $\varphi_i(x(t_i;x_0)) = \varphi_i(x_i(t_i;x_{i-1}^+)) = 0$, denote $x_i = x(t_i;x_0)$, then we define the equality $x(t_i+0) = x(t_i) + I_i(x(t_i;x_0)) = x(t_i) + I_i(x_i) = x_i^+$. Especially, $0 = t_0 \leq t < t_1$, we set $x_0 = x_0^+$.

In the arbitrary interval of continuity (t_{i-1},t_i), the solution of the problem (1.3) starts at the point $x_{i-1}^+ \in D$. The switching moment t_i is the first type of discontinuous point of the function $x(t;x_0)$, and the right hand side of system coincides with function $f_{i+1}(t,x)$ in the next interval of continuity of the solution. See Fig.1, showing the trajectory of the problem considered.

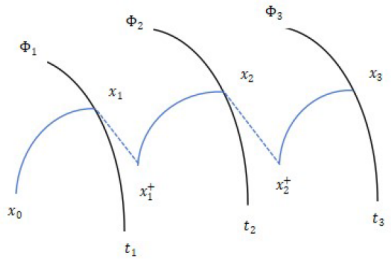

Fig. 1

Theorem 1. *Assume that the conditions* $[H_1]-[H_5]$ *hold. If the series*

$$\sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_i}{L_{i+1}K_{i+1}}$$

is divergent, then the switching moments for problem (1.3) *have no compression point.*

Proof. We will evaluate the difference between t_i and t_{i+1} for each $i=1,2,\cdots$.

For the interval $t\in[t_0,t_1)$, we obtain

$$\begin{aligned}\|x(t_1;x_0)-x(t_0;x_0)\| &= \|x_1(t_1;x_0)-x_0\| \le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}\|f_1(s,x_1(s;x_0))\|ds\\ &\le \frac{L_1}{\Gamma(\alpha+1)}(t_1-t_0)^{\alpha}.\end{aligned}$$

Note that

$$\begin{aligned}(t_1-t_0)^{\alpha} &\ge \frac{\Gamma(\alpha+1)}{L_1}\|x(t_1;x_0)-x(t_0;x_0)\| \\ &\ge \frac{\Gamma(\alpha+1)}{L_1K_1}|\varphi_1(x(t_1;x_0))-\varphi_1(x(t_0;x_0))|.\end{aligned} \tag{3.1}$$

In the interval $t\in[t_0,t_1)$, $\varphi_1(x(t_0;x_0))>0$, for $t=t_1$, $\varphi_1(x(t_1;x_0))=0$. We get $(t_1-t_0)^{\alpha}\ge 0$.

Next, we consider the interval (t_1,t_2), we have

$$\begin{aligned}\|x(t_2;x_0)-x(t_1+0;x_0)\| &= \left\|x_2(t_2;x_1^+)-x_2(t_1;x_1^+)\right\|\\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|f_2(s,x_2(s;x_1^+))\|ds\\ &\le \frac{L_2}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.\end{aligned}$$

Note that the conditions $[H_4]$ and $[H_5]$, we get

$$\begin{aligned}(t_2-t_1)^{\alpha} &\ge \frac{\Gamma(\alpha+1)}{L_2}\|x(t_2;x_0)-x(t_1+0;x_0)\| \\ &\ge \frac{\Gamma(\alpha+1)}{L_2K_2}|\varphi_2(x(t_2;x_0))-\varphi_2(x(t_1+0;x_0))| \ge \frac{\Gamma(\alpha+1)}{L_2K_2}|\varphi_2(x(t_1+0;x_0))|\\ &= \frac{\Gamma(\alpha+1)}{L_2K_2}|\varphi_2(x_1+I_1(x_1))| \ge \frac{C_1}{L_2K_2}\Gamma(\alpha+1).\end{aligned} \tag{3.2}$$

