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OSCILLATORY PROPERTIES OF FOURTH-ORDER NEUTRAL DIFFERENTIAL EQUATIONS USING NEW MONOTONIC PROPERTIES

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Abstract. The authors investigate the asymptotic behavior of solutions of the fourth-order half-linear neutral differential equation

$$\left(\alpha(t)\left((\mu(t)+d(t)\mu(a(t)))'''\right)^b\right)'+m(t)\mu^b(\delta(t))=0 \tag{e}$$

without assuming $\alpha'(t)\geq 0$. By using a linearization method and deriving some new monotonic properties of the nonoscillatory solutions, they analyze the oscillatory behavior of solutions of (e). They use two different techniques, namely, a comparison with second-order delay differential inequalities, and a generalization of very effective Koplatadze's method. They illustrate the improvements over known results by providing specific examples.

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1. INTRODUCTION

We are concerned with the oscillatory behavior of solutions of a class of fourth-order half-linear neutral differential equations of the type

$$\left(\alpha(t)\left(\eta'''(t)\right)^b\right)'+m(t)\mu^b(\delta(t))=0,\quad t\geq t_0\geq 0, \tag{E}$$

where

$$\eta(t)=\mu(t)+d(t)\mu(a(t)),$$

subject to the conditions:

(D1) $\alpha\in C'([t_0,\infty),\mathbb{R})$, $\alpha(t)>0$, $d, m\in C([t_0,\infty),\mathbb{R})$, $m(t)>0$, and

$$0\leq d(t)\leq d_0<1;$$

(D2) $a, \delta \in C([t_0,\infty),\mathbb{R})$, $a(t) \le t$, $\delta(t) \le t$, $\delta'(t) > 0$, and

$$\lim_{t\to\infty} a(t) = \lim_{t\to\infty} \delta(t) = \infty;$$

(D3) b is a ratio of odd positive integers.

Let $t^* = \min\{\min_{t\ge t_0} a(t),\ \min_{t\ge t_0} \delta(t)\}$. By a *solution* of (E), we mean a function $\mu \in C([t^*,\infty),\mathbb{R})$ such that $\alpha(\eta'')^b \in C'[t^*,\infty),\mathbb{R})$, and μ satisfies (E) on $[t_0,\infty)$. We consider only those solutions of (E) that are continuable, that is, they satisfy $\sup\{|\mu(t)| : t \ge T\} > 0$ for all $T \ge t_0$, and we tacitly assume that (E) possess such solutions. A solution $\mu(t)$ of (E) is said to be *oscillatory* if it has infinitely many zeros on $[t_\mu,\infty)$ for some $t_\mu \ge t_0$; otherwise, it is called *nonoscillatory*. Equation (E) itself is called oscillatory if all its solutions are oscillatory.

Fourth-order functional differential equations are often encountered in mathematical models of various physical, biological, and chemical phenomena, for example, see [4,10,14]. In mechanics, engineering, economics and population dynamics, questions related to the existence of oscillatory and nonoscillatory solutions play a significant role; see [13], for details.

In recent decades, several researchers have studied the oscillatory behavior of solutions to various classes of fourth order functional differential equations. See for instance, the monographs [2, 16], the papers [1, 3–13, 17, 18], and the references cited therein. Under the conditions

$$\int_{t_o}^{\infty} \alpha^{-\frac{1}{b}}(t)\mathrm{d}t = \infty, \tag{1.1}$$

or

$$\int_{t_o}^{\infty} \alpha^{-\frac{1}{b}}(t)\mathrm{d}t < \infty, \tag{1.2}$$

with

$$\alpha'(t) \ge 0 \tag{1.3}$$

and

$$\eta(t) \ge \theta t\eta'(t), \quad \theta \in (0,1), \tag{1.4}$$

the authors in [5, 8–10, 12, 15–17, 20–23, 25] established several sufficient conditions that guarantee that all solutions of (E) are oscillatory.

Our aim of this paper is to analyze the oscillatory properties of solutions of (E) in the case where condition (1.1) is satisfied without the restrictive assumption (1.3) and the relation (1.4) which contains the arbitrary constant θ. Therefore, the results obtained in this paper improve and generalize other oscillation criteria reported for (E) or its special cases. Our oscillation results will be obtained by using some new monotonic properties of positive solutions and the linearization of (E). Examples are provided to show the significance of our results over previously known ones.

2. MAIN RESULTS

We begin with a lemma that determines the signs of the derivatives of the positive solutions of (E); it can be obtained from a direct application of [2, Lemma 2.2.1] to (E).

Lemma 1. *Let $\mu(t)$ be a positive solution of (E). Then the companion function $\eta(t)$ is positive and satisfies either*

$$\eta(t) \in E_1 = \{\eta(t) : \eta'(t) > 0,\ \eta''(t) < 0,\ \eta'''(t) > 0,\ \left(\alpha(t)(\eta'''(t))^b\right)' < 0\}$$

or

$$\eta(t) \in E_3 = \{\eta(t) : \eta'(t) > 0,\ \eta''(t) > 0,\ \eta'''(t) > 0,\ \left(\alpha(t)(\eta'''(t))^b\right)' < 0\},$$

eventually.

