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HERMITE-HADAMARD TYPE INEQUALITIES FOR THE RIGHT RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS WITH VARIABLE ORDER

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Abstract. In this paper, we prove significant results related to the Hermite-Hadamard inequality for convex functions in the framework of Riemann-Liouville fractional integrals with variable order. Two novel equalities are proved to have new fractional trapezoid and midpoint-type inequalities for the differentiable convex functions. This work provides a computational analysis of these inequalities, with numerical examples for convex functions. Additionally, it is demonstrated that the newly identified inequalities represent a generalisation of the comparable inequalities in the literature.

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1. INTRODUCTION

Convexity is a basic math concept important for learning function geometry and finding the optimal solutions. It has roots in ancient cultures; however, to the best of our knowledge, in the late 19th century, the German mathematician Karl Hermann Amandus Schwarz made a revolutionary contribution by introducing the notion of convex functions [9]. It can be used to describe real-world phenomena in optimization, economics, and physics because it has both theoretical and practical applications [11, 22].

The Hermite-Hadamard (H-H) inequality is a key concept in convex function theory and was independently established by two renowned mathematicians, C. Hermite and J. Hadamard [7, 17]. The inequality is stated as follows: If a function $\psi\colon \mathbb{I} \to \mathbb{R}$ is a convex on the interval $\mathbb{I}$, then for any $\mu, \upsilon \in \mathbb{I}$ with $\mu \neq \upsilon$, we have the following

double inequality:

$$\psi\left(\frac{\mu+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon}\psi(z)dz \leq \frac{\psi(\mu)+\psi(\upsilon)}{2}. \tag{1.1}$$

If a function is convex, the H-H inequality states that the average of the function values at the endpoints of an interval, weighted by their respective distances from the midpoint, is greater than or equal to the value of the function at the midpoint. This inequality finds extensive utility across various domains of mathematics and economics, where convexity also plays a significant role. The fractional integral variants of the inequality (1.1) given by Sarikaya et al. [21] using the integral identity as follows:

$$\psi\left(\frac{\mu+\upsilon}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\upsilon-\mu)^{\alpha}}\left[J^{\alpha}_{\mu^{+}}\psi(\upsilon)+J^{\alpha}_{\upsilon^{-}}\psi(\mu)\right] \leq \frac{\psi(\mu)+\psi(\upsilon)}{2}, \tag{1.2}$$

with $\alpha > 0$. It is obvious that the inequality (1.2) turn into (1.1) as $\alpha = 1$.

In order to establish various bounds for the right-hand side of the inequality (1.1), Dragomir and Agarwal [8] initially derive a specific identity.

Lemma 1. *Let $\psi\colon \mathbb{I}^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^{\circ}, \mu, \upsilon \in \mathbb{I}^{\circ}$ with $\mu < \upsilon$. If $\psi' \in L[\mu,\upsilon]$, then the following equality holds:*

$$\frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon}\psi(z)\mathrm{d}z - \frac{\psi(\mu)+\psi(\upsilon)}{2} = \frac{\upsilon-\mu}{2}\int_{0}^{1}(1-2\tau)\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau. \tag{1.3}$$

Kirmaci [15], introduces a particular identity and then uses it to establish a variety of bounds for the left-hand side of the inequality (1.1).

Lemma 2. *If $\psi\colon \mathbb{I}^{\circ} \to \mathbb{R}$ is differentiable on $\mathbb{I}^{\circ}$ and $\psi' \in L[\mu,\upsilon]$, then we obtain*

$$\begin{aligned}&\psi\left(\frac{\mu+\upsilon}{2}\right) - \frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon}\psi(z)\mathrm{d}z \\ &= (\upsilon-\mu)\left[\int_{0}^{1/2}\tau\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau + \int_{1/2}^{1}(\tau-1)\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right], \forall\, \mu,\upsilon \in \mathbb{I}^{\circ}.\end{aligned} \tag{1.4}$$