According to the discussion, for (t_i,t_{i+1}), we have

$$\begin{aligned}\|x(t_{i+1};x_0)-x(t_i+0;x_0)\| &= \|x_{i+1}(t_{i+1};x_i^+)-x_{i+1}(t_i;x_i^+)\|\\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}\|f_{i+1}(s,x_{i+1}(s;x_i^+))\|ds\end{aligned}$$

$$\leq \frac{L_{i+1}}{\Gamma(\alpha+1)}(t_{i+1}-t_i)^{\alpha}.$$

Then

$$\begin{aligned}(t_{i+1}-t_i)^{\alpha} &\geq \frac{\Gamma(\alpha+1)}{L_{i+1}}\|x(t_{i+1};x_0)-x(t_i+0;x_0)\| \\ &\geq \frac{\Gamma(\alpha+1)}{L_{i+1}K_{i+1}}|\varphi_{i+1}(x(t_{i+1};x_0))-\varphi_{i+1}(x(t_i+0;x_0))| \\ &\geq \frac{\Gamma(\alpha+1)}{L_{i+1}K_{i+1}}|\varphi_{i+1}(x(t_i+0;x_0))| \\ &= \frac{\Gamma(\alpha+1)}{L_{i+1}K_{i+1}}|\varphi_{i+1}(x_i+I_i(x_i))| \geq \frac{C_i}{L_{i+1}K_{i+1}}\Gamma(\alpha+1).\end{aligned} \tag{3.3}$$

It is fulfilled

$$\begin{aligned}\lim_{i\to\infty} t_i &= \lim_{i\to\infty}\Big((t_i-t_{i-1})+(t_{i-1}-t_{i-2})+\cdots+(t_2-t_1)+(t_1-t_0)\Big)+t_0 \\ &\geq \lim_{i\to\infty}\Big((t_i-t_{i-1})^{\alpha}+(t_{i-1}-t_{i-2})^{\alpha}+\cdots+(t_2-t_1)^{\alpha}+(t_1-t_0)^{\alpha}\Big) \\ &\geq \lim_{i\to\infty}\Big(\frac{C_{i-1}}{L_iK_i}\Gamma(\alpha+1)+\frac{C_{i-2}}{L_{i-1}K_{i-1}}\Gamma(\alpha+1)+\cdots+\frac{C_1}{L_2K_2}\Gamma(\alpha+1)\Big) \\ &= \sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_i}{L_{i+1}K_{i+1}}=\infty.\end{aligned} \tag{3.4}$$

Therefore, the switching moments do not have a compression point. The proof is completed. □

Theorem 2. *Assume that the conditions* $[H_1]-[H_3]$, $[H_5]-[H_6]$ *hold. If the series*

$$\sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_{i-1}}{L_iK_{\varphi_i}}$$

is divergent, then the switching moments for problem (1.3) *have no compression point.*

Proof. We will consider the difference $t_{i+1}-t_i$ for each interval.

For $t\in[t_0,t_1)$, from (3.1), we get

$$\begin{aligned}(t_1-t_0)^{\alpha} &\geq \frac{\Gamma(\alpha+1)}{L_1}\|x(t_1;x_0)-x(t_0;x_0)\| \\ &\geq \frac{\Gamma(\alpha+1)}{L_1}\rho(x(t_0;x_0),\Phi_1) \geq \frac{\Gamma(\alpha+1)}{L_1K_{\varphi_1}}|\varphi_1(x(t_0;x_0))|.\end{aligned}$$

For $t\in[t_0,t_1)$, $\varphi_1(x(t_0;x_0))>0$, we get $(t_1-t_0)^{\alpha}\geq 0$.

From (3.2), we get

$$(t_2-t_1)^{\alpha}\geq\frac{\Gamma(\alpha+1)}{L_2}\|x(t_2;x_0)-x(t_1+0;x_0)\|\geq\frac{\Gamma(\alpha+1)}{L_2}\rho(x(t_1+0;x_0),\Phi_2)$$

$$\geq \frac{\Gamma(\alpha+1)}{L_2 K_{\varphi_2}}|\varphi_2(x(t_1+0;x_0))| = \frac{\Gamma(\alpha+1)}{L_2 K_{\varphi_2}}|\varphi_2(x_1+I_1(x_1))|$$
$$\geq \frac{C_1}{L_2 K_{\varphi_2}}\Gamma(\alpha+1).$$