For convenience, we introduce the following notation:

$$\Omega(t) = \int_{t_*}^{t} \alpha^{-\frac{1}{b}}(s)\mathrm{d}s, \quad \Omega_1(t) = \int_{t_*}^{t} (t-s)\alpha^{-\frac{1}{b}}(s)\mathrm{d}s, \quad \Omega_2(t) = \int_{t_*}^{t} \frac{(t-s)^2}{2}\alpha^{-\frac{1}{b}}(s)\mathrm{d}s,$$

$$M(t) = m(t)(1-d(\delta(t)))^b, \quad M_1(t) = \int_t^{\infty} \left(\frac{1}{\alpha(s)}\int_s^{\infty} M(s_1)\mathrm{d}s_1\right)^{\frac{1}{b}}\mathrm{d}s,$$

where $t_* \geq t_0$.

The next three lemmas are concerned with the asymptotic behavior of the companion function $\eta(t)$.

Lemma 2. *Assume that $\mu(t)$ is an eventually positive solution of (E). Then $\eta(t)$ eventually satisfies the inequality*

$$\left(\alpha(t)(\eta'''(t))^b\right)' + M(t)\eta^b(\delta(t)) \leq 0. \tag{2.1}$$

Proof. Let μ be an eventually positive solution of (E). Then there exists $t_1 \geq t_0$ such that $\mu(a(t)) > 0$ and $\mu(\delta(t) > 0$ for all $t \geq t_1$. From the definition of $\eta(t)$, it is easy to see that $\eta(t) > 0$ for all $t \geq t_1$, and in view of Lemma 1, we have $\eta'(t) > 0$ for all $t \geq t_1$ in both cases E_1 and E_3.

Now

$$\mu(t) = \eta(t) - d(t)\mu(a(t)) \geq \eta(t) - d(t)\eta(a(t)) \geq (1-d(t))\eta(t),$$

and using this in (E) gives

$$\left(\alpha(t)(\eta'''(t))^b\right)' + M(t)\eta^b(\delta(t)) \leq 0.$$

This completes the proof. □

Note that from Lemma 1 and inequality (2.1), it is easy to see that the condition

$$\int_{t_0}^{\infty} M(t)\mathrm{d}t = \infty$$

implies the oscillation of (E), and so in the remainder of this paper, we assume that the opposite condition is satisfied, that is,

$$\int_{t_0}^{\infty} M(t)\mathrm{d}t < \infty.$$

Lemma 3. *Let $\mu(t)$ be a positive solution of* (E) *and assume that*

$$\int_{t_0}^{\infty} \delta(s)M_1(s)\mathrm{d}s = \infty. \tag{2.2}$$

If $\eta \in E_1$, then

$$\frac{\eta(t)}{t} \text{ is decreasing and } \lim_{t\to\infty} \frac{\eta(t)}{t} = 0. \tag{2.3}$$

Proof. Let $\mu(t)$ be a positive solution of (E). Then we see that $\eta(t)$ is positive and by Lemma 2, it satisfies (2.1). Since $\eta(t) \in E_1$, integrating (2.1) from t to ∞, we have

$$\eta'''(t) \geq \eta(\delta(t)) \left(\frac{1}{\alpha(t)} \int_t^{\infty} M(s)\mathrm{d}s \right)^{\frac{1}{b}}.$$

Integrating again from t to ∞ gives

$$\eta''(t) + M_1(t)\eta(\delta(t)) \leq 0. \tag{2.4}$$

From the last inequality, we obtain

$$-\left(t^2 \left(\frac{\eta(t)}{t} \right)' \right)' \geq tM_1(t)\eta(\delta(t)). \tag{2.5}$$

Letting $k = \eta(t_0) - t_0\eta'(t_0)$ and integrating (2.5) from t_0 to t yields

$$\begin{aligned} -t^2 \left(\frac{\eta(t)}{t} \right)' &\geq k + \int_{t_0}^{t} sM_1(s)\eta(\delta(s))\mathrm{d}s \\ &\geq k + \eta(\delta(t_0)) \int_{t_0}^{t} sM_1(s)\mathrm{d}s \to \infty \text{ as } t \to \infty \end{aligned} \tag{2.6}$$

by (2.2). Hence,

$$\left(\frac{\eta(t)}{t} \right)' < 0,$$

eventually, and so $\frac{\eta(t)}{t}$ is decreasing, which also implies $k > 0$ for t large enough.

Now suppose that

$$\lim_{t\to\infty} \frac{\eta(t)}{t} = l > 0.$$

Using (2.6), we see that $z(t)=\frac{\eta(t)}{t}$ satisfies

$$z'(t)+\frac{1}{t^2}\int_{t_0}^{t} s\delta(s)M_1(s)z(\delta(s))\mathrm{d}s \le -\frac{k}{t^2}<0. \tag{2.7}$$

Integrating (2.7) from t_0 to ∞ and then integrating by parts, we obtain

$$z(t_0)-l \ge l\int_{t_0}^{\infty}\frac{1}{x^2}\left(\int_{t_0}^{x} s\delta(s)M_1(s)\mathrm{d}s\right)\mathrm{d}x = l\int_{t_0}^{\infty}\delta(s)M_1(s)\mathrm{d}s,$$

which contradicts (2.2) and so we conclude that $l=0$. This completes the proof of the lemma. □

Next, we set

$$\alpha_1(t)=\frac{1}{t^2}\int_{t_0}^{t} s\delta(s)M_1(s)\mathrm{d}s, \quad \beta_1'(t)=\alpha_1(t), \quad \text{and} \quad \gamma_1(t)=e^{\beta_1(t)}.$$

Lemma 4. *Let (2.2) hold. If $\eta\in E_1$, then $z(t)=\frac{\eta(t)}{t}$ satisfies*

$$z(t)\gamma_1(t) \text{ is decreasing}. \tag{2.8}$$

Proof. Let $\eta\in E_1$; then $\eta(t)$ is a positive solution of (2.1) and $z(t)$ satisfies

$$z'(t)+\alpha_1(t)z(t)\le 0$$

by (2.7). Therefore,

$$(z(t)\gamma_1(t))'\le 0,$$

which proves the lemma. □

The next two lemmas give conditions under which the class E_1 is empty.