In 1998, Dragomir and Agarwal [8] presented two inequalities applicable to differentiable convex functions. They established a link with the well-known H-H integral inequality that holds for convex functions and addressed error estimation concerns related to the trapezoidal formula. Aamir et al. [3] have presented two new fractional H-H type inequalities for convex functions. They also derived new Simpson's type inequalities for differentiable convex functions and proved two more Simpson's type inequalities using power mean and Hölder's inequalities. Xie et al. [24] have established H-H type inequalities for Riemann-Liouville fractional integrals and derived some new Simpson's and Ostrowski's type inequalities for differentiable convex mapping. Abbas et al. [1] have presented H-H type inequalities for exponentially sub-additive functions by utilizing Riemann-Liouville fractional integrals. Budak et al. [5] have derived certain q-H-H type inequalities and build a new q-H-H type

inequalities using the product of two coordinate convex functions. You et al. [25] have introduced a new version of H-H-Mercer inequalities for harmonically convex functions, and discusses special means for addressing newly discovered inequalities. Jleli and Sameet [14] have established new inequalities of the H-H type, these inequalities involve fractional integrals in relation to another function. Such fractional integrals are generalisations of the Riemann-Liouville and Hadamard fractional integrals. Iscan has formulated the H-H-Fejér type inequality for fractional integrals. As well as derived an integral identity for H-H-Fejér type integral, which is associated with fractional integrals [13]. By using fractional Riemann-Liouville integrals for functions whose absolute values of the second derivatives are convex, Bayraktar has defined the identity and derived several generalised inequalities of the H-H type [4]. Kunt et al. [16] presented a novel method for establishing new fractional H-H type inequalities using only the right Riemann-Liouville fractional integral. They also proved two additional equalities for fractional trapezoid and midpoint-type inequalities for differentiable convex functions. For an in-depth analysis of the various applications arising from the H-H inequality, readers are encouraged to explore references [2, 6, 12, 19, 23].

Definition 1 ([20])**.** A function ψ defined on the interval $[\mu,\upsilon]$, where $\mu \leq \upsilon$, and assuming $\psi \in L_1[\mu,\upsilon]$, the Riemann-Liouville fractional integral of order $\alpha > 0$ with $\mu \geq 0$ is defined as follows:

$$\mathfrak{J}^{\alpha}_{\mu+}\psi(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\mu}^{\varkappa}(\varkappa-\tau)^{\alpha-1}\psi(\tau)\mathrm{d}\tau, \quad \varkappa > \mu$$

and

$$\mathfrak{J}^{\alpha}_{\upsilon-}\psi(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\upsilon}(\tau-\varkappa)^{\alpha-1}\psi(\tau)\mathrm{d}\tau, \quad \varkappa < \upsilon.$$

Here, $\Gamma(\alpha)$ is the well-known Gamma function and $\mathfrak{J}^{0}_{\mu+}\psi(\varkappa) = \mathfrak{J}^{0}_{\upsilon-}\psi(\varkappa) = \psi(\varkappa)$.

Definition 2 ([10])**.** Let $0 < \alpha(\varkappa) < 1$ for all $\varkappa \in [\mu,\upsilon]$ and $\psi \in L_1[\mu,\upsilon]$. Then the left and right Riemann-Liouville integrals of variable fractional order $\alpha(\varkappa)$ defined by

$$\mathfrak{L}^{\alpha(\varkappa)}_{\mu+}\psi(\varkappa) = \frac{1}{\Gamma[\alpha(\varkappa)]}\int_{\mu}^{\varkappa}(\varkappa-\tau)^{\alpha(\varkappa)-1}\psi(\tau)\mathrm{d}\tau, \quad \Re\alpha(x) > 0$$

and

$$\mathfrak{L}^{\alpha(\varkappa)}_{\upsilon-}\psi(\varkappa) = \frac{1}{\Gamma[\alpha(\varkappa)]}\int_{\varkappa}^{\upsilon}(\tau-\varkappa)^{\alpha(\varkappa)-1}\psi(\tau)\mathrm{d}\tau, \quad \Re\alpha(\varkappa) > 0.$$

The classical Riemann-Liouville fractional integral has a fixed constant order of integration α, while the variable order definition allows the order of integration $\alpha(\varkappa)$ to vary as a function of another variable $\varkappa$. In this case $\alpha(\varkappa)$ is depend on the integral domain, resulting in different values at distinct points within this domain. This dynamic interplay between α and the position within the interval provides increased

flexibility and paves the way for graphical analysis, ultimately enhancing our understanding of the applicability of newly introduced generalized inequalities. The variability of the function $\alpha(\varkappa)$ is greatly impacted by the integral domain, resulting in a wide range of values for α over the domain. Consequently, the interplay between the integration order and the place inside the interval offers a degree of versatility that is absent in fixed-order fractional integrals.