From (3.3), we get

$$(t_{i+1}-t_i)^\alpha \geq \frac{\Gamma(\alpha+1)}{L_{i+1}}\|x(t_{i+1};x_0)-x(t_i+0;x_0)\| \geq \frac{\Gamma(\alpha+1)}{L_{i+1}}\rho(x(t_i+0;x_0),\Phi_{i+1})$$
$$\geq \frac{\Gamma(\alpha+1)}{L_{i+1}K_{\varphi_{i+1}}}|\varphi_{i+1}(x(t_i+0;x_0))| = \frac{\Gamma(\alpha+1)}{L_{i+1}K_{\varphi_{i+1}}}|\varphi_{i+1}(x_i+I_i(x_i))|$$
$$\geq \frac{C_i}{L_{i+1}K_{\varphi_{i+1}}}\Gamma(\alpha+1).$$

Analogously to (3.4), we obtain

$$\lim_{i\to\infty} t_i \geq \sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_{i-1}}{L_i K_{\varphi_i}} = \infty.$$

The theorem is proved. □

Theorem 3. *Assume that the conditions* $[H_1]-[H_3]$, $[H_6]-[H_8]$ *hold. If the series*

$$\sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_{N_i}}{L_i K_{\varphi_i}}$$

is divergent, then the switching moments for problem (1.3) *have no compression point.*

Proof. We consider the point $x_i^+ = x(t_i+0;x_0) = x(t_i)+I_i(x(t_i;x_0)) = x(t_i)+I_i(x_i)$. Since $x_i \in \Phi_i$ and under condition $[H_8]$, we have $x_i^+ \in D\backslash N_{i+1}$. We go into the solution $x_{i+1}(t;x_i^+)$ of initial value problem

$$\begin{cases} {}^cD^{\alpha}_{t_i,t}x(t) = f_{i+1}(t,x),\ \alpha\in(0,1), \\ x(t_i) = x_i^+. \end{cases}$$

Since

$$x_{i+1}(t;x_i^+) = x_i^+ + \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-s)^{\alpha-1}f_{i+1}(s,x_{i+1}(s;x_i^+))ds,$$

then $x_{i+1}(t_i;x_i^+) = x_i^+ \in D\backslash N_{i+1}$, $x_{i+1}(t_{i+1};x_i^+) = x(t_{i+1};x_0) = x_{i+1} \in \Phi_{i+1} \subset N_{i+1}$.

According to the continuity of the curve $x_{i+1}(t;x_i^+)$, then $\exists\ t_{\partial N_{i+1}}$, for $t_i < t_{\partial N_{i+1}} < t_{i+1}$, such that

$$x_{i+1}(t_{\partial N_{i+1}};x_i^+) = x(t_{\partial N_{i+1}};x_0) \in \partial N_{i+1}\cap D.$$

By the condition $[H_7]$, we obtain

$$|\varphi_{i+1}(x(t_{\partial N_{i+1}};x_0))| \geq C_{N_{i+1}}.$$

On the other hand, we have

$$\|x(t_{i+1};x_0)-x(t_{\partial N_{i+1}};x_0)\| \le \frac{L_{i+1}}{\Gamma(\alpha+1)}(t_{i+1}-t_{\partial N_{i+1}})^{\alpha},$$

then

$$\begin{aligned}(t_{i+1}-t_{\partial N_{i+1}})^{\alpha} &\ge \frac{\Gamma(\alpha+1)}{L_{i+1}}\|x(t_{i+1};x_0)-x(t_{\partial N_{i+1}};x_0)\| \\ &\ge \frac{\Gamma(\alpha+1)}{L_{i+1}}\rho(x(t_{\partial N_{i+1}};x_0),\Phi_{i+1}) \ge \frac{\Gamma(\alpha+1)}{L_{i+1}K_{\varphi_{i+1}}}|\varphi_{i+1}(x(t_{\partial N_{i+1}};x_0))| \\ &\ge \Gamma(\alpha+1)\frac{C_{N_{i+1}}}{L_{i+1}K_{\varphi_{i+1}}}.\end{aligned}$$

From the estimate above, we find

$$(t_{i+1}-t_i)^{\alpha} > (t_{i+1}-t_{\partial N_{i+1}})^{\alpha} \ge \Gamma(\alpha+1)\frac{C_{N_{i+1}}}{L_{i+1}K_{\varphi_{i+1}}},\ \ i=1,2,\cdots$$

From (3.4), we obtain

$$\lim_{i\to\infty} t_i \ge \sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_{N_i}}{L_iK_{\varphi_i}} = \infty.$$

The theorem is proved. □

3.2. *Stability*

In the section, we will find some sufficient conditions for stability of the zero solution for the problem (1.3).