Lemma 5. *Let (2.2) hold. If*

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t}\frac{\gamma_1(\delta(x))}{x^2}\left(\int_{t_0}^{x}\frac{s\delta(s)M_1(s)}{\gamma_1(\delta(s))}\mathrm{d}s\right)\mathrm{d}x>\frac{1}{e}, \tag{2.9}$$

then the class E_1 is empty.

Proof. Assume, to the contrary, that $\eta(t)\in E_1$. It follows from (2.7) that

$$0\ge z'(t)+\frac{1}{t^2}\left(\int_{t_0}^{t} s\delta(s)M_1(s)z(\delta(s))\mathrm{d}s\right),$$

which in view of (2.8) implies that $z(t)$ is a positive solution of the differential inequality

$$z'(t)+\left(\frac{\gamma_1(\delta(t))}{t^2}\int_{t_0}^{t}\frac{s\delta(s)M_1(s)}{\gamma_1(\delta(s))}\mathrm{d}s\right)z(\delta(t))\le 0.$$

But by [16, Theorem 2.1.1], and (2.9), we see that the last inequality has no positive solution. This contradiction completes the proof. □

Lemma 6. *Let (2.2) hold. If*

$$\liminf_{t\to\infty} t\int_t^\infty \frac{M_1(s)\delta(s)\gamma_1(s)}{s\gamma_1(\delta(s))}\mathrm{d}s > \frac{1}{4}, \tag{2.10}$$

then the class E_1 is empty.

Proof. Assume that $\eta(t)\in E_1$. Then, from Lemma 3, we have

$$\eta''(t)+M_1(t)\eta(\delta(t))\le 0. \tag{2.11}$$

Using (2.8) in (2.11) gives

$$\eta''(t)+M_1(t)\frac{\delta(t)}{t}\frac{\gamma_1(t)}{\gamma_1(\delta(t))}\eta(t)\le 0.$$

But by [15, Lemma 1], we see that $\eta(t)$ is a positive solution of the differential equation

$$\eta''(t)+M_1(t)\frac{\delta(t)}{t}\frac{\gamma_1(t)}{\gamma_1(\delta(t))}\eta(t)=0, \tag{2.12}$$

and in view of [15, Theorem 3] and (2.10), $\eta(t)$ is an oscillatory solution of (2.12). This contradiction proves the lemma. □

Next, we recall the following result from [25],

Lemma 7. *Let (2.2) hold. Then every positive solution $\eta(t)$ of (2.11) satisfies*

$$\eta(t)\ge t\int_t^\infty M_1(s)\eta(\delta(s))\mathrm{d}s+\int_{t_0}^t sM_1(s)\eta(\delta(s))\mathrm{d}s. \tag{2.13}$$

We have one more result that shows the class E_1 is empty. Set $\gamma_0(t)=\frac{\gamma_1(t)}{t}$.

Lemma 8. *Let (2.2) hold. If*

$$\limsup_{t\to\infty}\Big\{\delta(t)\int_t^\infty M_1(s)\mathrm{d}s+\gamma_1(\delta(t))\int_{\delta(t)}^t\frac{M_1(s)}{\gamma_0(\delta(s))}\mathrm{d}s+\gamma_0(\delta(t))\int_{t_0}^{\delta(t)}\frac{sM_1(s)}{\gamma_0(\delta(s))}\mathrm{d}s\Big\}>1, \tag{2.14}$$

then the class E_1 is empty.

Proof. Assume, to the contrary, that $\eta(t)\in E_1$. Then, as in the proof of from Lemma 6, we obtain equation (2.11). It follows from (2.13) that

$$\eta(\delta(t))\ge\delta(t)\int_t^\infty M_1(s)\eta(\delta(s))\mathrm{d}s+\delta(t)\int_{\delta(t)}^t M_1(s)\eta(\delta(s))\mathrm{d}s+\int_{t_0}^{\delta(t)} sM_1(s)\eta(\delta(s))\mathrm{d}s.$$

Taking into account that $\delta(t)\le t$, $\eta(\delta(t))$ is increasing and $z(t)\gamma_1(t)=\frac{\eta(t)}{t}\gamma_1(t)$ is decreasing, we obtain

$$\eta(\delta(t))\ge\eta(\delta(t))\Big\{\delta(t)\int_t^\infty M_1(s)\mathrm{d}s+\gamma_1(\delta(t))\int_{\delta(t)}^t\frac{M_1(s)}{\gamma_0(\delta(s))}\mathrm{d}s+\gamma_0(\delta(t))\int_{t_0}^{\delta(t)}\frac{sM_1(s)}{\gamma_0(\delta(s))}\mathrm{d}s\Big\},$$

where $\gamma_0(t)=\frac{\gamma_1(t)}{t}$. This contradicts (2.14), so the class E_1 is empty, and this completes the proof. □

Next, we turn our attention to the behavior of solutions in the class E_3.