The main goal of this study is to generalize H-H inequality by using Riemann-Liouville fractional integral with variable order $\alpha(\varkappa)$, depending upon the function and interval point, which vary simultaneously. Our method precisely explains how functions behave at specific points within intervals. Additionally, we introduce two novel identities applicable to differentiable convex functions, which give fractional trapezoid and midpoint-type inequalities.

2. RESULTS

The following lemma is required to prove the next result.

Lemma 3. *For any integrable mapping $\psi\colon [\mu,\upsilon]\to\mathbb{R}$ and let f be a convex mapping, the following inequality is true:*

$$\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa\geq f\left(\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\psi(\varkappa)\mathrm{d}\varkappa\right) \tag{2.1}$$

with $\alpha(\mu)>0$.

Proof. Since f is a convex mapping, therefore for every point $(\varkappa_0,f(\varkappa_0))$ on the graph of f there is a line $y=\beta(\varkappa-\varkappa_0)+f(\varkappa_0)$ such that $f(\varkappa)\geq\beta(\varkappa-\varkappa_0)+f(\varkappa_0)$ for all $\varkappa$ in the domain of f. Now, we assume that

$$x_0=\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\psi(\varkappa)\mathrm{d}\varkappa$$

and integrate the inequality

$$f(\psi(\varkappa))\geq\beta(\psi(\varkappa)-\varkappa_0)+f(\varkappa_0),$$

$$\begin{aligned}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa\geq{}&\beta\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa\\&-\beta\varkappa_0\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\mathrm{d}\varkappa\\&+f(\varkappa_0)\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\mathrm{d}\varkappa.\end{aligned}$$

By utilizing the value of $\varkappa_0$, we observe that

$$\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa\geq\beta(\varkappa_0-\varkappa_0)\frac{(\upsilon-\mu)^{\alpha(\mu)}}{\alpha(\mu)}+\frac{(\upsilon-\mu)^{\alpha(\mu)}}{\alpha(\mu)}f(\varkappa_0).$$

Thus,

$$\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa \geq f\left(\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\psi(\varkappa)\mathrm{d}\varkappa\right)$$

and the proof is completed. □

Theorem 1. *Let* $\psi\colon [\mu,\upsilon]\to\mathbb{R}$ *be a positive function with* $0\leq\mu<\upsilon$, $\alpha(\mu)>0$ *and* $\psi\in L_1[\mu,\upsilon]$. *If* ψ *is a convex function on* $[\mu,\upsilon]$, *then the following inequalities for fractional integrals hold:*

$$\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)\leq\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon^-}^{\alpha(\mu)}\psi(\mu)\leqslant\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}. \tag{2.2}$$

Proof. Since ψ is convex, therefore by using Lemma 3, we have

$$\psi\left(\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\varkappa\mathrm{d}\varkappa\right)\leq\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\psi(\varkappa)\mathrm{d}\varkappa. \tag{2.3}$$

One can easily observe that

$$\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)\leq\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon^-}^{\alpha(\mu)}\psi(\mu). \tag{2.4}$$

Thus the first inequality of (2.2) is proved.

To prove the second inequality, we again use the convexity and we have

$$\psi(\varkappa)\leq\psi(\upsilon)+\frac{\psi(\upsilon)-\psi(\mu)}{\upsilon-\mu}(\varkappa-\upsilon). \tag{2.5}$$

Multiplying both sides of (2.5) by $\frac{1}{\Gamma(\alpha(\mu))}(\varkappa-\mu)^{\alpha(\mu)-1}$, then integrating the resulting inequality with respect to $\varkappa$ over $[0,1]$, we obtain

$$\mathbb{J}_{\upsilon^-}^{\alpha(\mu)}\psi(\mu)\leq\frac{(\upsilon-\mu)^{\alpha(\mu)}}{\Gamma(\alpha(\mu)+1)}\left[\psi(\upsilon)-\frac{\psi(\upsilon)-\psi(\mu)}{\alpha(\mu)+1}\right],$$

after simplification, we have

$$\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon^-}^{\alpha(\mu)}\psi(\varkappa)\leq\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}, \tag{2.6}$$

by adding equations (2.4) and (2.6), we get the desired inequality (2.2). □

Remark 1. If we assign $\alpha(\varkappa)=1$, in Theorem 1, then we find inequality (1.1).