We will use the following conditions:

$\Lambda = \{\omega \in C(R^+,R^+),\ \omega \text{ is strictly increasing and } \omega(0)=0\}$.

H$_9$ $0\in D$ and $f_i(t,0)=0, t\in R^+, i=1,2,\cdots,\ \varphi_i(0)=\varphi_i(0,0,\cdots,0)\ne 0, i=1,2,\cdots$.

H$_{10}$ There exists a sequence of functions $V_i\in C(R^+\times D,R^+)$ such that $V_i(t,0)=0,\ i=1,2,\cdots$.

H$_{11}$ $\omega(\|x\|)\le V_i(t,x)$, for $(t,x)\in R^+\times D,\ i=1,2,\cdots$, where $\omega\in\Lambda$;

H$_{11'}$ $\omega(\|x\|)\le V_i(t,x)\le\mu(\|x\|)$, for $(t,x)\in R^+\times D,\ i=1,2,\cdots$, where $\omega,\mu\in\Lambda$;

H$_{12}$ $V_{i+1}(t+0,x+I_i(x))=V_{i+1}(t,x+I_i(x))\le V_i(t,x)$, for $(t,x)\in R^+\times\Phi_i,\ i=1,2,\cdots$;

H$_{13}$ ${}^{C}_{(1.3)}D^{\alpha}_{+}V_i(t,x)\le 0$, for $(t,x)\in R^+\times D\backslash\Phi_i,\ i=1,2,\cdots$;

H$_{13'}$ ${}^{C}_{(1.3)}D^{\alpha}_{+}V_i(t,x)\le -\kappa(\|x\|)$, for $(t,x)\in R^+\times D\backslash\Phi_i,\ i=1,2,\cdots$, where $\kappa\in\Lambda$.

Remark 3. The conditions $[H_1]$ and $[H_9]$ are fulfilled. Then the problem (1.3) possesses a zero solution, i.e. $x(t;0)=0$, for $0\le t_0\le t<\infty$.

Lemma 1. *(see[10]). Let* ${}^{c}D^{\alpha}_{t_0,t}x(t)\le {}^{c}D^{\alpha}_{t_0,t}y(t)$, *where* $\alpha\in(0,1)$, *and* $x(t_0)=y(t_0)$, *then* $x(t)\le y(t)$.

Theorem 4. *Assume that the conditions* $[H_1]-[H_5]$, $[H_9]-[H_{13}]$ *hold. Then the zero solution of the problem* (1.3) *is stable.*

Proof. Obviously, the problem (1.3) has a zero solution for $t \in [t_0,\infty)$.

Since $V_1(t_0,0)=0$, set $\varepsilon>0$, $\exists\delta(t_0,\varepsilon)>0$ such that $V_1(t_0,x)<\omega(\varepsilon)$, for $\|x\|<\delta$. For $x_0 \in B_\delta(0)$, then $V_1(t_0,x_0)<\omega(\varepsilon)$.

We say the problem (1.3) have infinity many switching moments, i.e. $t_0<t_1<t_2<t_3<\cdots$.

For $t_{i-1}<t<t_i$, the solution of the problem (1.3) is continuous and coincides with the solution of the problem without impulses.

$$\begin{cases} {}^cD^{\alpha}_{t_{i-1},t}x(t)=f_i(t,x), \\ x(t_{i-1}+0)=x^+_{i-1}. \end{cases}$$

By conditions $[H_{11}]-[H_{13}]$, we have

$$\omega(\|x(t;x_0)\|)\le V_i(t,x(t;x_0))\le V_i(t_{i-1}+0,x^+_{i-1})\le V_1(t_0,x_0)<\omega(\varepsilon).$$

Since $\omega\in\Lambda$, from the last inequality, we get

$$\|x(t;x_0)\|<\varepsilon, \ \text{ for } \ t_{i-1}<t<t_i, \ i=1,2,\cdots.$$

Apply (3.4), we have $\lim_{i\to\infty}t_i=\infty$, use the equalities $x(t_i-0;x_0)=x(t_i;x_0)$, we obtain that

$$\|x(t;x_0)\|<\varepsilon, \ \text{ for } \ t_0\le t<\infty.$$

The theorem is proved. □

Theorem 5. *Assume that the conditions* $[H_1]-[H_5]$, $[H_9]-[H_{10}]$, $[H_{11'}]$, $[H_{12}]-[H_{13}]$ *hold. Then the zero solution of the problem* (1.3) *is uniformly stable.*