Lemma 9. *Let $\eta(t) \in E_3$. Then,*

$$\eta(t) \geq \frac{\Omega_2(t)}{\Omega(t)}\eta''(t). \tag{2.15}$$

Proof. Since $\eta \in E_3$, we see that

$$\eta''(t) = \eta''(t_1) + \int_{t_1}^{t} \frac{\alpha^{\frac{1}{b}}(s)\eta'''(s)}{\alpha^{\frac{1}{b}}(s)} ds \geq \Omega(t)\alpha^{\frac{1}{b}}(t)\eta'''(t), \tag{2.16}$$

and so

$$\left(\frac{\eta''(t)}{\Omega(t)}\right)' = \frac{\Omega(t)\alpha^{\frac{1}{b}}(t)\eta'''(t) - \eta''(t)}{\alpha^{\frac{1}{b}}(t)\Omega^2(t)} < 0.$$

Hence, $\frac{\eta''(t)}{\Omega(t)}$ is decreasing and

$$\eta'(t) = \eta'(t_1) + \int_{t_1}^{t} \frac{\Omega(s)\eta''(s)}{\Omega(s)} ds \geq \Omega_1(t)\frac{\eta''(t)}{\Omega(t)}. \tag{2.17}$$

From (2.17), we have

$$\left(\frac{\eta'(t)}{\Omega_1(t)}\right)' = \frac{\Omega_1(t)\eta''(t) - \Omega(t)\eta'(t)}{\Omega_1^2(t)} < 0,$$

so $\frac{\eta'(t)}{\Omega_1(t)}$ is decreasing. On the other hand,

$$\eta(t) = \eta(t_1) + \int_{t_1}^{t} \Omega_1(s)\frac{\eta'(s)}{\Omega_1(s)} ds \geq \Omega_2(t)\frac{\eta'(t)}{\Omega_1(t)}. \tag{2.18}$$

Using (2.17) in (2.18), we obtain (2.15). This completes the proof. □

Lemma 10. *Let $b \geq 1$. Assume that $\mu(t)$ is a positive solution of (E) and $\eta(t) \in E_3$. Then $w(t) = \eta''(t)$ is a positive solution of the second-order differential equation*

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)' + \frac{1}{b}M_2(t)w(\delta(t)) = 0, \tag{2.19}$$

where $M_2(t) = M(t)\frac{\Omega_2^b(\delta(t))}{\Omega(\delta(t))}$.

Proof. Let $\mu(t)$ be an eventually positive solution of (E). Then there exists $t_1 \geq t_0$ such that $\mu(a(t)) > 0$ and $\mu(\delta(t)) > 0$ for all $t \geq t_1$. Then $\eta(t) > 0$ and satisfies inequality (2.1). Now

$$\left(\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)^b\right)' = b\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)^{b-1}\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)'$$

and using (2.1), we obtain

$$\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)' + \frac{1}{b}\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)^{1-b}M(t)\eta^b(\delta(t)) \leq 0. \tag{2.20}$$

From (2.16)–(2.18), we have

$$\eta(t) \geq \Omega_2(t)\alpha^{\frac{1}{b}}(t)\eta'''(t)$$

and so

$$\eta(\delta(t)) \geq \Omega_2(\delta(t))\alpha^{\frac{1}{b}}(\delta(t))\eta'''(\delta(t)) \geq \Omega_2(\delta(t))\alpha^{\frac{1}{b}}(t)\eta'''(t), \tag{2.21}$$

since $\alpha^{\frac{1}{b}}(t)\eta'''(t)$ is positive and decreasing. Also, since $b \geq 1$, from (2.21)

$$\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)^{1-b} \geq \Omega_2^{b-1}(\delta(t))\eta^{1-b}(\delta(t)).$$

Using the last inequality in (2.20) yields

$$\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)' + \frac{1}{b}M(t)\Omega_2^{b-1}(\delta(t))\eta(\delta(t)) \leq 0. \tag{2.22}$$

Let $w(t) = \eta''(t)$ and using (2.15) in (2.22), we obtain that $w(t)$ is a positive solution of the differential inequality

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)' + \frac{1}{b}M_2(t)w(\delta(t)) \leq 0.$$

But by [15, Lemma1 1], equation (2.19) also has a positive solution. This proves the lemma. □

Lemma 11. *Let $b \geq 1$ and $\eta \in E_3$. If*

$$\int_{t_0}^{\infty} \Omega(\delta(s))M_2(s)\mathrm{d}s = \infty, \tag{2.23}$$

then every positive solution $w(t)$ of (2.19) satisfies

$$\frac{w(t)}{\Omega(t)} \text{ is decreasing and } \lim_{t\to\infty}\frac{w(t)}{\Omega(t)} = 0. \tag{2.24}$$

Proof. Let $w(t) = \eta''(t)$; then from the proof of Lemma 9, $\frac{w(t)}{\Omega(t)}$ is decreasing. From (2.19), it is easy to see that

$$-\left(\alpha^{\frac{1}{b}}(t)\Omega^2(t)\left(\frac{w(t)}{\Omega(t)}\right)'\right)' = \frac{\Omega(t)}{b}M_2(t)w(\delta(t)). \tag{2.25}$$

Set $k = w(t_0) - \Omega(t_0)\alpha^{\frac{1}{b}}(t_0)w'(t_0)$; then integrating (2.25) from t_0 to t gives

$$\begin{aligned} -\alpha^{\frac{1}{b}}(t)\Omega^2(t)\left(\frac{w(t)}{\Omega(t)}\right)' &= k + \frac{1}{b}\int_{t_0}^{t}\Omega(s)M_2(s)w(\delta(s))\mathrm{d}s \\ &\geq k + \frac{w(\delta(t_0))}{b}\int_{t_0}^{t}\Omega(s)M_2(s)\mathrm{d}s \to \infty \text{ as } t\to\infty, \end{aligned} \tag{2.26}$$

by (2.23). Hence, $\left(\frac{w(t)}{\Omega(t)}\right)' < 0$ eventually, and we conclude that $\frac{w(t)}{\Omega(t)}$ is decreasing, which also implies $k > 0$ for t_0 large enough.