Remark 2. If we choose $\alpha(\varkappa)=\beta$ (constant), then we have Theorem 2.1 proved by Kunt et al. in [16].

Lemma 4. *Let $\psi\colon \mathbb{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ$, $\mu,\upsilon \in \mathbb{I}^\circ$ with $\mu<\upsilon$ and $\alpha(\mu)>0$. If $\psi' \in L[\mu,\upsilon]$, then the following equality for the right Riemann-Liouville fractional integral holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\\&=\frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left[1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right]\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau.\end{aligned}\tag{2.7}$$

Proof. If we apply the integration by parts to the right hand side of (2.7), we have

$$\begin{aligned}&\frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left[1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right]\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\\&=(\upsilon-\mu)\left[\frac{1}{\alpha(\mu)+1}\int_0^1\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau-\int_0^1\tau^{\alpha(\mu)}\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right]\\&=\left[\frac{\psi(\mu)-\psi(\upsilon)}{\alpha(\mu)+1}+\psi(\upsilon)-\alpha(\mu)\int_0^1\tau^{\alpha(\mu)-1}\psi((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right]\\&=\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}.\end{aligned}$$

This completes the proof. □

Remark 3. If we take $\alpha(\varkappa)=1$, in Lemma 4, then we obtain Lemma 1.

Remark 4. If we choose $\alpha(\varkappa)=\beta$ (constant), then we will get [16, Lemma 3.1].

Lemma 5. *Let $\psi\colon \mathbb{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ, \mu,\upsilon \in \mathbb{I}^\circ$ with $\mu<\upsilon$ and $\alpha(\mu)>0$. If $\psi' \in L[\mu,\upsilon]$, then the following equality for the right Riemann-Liouville fractional integral holds:*

$$\begin{aligned}&\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)\\&=(\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau+\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(\tau^{\alpha(\mu)}-1\right)\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right].\end{aligned}\tag{2.8}$$

Proof. If we apply the integration by parts to the right hand side of (2.8), we find

$$\begin{aligned}&(\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau+\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(\tau^{\alpha(\mu)}-1\right)\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right]\\&=(\upsilon-\mu)\left[\int_0^1\tau^{\alpha(\mu)}\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau-\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right]\\&=\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\nu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu).\end{aligned}$$

This completes the proof. □

Remark 5. If we take $\alpha(\varkappa)=1$, in Lemma 5, it becomes Lemma 2.

Remark 6. If we assign $\alpha(\varkappa)=\beta$ (constant), then we will get [16, Lemma 3.2].

3. Midpoint and trapezoid inequalities

By using Lemma 4 and Lemma 5, we will obtain some new right Riemann-Liouville fractional trapezoid and midpoint-type inequalities in this section.

Theorem 2. *Let $\psi\colon \mathbb{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ, \mu, \upsilon \in \mathbb{I}^\circ$ with $\mu<\upsilon$ and $\alpha(\mu)>0$. If $|\psi'|$ is convex on $[\mu,\upsilon]$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|$$
$$\leq \frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\mathbb{A}_1(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_2(\alpha(\mu))\left|\psi'(\upsilon)\right|+\mathbb{A}_3(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_4(\alpha(\mu))\left|\psi'(\upsilon)\right|\right],$$

where

$$\mathbb{A}_1(\alpha(\mu))=\frac{\alpha(\mu)\left(2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{1}{\alpha(\mu)}-1}-1\right)}{2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}},$$
$$\mathbb{A}_2(\alpha(\mu))=\frac{\alpha(\mu)}{2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}},$$
$$\mathbb{A}_3(\alpha(\mu))=\frac{\alpha(\mu)\left(2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{1}{\alpha(\mu)}-1}-1-(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}\right)}{2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}},$$
$$\mathbb{A}_4(\alpha(\mu))=\frac{\alpha(\mu)\left(1+(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}\right)}{2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}}.$$

Proof. Using Lemma 4 and the convexity of $|\psi'|$, we acquire

$$\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|$$
$$\leq \frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau$$
$$\leq \frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\int_0^{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}\left(1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right)\left[(1-\tau)\left|\psi'(\mu)\right|+\tau\left|\psi'(\upsilon)\right|\right]\mathrm{d}\tau\right.$$
$$\left.+\int_{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)}}}^1\left((\alpha(\mu)+1)\tau^{\alpha(\mu)}-1\right)\left[(1-\tau)\left|\psi'(\mu)\right|+\tau\left|\psi'(\upsilon)\right|\right]\mathrm{d}\tau\right]$$
$$\leq \frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\mathbb{A}_1(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_2(\alpha(\mu))\left|\psi'(\upsilon)\right|+\mathbb{A}_3(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_4(\alpha(\mu))\left|\psi'(\upsilon)\right|\right].$$

Finally, the proof of Theorem 2 is completed. □

Remark 7. If we take $\alpha(\varkappa)=1$, in Theorem 2, then we obtain [8, Theorem 2.2].