Proof. For $\forall t>t_0$, $t\neq t_i$, $i=1,2,\cdots$. Set k be the biggest number and $t>t_k$. Let $\varepsilon>0$, $\exists\delta=\delta(\varepsilon)$ satisfied the inequality $\mu(\delta)<\omega(\varepsilon)$. For $x_0\in B_\delta(0)$, then

$$\omega(\|x(t;x_0)\|)\le V_k(t,x(t;x_0))\le V_1(t_0,x_0)\le\mu(\|x_0\|)\le\mu(\delta)<\omega(\varepsilon).$$

Based on $\omega\in\Lambda$ and Theorem 1, we obtain that

$$\|x(t;x_0)\|\le\varepsilon, \ \ for \ \ t_0\le t<\infty.$$

The theorem is proved. □

Theorem 6. *Assume that the conditions* $[H_1]-[H_5]$, $[H_9]-[H_{10}]$, $[H_{11'}]$, $[H_{12}]$, $[H_{13'}]$ *hold. Then the zero solution of the problem* (1.3) *is uniformly asymptotically stable.*

Proof. From Theorem 5, the zero solution of the problem (1.3) is uniformly stable.

For $x_0\in\Delta$, we get $V_k(t,x(t;x_0))\le\omega(\sigma)$. Let a constant $\lambda>0$ such that $\mu(\lambda)<\omega(\sigma)$, then for $x_0\in B_\lambda(0)$, it is fulfilled

$$V_1(t_0,x_0)\le\mu(\|x_0\|)\le\mu(\lambda)<\omega(\sigma).$$

It means that the solution $x(t;x_0)$ of problem (1.3) is defined for $t\in[t_0,\infty)$.

Let $\varepsilon > 0$, choose the constant $\eta = \eta(\varepsilon) > 0$ such that $\mu(\eta) < \omega(\varepsilon)$. Set the constant T satisfies the inequality

$$T > \sqrt[\alpha]{\alpha\Gamma(\alpha)\frac{\mu(\sigma)}{\kappa(\eta)}}.$$

We now prove that $\|x(t;x_0)\| < \varepsilon$, for $t \geq t_0 + T$. Assume $\|x(t;x_0)\| \geq \eta$, for $t \in [t_0, t_0 + T]$.

From the conditions ${}^{C}_{(1.3)}D^{\alpha}_{+}V_i(t,x) \leq -\kappa(\|x\|)$ and $V_{i+1}(t+0, x+I_i(x)) \leq V_i(t,x)$, then for $t \in [t_0, T]$, we obtain

$$V_i(t, x(t;x_0)) \leq V_1(t_0,x_0) - \frac{1}{\Gamma(\alpha)}\left(\sum_{m=0}^{i-1}\int_{t_m}^{t_{m+1}}(t_{m+1}-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds + \int_{t_i}^{t} t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds\right).$$

Then from the conditions and inequality $A^{\alpha} + B^{\alpha} \geq (A+B)^{\alpha}$ for $A, B > 0$, we get successively

$$\begin{aligned}
\omega(\|x(t;x_0)\|) &\leq V_k(t, x(t;x_0)) \\
&\leq V_k(t_k + 0, x(t_k+0;x_0)) - \frac{1}{\Gamma(\alpha)}\int_{t_k+0}^{t}(t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \\
&\leq V_k(t_k, x(t_k-0;x_0) + I_{k-1}(x(t_k-0;x_0))) \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_{t_k+0}^{t}(t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \\
&\leq V_{k-1}(t_{k-1}+0, x(t_{k-1}+0;x_0)) \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_{t_{k-1}+0}^{t_k}(t_k-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_{t_k+0}^{t}(t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \\
&\vdots \\
&\leq V_1(t_0,x_0) - \frac{1}{\Gamma(\alpha)}\left(\sum_{m=0}^{k-1}\int_{t_m}^{t_{m+1}}(t_{m+1}-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \right. \\
&\quad \left. + \int_{t_k}^{t}(t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds\right) \\
&\leq \mu(\|x_0\|) - \frac{\kappa(\eta)}{\Gamma(\alpha)}\left(\sum_{m=0}^{k-1}\int_{t_m}^{t_{m+1}}(t_{m+1}-s)^{\alpha-1}ds + \int_{t_k}^{t}(t-s)^{\alpha-1}ds\right) \\
&\leq \mu(\sigma) - \frac{\kappa(\eta)}{\alpha\Gamma(\alpha)}\left((t_1-t_0)^{\alpha} + \sum_{m=1}^{k-1}(t_{m+1}-t_1)^{\alpha} + (t-t_k)^{\alpha}\right)
\end{aligned}$$