Now assume that

$$\lim_{t\to\infty}\frac{w(t)}{\Omega(t)}=l>0.$$

The function $p(t)=\frac{w(t)}{\Omega(t)}\geq 0$, and in view of (2.26), satisfies

$$p'(t)+\frac{1}{b\alpha^{\frac{1}{b}}(t)\Omega^2(t)}\int_{t_0}^{t}\Omega(s)\Omega(\delta(s))M_2(s)p(\delta(s))\mathrm{d}s=-\frac{k}{\alpha^{\frac{1}{b}}(t)\Omega^2(t)}<0. \quad (2.27)$$

Integrating (2.27) from t_0 to ∞ and then using integration by parts, gives

$$\begin{aligned}p(t_0)-l&\geq\frac{l}{b}\int_{t_0}^{\infty}\frac{1}{\alpha^{\frac{1}{b}}(s)\Omega^2(s)}\left(\int_{t_0}^{s}\Omega(x)\Omega(\delta(x))M_2(x)\mathrm{d}x\right)\mathrm{d}s\\&=\frac{l}{b}\int_{t_0}^{\infty}\Omega(\delta(s))M_2(s)\mathrm{d}s,\end{aligned}$$

which contradicts (2.23), and so $l=0$. This proves the lemma. □

Next, we introduce the notation

$$\alpha_2(t)=\frac{1}{b\alpha^{\frac{1}{b}}(t)\Omega^2(t)}\int_{t_0}^{t}\Omega(s)\Omega(\delta(s))M_2(s)\mathrm{d}s,\quad \beta_2'(t)=\alpha_2(t),\ \text{and}\ \ \gamma_2(t)=e^{\beta_2(t)}.$$

Lemma 12. *Let $b\geq 1$, $\eta\in E_3$ and (2.23) hold. Then for every positive solution $w(t)$ of (2.19), the function $p(t)=\frac{w(t)}{\Omega(t)}$ satisfies*

$$p(t)\gamma_2(t)\ \textit{is decreasing}. \quad (2.28)$$

Proof. From (2.27), the function $p(t)$ satisfies

$$p'(t)+\alpha_2(t)p(t)\leq 0.$$

Therefore,

$$(p(t)\gamma_2(t))'\leq 0,$$

which proves the result. □

In our next four lemmas we give conditions to ensure that the class E_3 is empty.

Lemma 13. *Let $b\geq 1$ and (2.23) hold. If*

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t}\left(\frac{\gamma_2(\delta(x))}{\alpha^{\frac{1}{b}}(x)\Omega^2(x)}\int_{t_0}^{x}\frac{\Omega(s)\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s\right)\mathrm{d}x>\frac{b}{e}, \quad (2.29)$$

then the class E_3 is empty.

Proof. Assume there exists $\eta(t)\in E_3$. It follows from (2.27) that

$$0\geq p'(t)+\frac{1}{b\alpha^{\frac{1}{b}}(t)\Omega^2(t)}\int_{t_0}^{t}\Omega(s)\Omega(\delta(s))M_2(s)p(\delta(s))\mathrm{d}s,$$

which in view of (2.28) implies that $p(t)$ is a positive solution of the differential inequality

$$p'(t)+\left(\frac{\gamma_2(\delta(t))}{b\alpha^{\frac{1}{b}}(t)\Omega^2(t)}\int_{t_0}^{t}\frac{\Omega(s)\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s\right)p(\delta(t))\le 0.$$

But by [16, Theorem 2.1.1] and (2.29), we see that the last inequality has no positive solutions. This contradiction completes the proof of the lemma. □

Lemma 14. *Let $b\ge 1$ and (2.23) hold. If*

$$\liminf_{t\to\infty}\Omega(t)\int_t^{\infty}M_3(s)\mathrm{d}s>\frac{b}{4}, \tag{2.30}$$

where $M_3(t)=M_2(t)\frac{\Omega(\delta(t))}{\Omega(t)}\frac{\gamma_2(t)}{\gamma_2(\delta(t))}$, then the class E_3 is empty.