Remark 8. If we choose $\alpha(\varkappa)=\beta$ (constant), then we find Theorem 4.1 in [16].

Theorem 3. *Let $\psi\colon \mathbb{I}^{\circ}\to\mathbb{R}$ be a differentiable mapping on $\mathbb{I}^{\circ},\mu,\upsilon\in\mathbb{I}^{\circ}$ with $\mu<\upsilon$. If $|\psi'|^{q}$ is convex on $[\mu,\upsilon]$ for $q\geq 1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\left(\frac{2\alpha(\mu)}{(\alpha(\mu)+1)^{1+\frac{1}{\alpha(\mu)}}}\right)^{1-\frac{1}{q}}$$

$$\times\left(\mathbb{A}_1(\alpha(\mu))\left|\psi'(\mu)\right|^{q}+\mathbb{A}_2(\alpha(\mu))\left|\psi'(\upsilon)\right|^{q}+\mathbb{A}_3(\alpha(\mu))\left|\psi'(\mu)\right|^{q}+\mathbb{A}_4(\alpha(\mu))\left|\psi'(\upsilon)\right|^{q}\right)^{\frac{1}{q}},$$

where $\mathbb{A}_1(\alpha(\mu))-\mathbb{A}_4(\alpha(\mu))$ are the same as in Theorem 2 and $\alpha(\mu)>0$.

Proof. Using Lemma 4, power mean inequality and the convexity of $|\psi'|^{q}$, we establish

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|\\
&\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau\\
&\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\left(\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\mathrm{d}\tau\right)^{1-\frac{1}{q}}\right.\\
&\quad\left.\times\left(\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^{q}\,\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\left(\frac{2\alpha(\mu)}{(\alpha(\mu)+1)^{1+\frac{1}{\alpha(\mu)}}}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\mathbb{A}_1(\alpha(\mu))\left|\psi'(\mu)\right|^{q}+\mathbb{A}_2(\alpha(\mu))\left|\psi'(\upsilon)\right|^{q}+\mathbb{A}_3(\alpha(\mu))\left|\psi'(\mu)\right|^{q}+\mathbb{A}_4(\alpha(\mu))\left|\psi'(\upsilon)\right|^{q}\right)^{\frac{1}{q}}.
\end{aligned}$$

Finally, the proof of Theorem 3 is accomplished. □

Remark 9. If we assign $\alpha(\varkappa)=1$, in Theorem 3, then we have Theorem 1 in [18].

Remark 10. If we choose $\alpha(\varkappa)=\beta$ (constant), then we find Theorem 4.2 in [16].

Theorem 4. *Let $\psi\colon \mathbb{I}^{\circ}\to\mathbb{R}$ be a differentiable mapping on $\mathbb{I}^{\circ},\mu,\upsilon\in\mathbb{I}^{\circ}$ with $\mu<\upsilon$. If $|\psi'|^{q}$ is convex on $[\mu,\upsilon]$ for $q>1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|\\
&\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\left(\mathbb{A}_5(\alpha(\mu),p)+\mathbb{A}_6(\alpha(\mu),p)\right)^{\frac{1}{p}}\left(\frac{|\psi'(\mu)|^{q}+|\psi'(\upsilon)|^{q}}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\mathbb{A}_5(\alpha(\mu),p)=\int_0^{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}\left(1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right)^p\,\mathrm{d}\tau,$$

$$\mathbb{A}_6(\alpha(\mu),p)=\int_{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}^1\left((\alpha(\mu)+1)\tau^{\alpha(\mu)}-1\right)^p\,\mathrm{d}\tau,$$

with $\frac{1}{p}+\frac{1}{q}=1$ *and* $\alpha(\mu)>0$.