$$\leq \mu(\sigma) - \frac{\kappa(\eta)}{\alpha\Gamma(\alpha)}\left((t_1 - t_0) + \sum_{m=1}^{k-1}(t_{m+1} - t_1) + (t - t_k)\right)^{\alpha}$$
$$\leq \mu(\sigma) - \frac{\kappa(\eta)}{\alpha\Gamma(\alpha)}(t - t_0)^{\alpha} < 0.$$

The above contradiction proves there exists $t^* \in [t_0, t_0 + T]$ such that $\|x(t^*; x_0)\| < \eta$.

Let $t^* \leq t_0 + T \leq t$, satisfies the inequality $t_{k-p-1} < t^* < t_{k-p} < \cdots < t_{k-1} < t_k < t$, and k is the biggest number. In accordance with $[H_{11'}]$, $[H_{12}]$, $[H_{13'}]$, then

$$\omega(\|x(t; x_0)\|) \leq V_k(t, x(t; x_0)) \leq V_{k-p-1}(t^*, x(t^*; x_0)) \leq \mu(\|x(t^*; x_0)\|) = \mu(\eta) < \omega(\varepsilon).$$

Since $\omega \in \Lambda$, we have $\|x(t; x_0)\| < \varepsilon$, for $t \geq t_0 + T$.

The theorem is proved. □

4. Examples

In this section, two examples are given to illustrate our above theoretical results respectively.

Example 1. Consider the following initial problem for fractional order differential equations

$$\begin{cases} {}^{c}D^{\frac{1}{2}}_{t_{i-1},t} W(t) = -PW(t), \ \varphi_i(W(t)) = W_{\max i} - W(t) > 0, \ t_{i-1} < t < t_i, \ P \geq 0, \\ W(t_i + 0) = W(t_i) + \frac{D_i}{i^2} W(t_i), \varphi_i(W(t_i)) = W_{\max i} - W(t_i) = 0, \ D_i > 0, \ i = 1, 2, \ldots \\ W(t_0) = W_0. \end{cases} \tag{4.1}$$

Set $\alpha = \frac{1}{2}$, define $f_i(W) = f(W) = -PW(t)$, $I_i(W) = \frac{D_i}{i^2} W(t)$, $i = 1, 2 \cdots$. Obviously, the function $f : D \to R^n$, $I_i : D \to R^n$ is continuous for each $W \in D$. We have $|f(W)| \leq P \sup_{W_i, W_j \in D} |W_i - W_j| = L_i = L$. Thus, $[H_1] - [H_2]$ hold.

Consider the system associated with (4.1) without impulse

$$\begin{cases} {}^{c}D^{\frac{1}{2}}_{t_0,t} W(t) = -PW(t), \ P \geq 0, \\ W(t_0) = W_0. \end{cases} \tag{4.2}$$

Obviously, (4.2) has a unique solution $W(t) = E_{\frac{1}{2}}(-P(t - t_0)^{\frac{1}{2}}) W_0$ in interval $[t_0, \infty)$. Thus, $[H_3]$ holds.

$W_{\max i} > 0$, $i = 1, 2 \cdots$, are the upper barrier constants that are specific for each interval of continuity of the solution. It is fulfilled $W_{\max i} \leq \sup_{W_i, W_j \in D} |W_i - W_j|$. The functions $\varphi_i(W) = W_{\max i} - W$, $\varphi_i : D \to R$, $i = 1, 2 \cdots$. Lipschitz constants for condition $[H_4]$ is $K_i = 1$.

Furthermore, we assume that $I_i(W) = \frac{D_i}{i^2} W(t) = \frac{1}{2} W_{\max_{i+1}} - W_{\max i}$, $i = 1, 2 \cdots$. Then $|\varphi_{i+1}((Id + I_i)(x))| = \frac{1}{2} W_{\max_{i+1}} = C_i$, for any $W \in \Phi_i$. The $\Phi_i = \{W_{\max i}\}$, $i =$

$1,2,\cdots$ each one of the switching sets consists of one point. Thereby, it is shown that $[H_5]$ is valid. Then the problem (4.1) have no compression point.