Proof. Assume that $\eta(t)\in E_3$. Then $w(t)=\eta''(t)$ satisfies (2.19), that is,

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)'+\frac{1}{b}M_2(t)w(\delta(t))=0,$$

by Lemma 10. By Lemma 12, the function $p(t)\gamma_2(t)=\frac{w(t)}{\Omega(t)}\gamma_2(t)$ is decreasing; using this in (2.19) yields

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)'+\frac{1}{b}M_2(t)\frac{\Omega(\delta(t))}{\gamma_2(\delta(t))}\frac{\gamma_2(t)}{\Omega(t)}w(t)\le 0.$$

But by [21, Lemma 1], $w(t)$ is a positive solution of the differential equation

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)'+\frac{1}{b}M_3(t)w(t)=0. \tag{2.31}$$

In view of [11, Theorem 3] and condition (2.30), equation (2.31) is oscillatory. This contradiction proves the lemma. □

Lemma 15. *Let $b\ge 1$ and (2.23) hold. If*

$$\limsup_{t\to\infty}\left\{\frac{\gamma_2(\delta(t))}{\Omega(\delta(t))}\int_{t_0}^{\delta(t)}\frac{\Omega(s)\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s+\gamma_2(\delta(t))\int_{\delta(t)}^{t}\frac{\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s+\Omega(\delta(t))\int_t^{\infty}M_2(s)\mathrm{d}s\right\}>b, \tag{2.32}$$

then the class E_3 is empty.

Proof. Assume that $\eta(t)\in E_3$. Then $\eta''(t)=w(t)$ satisfies (2.19), and integrating it from t to ∞ yields

$$w'(t)\ge\frac{1}{b}\frac{1}{\alpha^{\frac{1}{b}}(t)}\int_t^{\infty}M_2(s)w(\delta(s))\mathrm{d}s.$$

Integrating from $t_1 \geq t_0$ to t, gives

$$w(t) \geq \frac{1}{b}\int_{t_1}^{t} \frac{1}{\alpha^{\frac{1}{b}}(s)}\left(\int_s^{\infty} M_2(x)w(\delta(x))\mathrm{d}x\right)\mathrm{d}s,$$

and integrating by parts, we obtain

$$bw(t) \geq \Omega(t)\int_t^{\infty} M_2(s)w(\delta(s))\mathrm{d}s + \int_{t_1}^{t} \Omega(s)M_2(s)w(\delta(s))\mathrm{d}s.$$

Hence,

$$\begin{aligned} bw(\delta(t)) \geq{}& \Omega(\delta(t))\int_{\delta(t)}^{t} M_2(s)w(\delta(s))\mathrm{d}s + \Omega(\delta(t))\int_t^{\infty} M_2(s)w(\delta(s))\mathrm{d}s \\ &+ \int_{t_1}^{\delta(t)} \Omega(s)M_2(s)w(\delta(s))\mathrm{d}s. \end{aligned}$$

Taking into account that $w(t)$ is increasing, $\frac{w(t)}{\Omega(t)}\gamma_2(t)$ is decreasing, $\delta(t) \leq t$, and $\delta(t)$ is nondecreasing, we have

$$\begin{aligned} bw(\delta(t)) \geq w(\delta(t))\Bigg\{&\gamma_2(\delta(t))\int_{\delta(t)}^{t} \frac{\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s \\ &+ \Omega(\delta(t))\int_t^{\infty} M_2(s)\mathrm{d}s + \frac{\gamma_2(\delta(t))}{\Omega(\delta(t))}\int_{t_1}^{\delta(t)} \Omega(s)\frac{\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s)))}\mathrm{d}s\Bigg\}. \end{aligned}$$

This contradicts (2.32) and completes the proof of the lemma. □

Without assuming $b \geq 1$, we obtain the following result.

Lemma 16. *Assume that μ is an eventually positive solution of (E). If*

$$\liminf_{t\to\infty} \Omega^b(\delta(t))\int_t^{\infty} M_4(s)\mathrm{d}s > \frac{b^b}{(b+1)^{(b+1)}}, \tag{2.33}$$

where $M_4(t) = M(t)\frac{\Omega_2^b(\delta(t))}{\Omega^b(\delta(t))}$, then the class E_3 is empty.

Proof. Assume, to the contrary, that $\eta(t) \in E_3$. From Lemma 2, we have (2.1). Now, using (2.15) from Lemma 9 in (2.1), we have

$$(\alpha(t)(\eta'''(t))^b)' + M_4(t)(\eta''(\delta(t)))^b \leq 0.$$

Let $w(t) = \eta''(t) > 0$. Then $w(t)$ is a positive solution of the differential inequality

$$(\alpha(t)(w'(t))^b)' + M_4(t)w^b(\delta(t)) \leq 0,$$

and by [15, Lemma 1], the differential equation

$$(\alpha(t)(w'(t))^b)' + M_4(t)w^b(\delta(t)) = 0$$

also has a positive solution. But [15, Theorem 3], and condition (2.33) imply that $w(t)$ is oscillatory. This contradiction completes the proof. □

Now we are ready to state and prove our oscillation theorems. Using Lemmas 5, 6, and 8 combined with Lemmas 13, 14 and 15, we can formulate several oscillation criteria for equation (E).

Theorem 1. *Let $b \geq 1$ and conditions (2.2) and (2.23) hold. If conditions (2.9) and (2.29) hold, then (E) is oscillatory.*

Proof. Let $\mu(t)$ be a positive solution of (E). Then there exists $t_1 \geq t_0$ such that $\mu(a(t)) > 0$ and $\mu(\delta(t)) > 0$ for all $t \geq t_1$. Then by Lemma 1, the function $\eta(t)$ is positive and either $\eta(t) \in E_1$ or $\eta(t) \in E_3$ for all $t \geq t_1$.