Proof. Using Lemma 4, Hölder inequality and the convexity of $|\psi'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|\\
&\le\frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau\\
&\le\frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\left(\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|^p\,\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&\le\frac{\upsilon-\mu}{\alpha(\mu)+1}\left(\int_0^{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}\left(1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right)^p\,\mathrm{d}\tau\right.\\
&\quad\left.+\int_{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}^1\left((\alpha(\mu)+1)\tau^{\alpha(\mu)}-1\right)^p\,\mathrm{d}\tau\right)^{\frac{1}{p}}\times\left(\int_0^1\left[\tau\left|\psi'(\mu)\right|^q+(1-\tau)\left|\psi'(\upsilon)\right|^q\right]\mathrm{d}\tau\right)^{\frac{1}{q}}\\
&\le\frac{\upsilon-\mu}{\alpha(\mu)+1}\left[(\mathbb{A}_5(\alpha(\mu),p)+\mathbb{A}_6(\alpha(\mu),p))^{\frac{1}{p}}\left(\frac{|\psi'(\mu)|^q+|\psi'(\upsilon)|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Thus, the proof of Theorem 4 is finished. □

Remark 11. If we assign $\alpha(\varkappa)=1$, then we obtain Theorem 2.3 in [8].

Remark 12. If we choose $\alpha(\varkappa)=\beta$ (constant), then we have Theorem 4.3 in [16].

Theorem 5. *Let* $\psi\colon\mathbb{I}^\circ\to\mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^\circ,\mu,\upsilon\in\mathbb{I}^\circ$ *with* $\mu<\upsilon$ *and with* $\alpha(\mu)>0$. *If* $|\psi'|$ *is convex on* $[\mu,\upsilon]$, *then the following right Riemann-Liouville fractional integral inequality holds:*

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)\right|\\
&\le(\upsilon-\mu)\left[\mathbb{A}_7(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_8(\alpha(\mu))\left|\psi'(\upsilon)\right|+\mathbb{A}_9(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_{10}(\alpha(\mu))\left|\psi'(\upsilon)\right|\right],
\end{aligned}$$

where

$$\mathbb{A}_7(\alpha(\mu))=\frac{2\left(\frac{\alpha(\mu)}{\alpha(\mu)+1}\right)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)(\alpha(\mu)+2)},\quad\mathbb{A}_8(\alpha(\mu))=\frac{\alpha(\mu)\left(\frac{\alpha(\mu)}{\alpha(\mu)+1}\right)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)(\alpha(\mu)+2)},$$

$$\mathbb{A}_9(\alpha(\mu))=\frac{4(\alpha(\mu))^{\alpha(\mu)+1}-\alpha(\mu)(\alpha(\mu)+1)^{\alpha(\mu)}}{2(\alpha(\mu)+1)^{\alpha(\mu)+2}(\alpha(\mu)+2)},$$

$$\mathbb{A}_{10}(\alpha(\mu)) = \frac{\alpha(\mu)\left(2(\alpha(\mu))^{\alpha(\mu)} + (\alpha(\mu)+1)^{\alpha(\mu)}\right)}{2(\alpha(\mu)+1)^{\alpha(\mu)+2}(\alpha(\mu)+2)}.$$