Example 2. Consider the following initial problem for fractional order differential equations

$$\begin{cases} {}^{c}D^{\frac{1}{2}}_{t_{i-1},t}x(t) = \dfrac{-\frac{1}{\sqrt{t\pi}}+\frac{1}{2}\sqrt{t}E_{2,\frac{3}{2}}(-t^2)}{2-\sin t}x, \ \ \varphi_i(x(t)) = x_{\max i} - x(t) > 0, \ \ t_{i-1} < t < t_i, \\ x(t+0) = J_i(t)x(t), \varphi_i(x(t)) = x_{\max i} - x(t) = 0, \ t = t_i, \ i = 1,2,\cdots, \\ x(t_0) = x_0, \end{cases} \tag{4.3}$$

where $f_i(t,x) = f(t,x) = \dfrac{-\frac{1}{\sqrt{t\pi}}+\frac{1}{2}\sqrt{t}E_{2,\frac{3}{2}}(-t^2)}{2-\sin t}x$, $E_{\alpha,\beta}$ is the Mittag-Leffler function defined by $E_{\alpha,\beta}(z) = \frac{z^k}{\Gamma(\alpha k+\beta)}$, $\sup_{t=t_i}|J_i(t)| \le 1, \ i = 1,2,\cdots$.

Let $V_i(t,x) = V(t,x) = (2-\sin t)x^2$. The inequality $\|x\| \le V(t,x) \le 2\|x\|$ is satisfied for any $x \in R$. Consider $\lim_{H\to\infty}\sum_{r=0}^{H}(-1)^r\binom{\alpha}{r} = 0$, where H is a natural number and $\lim_{h\to 0^+}[\frac{t-t_0}{h}] = \infty$, then

$$\lim_{h\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r} = -1,$$

and

$$\limsup_{h\to 0^+}\frac{1}{h^\alpha}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r} = D_0^\alpha(1) = \frac{1}{(t-t_0)^\alpha\Gamma(1-\alpha)}.$$

Now apply generalized Caputo fractional Dini derivative to the considered Lyapunov function, we obtain

$$\begin{aligned} &{}^{C}_{(4.3)}D^\alpha_+V(t,x;t_0,x_0) \\ &\quad = \limsup_{h\to 0^+}\frac{1}{h^\alpha}\Bigg((2-\sin t)x^2 - (2-\sin t_0)x_0^2 \\ &\qquad - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{\alpha}{r}\big((2-\sin(t-rh))(x-h^\alpha f(t,x))^2 - (2-\sin t_0)x_0^2\big)\Bigg) \\ &\quad = -\frac{(2-\sin t_0)x_0^2}{(t-t_0)^\alpha\Gamma(1-\alpha)} + \limsup_{h\to 0^+}\frac{1}{h^\alpha}\Bigg\{x^2\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r}(2-\sin(t-rh)) \\ &\qquad - 2xh^\alpha f(t,x)\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r}(2-\sin(t-rh)) \\ &\qquad + h^{2\alpha}f^2(t,x)\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r}(2-\sin(t-rh))\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{2x_0^2}{\sqrt{t\pi}} + x^2 \limsup_{h\to 0^+} \frac{1}{h^\alpha} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} (2-\sin(t-rh)) \\
&\quad - 2xf(t,x) \limsup_{h\to 0^+} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} (2-\sin(t-rh)) \\
&\quad + f^2(t,x) \limsup_{h\to 0^+} h^\alpha \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} (2-\sin(t-rh)) \Big\} \\
&= -\frac{2x_0^2}{\sqrt{t\pi}} + x^2 D_0^\alpha (2-\sin t) + 2x(2-\sin t) f(t,x) = -\frac{2x_0^2}{\sqrt{t\pi}} \le 0.
\end{aligned}
$$

We get $V(t,J_i(t)x) = (2-\sin t)(J_i(t)x)^2 \le (2-\sin t)x^2 = V(t,x)$, for $t = t_i$, $i = 1,2,\cdots$. Therefore the zero solution of the problem (4.3) is uniformly stable.

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