First assume that $\eta(t) \in E_1$. Then, by Lemma 5 we get a contradiction. So next assume $\eta(t) \in E_3$. Then by Lemma 13, we again have a contradiction. This completes the proof of the theorem. □

Theorem 2. *Let $b \geq 1$ and conditions (2.2) and (2.23) hold. If conditions (2.10) and (2.30) hold, then (E) is oscillatory.*

Proof. Let $\mu(t)$ be a positive solution of (E). Then either $\eta(t) \in E_1$ or $\eta(t) \in E_3$ for all $t \geq t_1$ for some $t_1 \geq t_0$.

First assume that $\eta(t) \in E_1$. Then by Lemma 6, we are led to a contradiction. Next, assume that $\eta(t) \in E_3$. Then by Lemma 14, we again obtain a contradiction. This completes the proof. □

Theorem 3. *Let $b \geq 1$ and conditions (2.2) and (2.23) hold. If conditions (2.14) and (2.32) are satisfied, then (E) is oscillatory.*

Proof. Let $\mu(t)$ be a positive solution of (E); then $\eta(t) \in E_1$ or $\eta(t) \in E_3$ for all $t \geq t_1 \geq t_0$. If $\eta(t) \in E_1$, then Lemma 8 gives a contradiction, and if $\eta(t) \in E_3$, then Lemma 15 yields a contradiction. This proves the theorem. □

Similar to those above, we can easily prove the following theorems.

Theorem 4. *Let $b \geq 1$ and conditions (2.2), (2.9), and (2.23) hold. If condition (2.30) or (2.32) is satisfied, then (E) is oscillatory.*

Theorem 5. *Let $b \geq 1$ and conditions (2.2), (2.10), and (2.23) hold. If condition (2.29) or (2.32) is satisfied, then (E) is oscillatory.*

Theorem 6. *Let $b \geq 1$ and conditions (2.2), (2.14), and (2.23) hold. If condition (2.29) or (2.30) is satisfied, then (E) is oscillatory.*

In our final theorem, we obtain a oscillation criterion without assuming that $b \geq 1$ or that (2.23) holds.

Theorem 7. *Let conditions (2.2) and (2.33) hold. If either condition (2.9), or (2.10), or (2.14) holds, then equation (E) is oscillatory.*

Proof. The proof follows from Lemma 16 along with either Lemma 5, or 6, or 8. This completes the proof of the theorem. □

3. EXAMPLES

In this section, we provide four examples to show how our results are improvements over previously known ones. Note that the covering assumptions (D1)–(D3) hold for all of these examples.

Example 1. Consider the fourth-order delay differential equation

$$\mu^{(4)}(t)+\frac{q_0}{t^4}\mu\left(\frac{9t}{10}\right)=0,\quad t\geq 1, \tag{3.1}$$

where $q_0>0$ is a constant. Comparing this to equation (E), we see that $\alpha(t)=1$, $d(t)=0$, $b=1$, $m(t)=\frac{q_0}{t^4}$, and $\delta(t)=\frac{9t}{10}$. Simple calculations show that $M(t)=\frac{q_0}{t^4}$, $M_1(t)=\frac{q_0}{6t^2}$, $\Omega(t)\approx t$, $\Omega_1(t)\approx\frac{t^2}{2}$, $\Omega_2(t)\approx\frac{t^3}{6}$, and $M_2(t)=\frac{27q_0}{200t^2}$. Conditions (2.2) and (2.23) become

$$\int_1^\infty \frac{9}{10}s\frac{q_0}{6s^2}\mathrm{d}s=\frac{3q_0}{20}\int_1^\infty\frac{1}{s}\mathrm{d}s=\infty$$

and

$$\int_1^\infty \frac{9s}{10}\frac{27}{200}\frac{q_0}{s^2}\mathrm{d}s=\frac{243}{2000}q_0\int_1^\infty\frac{1}{s}\mathrm{d}s=\infty,$$

respectively, so they both hold. Also, we see that $\gamma_1(t)=t^{\frac{3q_0}{20}}$ and $\gamma_2=t^{\frac{243q_0}{2000}}$. Condition (2.10) is satisfied if $q_0>3$, and condition (2.30) is satisfied if $q_0\geq 4$. Hence, by Theorem 2, equation (3.1) is oscillatory if $q_0\geq 4$.

Note that [6, Corrolary 2.2] shows that (3.1) is oscillatory if $q_0>6.17$. Hence, Theorem 2 improves Corollary 2.2 in [6]. Also, as indicated in [6], Theorem 2 would also improve the results in [12, 24].

Example 2. Consider the fourth-order neutral delay equation

$$\left(\frac{1}{t}(\eta'''(t)\right)'+\frac{q_0}{t^5}\mu\left(\frac{t}{2}\right)=0,\quad t\geq 1, \tag{3.2}$$

where $\eta(t)=\mu(t)+\frac{1}{2}\mu\left(\frac{t}{3}\right)$, $b=1$, and $q_0>0$. By comparing (E) to (3.2), we see that $\alpha(t)=\frac{1}{t}$, $a(t)=\frac{t}{3}$, $d(t)=\frac{1}{2}$, $\delta(t)=\frac{t}{2}$, and $m(t)=\frac{q_0}{t^5}$. In addition, $\Omega(t)\approx\frac{t^2}{2}$, $\Omega_1(t)\approx\frac{t^3}{6}$, $\Omega_2(t)\approx\frac{t^4}{24}$, $M(t)=\frac{q_0}{2t^5}$, $M_1(t)=\frac{q_0}{24t^2}$, and $M_2(t)=\frac{q_0}{96t^3}$. Conditions (2.2) and (2.23) are satisfied since they become

$$\int_1^\infty \frac{s}{2}\frac{q_0}{24s^2}\mathrm{d}s=\frac{q_0}{48}\int_1^\infty\frac{1}{s}\mathrm{d}s=\infty$$

and

$$\int_1^\infty \frac{s^2}{8}\frac{q_0}{96s^3}\mathrm{d}s=\frac{q_0}{768}\int_1^\infty\frac{1}{s}\mathrm{d}s=\infty.$$

Furthermore, it is easy to see that $\gamma_1(t)=t^{\frac{q_0}{48}}$ and $\gamma_2(t)=t^{\frac{q_0}{768}}$. Conditions (2.10) and (2.30) become $q_0(2)^{\frac{q_0}{48}}>12$ and $q_0(2)^{\frac{q_0}{768}}>384$, respectively. Hence for $q_0\geq 297$, all conditions of Theorem 2 are satisfied. Thus, equation (3.2) is oscillatory if $q_0\geq 297$.

Note that none of the results reported in [1, 5, 6, 9, 11, 17, 25] apply to equation (3.2) since $\alpha'(t) < 0$ for all $t \geq 1$. Hence, our results are new and more applicable than those mentioned above.

Example 3. Consider the fourth-order equation

$$\eta^{(4)}(t) + \frac{q_0}{t^4}\mu(\lambda t) = 0, \quad t \geq 1, \tag{3.3}$$

where $\eta(t) = \mu(t) + d_0\mu(\theta t)$, θ, λ, $d_0 \in (0,1)$, and $q_0 > 0$ is a constant. Here we have $\alpha(t) = 1$, $b = 1$, $d(t) = d_0$, $\delta(t) = \lambda t$, $a(t) = \theta t$, and $m(t) = \frac{q_0}{t^4}$. Calculation shows that $M(t) = \frac{q_0(1-d_0)}{t^4}$, $M_1(t) = \frac{q_0(1-d_0)}{6t^2}$, $\Omega(t) \approx t$, $\Omega_1(t) \approx \frac{t^2}{2}$, $\Omega_2(t) \approx \frac{t^3}{6}$, and $M_2(t) = \frac{q_0(1-d_0)\lambda^2}{6t^2}$. Conditions (2.2) and (2.23) are easily seen to be satisfied. We also have $\gamma_1(t) = t^{q_0(1-d_0)\frac{\lambda}{6}}$ and $\gamma_2(t) = t^{q_0(1-d_0)\frac{\lambda^3}{6}}$.

For $d_0 = \lambda = \frac{1}{2}$, we see that (2.9) holds if $q_0 > 8.3218$ and the (2.29) is satisfied if $q_0 > 30$. Hence, by Theorem 1, equation (3.3) is oscillatory if $q_0 > 30$, but [8, Theorem 4], shows that (3.3) is oscillatory provided $q_0 > 72$. Therefore, Theorem 1 improves Theorem 4 of [8].

For $d_0 = 0$ and $\lambda = \frac{1}{2}$, we see that condition (2.10) holds if $q_0 \geq 3$ and (2.30) holds if $q_0 \geq 12$. Hence, by Theorem 2, equation (3.3) with $d_0 = 0$ and $\lambda = \frac{1}{2}$ is oscillatory if $q_0 > 12$. However, by [21, Theorem 2], we see that the same equation (3.3) is oscillatory if $q_0 > 36$. Thus, our Theorem 2 improves Theorem 2 of [21].

Example 4. Consider the equation

$$\left(t(\mu'''(t))^3\right)' + \frac{q_0}{t^9}\mu^3\left(\frac{t}{2}\right) = 0, \quad t \geq 1, \tag{3.4}$$

where $q_0 > 0$ is a constant. We have $\alpha(t) = t$, $d(t) = 0$, $b = 3$, $m(t) = \frac{q_0}{t^9}$, and $\delta(t) = \frac{t}{2}$. In this case, $\Omega(t) \approx \frac{3}{2}t^{\frac{2}{3}}$, $\Omega_1(t) \approx \frac{9}{10}t^{\frac{5}{3}}$, $\Omega_2(t) \approx \frac{27}{80}t^{\frac{8}{3}}$, and $M_1(t) = q_0^{\frac{1}{3}}/4t^2$. Condition (2.2) becomes

$$\int_1^\infty \frac{t}{8}\frac{q_0^{\frac{1}{3}}}{t^2}\mathrm{d}t = \frac{q_0^{\frac{1}{3}}}{8}\int_1^\infty \frac{1}{t}\mathrm{d}t = \infty,$$

that is, (2.2) holds. Condition (2.33) is satisfied if $q_0 > 70.2332$. Furthermore, $\gamma_1(t) = t^{\frac{q_0^{\frac{1}{3}}}{8}}$ and (2.10) is satisfied if $q_0 > 8$. Hence, by Theorem 7, equation (3.4) is oscillatory if $q_0 > 70.2332$.

Note that [19, Theorem 3.1] gives oscillation of (3.4) provided $q_0 > 5832$. Therefore, Theorem 7 significantly improves Theorem 3.1 in [19].

4. Conclusion

In this paper, we obtained new oscillation criteria by first obtaining some new monotonic properties of the positive solutions of the equation. Our results are applicable to a broader class of fourth-order neutral type differential equations than

previously covered in the literature. Thus, the results obtained here generalize and improve many previously known results. This is demonstrated by some particular examples.

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