Proof. Using Lemma 5 and the convexity of $|\psi'|$, we acquire

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right) - \frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)\right| \\
&\leq (\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}} \tau^{\alpha(\mu)}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|d\tau + \int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1 \left(1-\tau^{\alpha(\mu)}\right)\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|d\tau\right] \\
&\leq (\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}} \tau^{\alpha(\mu)}\left[(1-\tau)\left|\psi'(\mu)\right|\right.\right. \\
&\quad \left.\left.+\tau\left|\psi'(\upsilon)\right|\right]d\tau + \int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1 \left(1-\tau^{\alpha(\mu)}\right)\left[(1-\tau)\left|\psi'(\mu)\right|+\tau\left|\psi'(\upsilon)\right|\right]d\tau\right] \\
&\leq (\upsilon-\mu)\left[\frac{\alpha(\mu)\left(\frac{\alpha(\mu)}{\alpha(\mu)+1}\right)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)(\alpha(\mu)+2)}\left|\psi'(\mu)\right| + \frac{2\left(\frac{\alpha(\mu)}{\alpha(\mu)+1}\right)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)(\alpha(\mu)+2)}\left|\psi'(\upsilon)\right|\right. \\
&\quad \left.+\frac{\alpha(\mu)\left(2\alpha^{\alpha(\mu)}+(\alpha(\mu)+1)^{\alpha(\mu)}\right)}{2(\alpha(\mu)+1)^{\alpha(\mu)+2}(\alpha(\mu)+2)}\left|\psi'(\mu)\right| + \frac{4\alpha(\mu)^{\alpha(\mu)+1}-\alpha(\mu)(\alpha(\mu)+1)^{\alpha(\mu)}}{2(\alpha(\mu)+1)^{\alpha(\mu)+2}(\alpha(\mu)+2)}\left|\psi'(\upsilon)\right|\right] \\
&\leq (\upsilon-\mu)\left[\mathbb{A}_7(\alpha(\mu))\left|\psi'(\mu)\right| + \mathbb{A}_8(\alpha\mu)\left|\psi'(\upsilon)\right| + \mathbb{A}_9(\alpha(\mu))\left|\psi'(\mu)\right| + \mathbb{A}_{10}(\alpha(\mu))\left|\psi'(\upsilon)\right|\right].
\end{aligned}$$

Thus, the proof of Theorem 5 is finished. □

Remark 13. If we choose $\alpha(\varkappa) = 1$, then we obtain Theorem 2.2 in [15].

Remark 14. If we choose $\alpha(\varkappa) = \beta$ (constant), then we get Theorem 4.4 in [16].

Theorem 6. *Let $\psi\colon \mathbb{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ, \mu, \upsilon \in \mathbb{I}^\circ$ with $\mu < \upsilon$. If $|\psi'|^q$ is convex on $[\mu,\upsilon]$ for $q \geq 1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right) - \frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)\right| \leq (\upsilon-\mu)\left(\frac{(\alpha(\mu))^{\alpha(\mu)+1}}{(\alpha(\mu)+1)^{\alpha(\mu)+2}}\right)^{1-\frac{1}{q}} \times \\
&\left[\left(\mathbb{A}_7(\alpha(\mu))\left|\psi'(\mu)\right|^q + \mathbb{A}_8(\alpha(\mu))\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}} + \left(\mathbb{A}_9(\alpha(\mu))\left|\psi'(\mu)\right|^q + \mathbb{A}_{10}(\alpha(\mu))\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\mathbb{A}_7(\alpha(\mu)) - \mathbb{A}_{10}(\alpha(\mu))$ are the same as in Theorem 5 and $\alpha(\mu) > 0$.

Proof. Using Lemma 5, power mean inequality and the convexity of $|\psi'|^q$, we have

$$\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right) - \frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi\mu\right|$$

$$\leq (\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|d\tau+\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|d\tau\right]$$

$$\leq (\upsilon-\mu)\left[\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}d\tau\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,d\tau\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,d\tau\right)^{\frac{1}{q}}\right]$$

$$\leq (\upsilon-\mu)\left(\frac{\alpha(\mu)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)^{\alpha(\mu)+2}}\right)^{1-\frac{1}{q}}\left[\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\left[(1-\tau)\left|\psi'(\mu)\right|^q+\tau\left|\psi'(\upsilon)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)\left[(1-\tau)\left|\psi'(\mu)\right|^q+\tau\left|\psi'(\upsilon)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right]$$

$$\leq (\upsilon-\mu)\left(\frac{(\alpha(\mu))^{\alpha(\mu)+1}}{(\alpha(\mu)+1)^{\alpha(\mu)+2}}\right)^{1-\frac{1}{q}}\times$$

$$\left[\left(\mathbb{A}_7(\alpha(\mu))\left|\psi'(\mu)\right|^q+\mathbb{A}_8(\alpha(\mu))\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}+\left(\mathbb{A}_9(\alpha(\mu))\left|\psi'(\mu)\right|^q+\mathbb{A}_{10}(\alpha(\mu))\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right].$$

Hence, the proof of Theorem 6 is completed. □

Remark 15. If we take $\alpha(\varkappa)=1$ in Theorem 6, we obtain the following midpoint-type inequality

$$\left|\psi\left(\frac{\mu+\upsilon}{2}\right)-\frac{1}{\upsilon-\mu}\int_\mu^\upsilon\psi(z)dz\right|$$

$$\leq\frac{\upsilon-\mu}{8}\left[\left(\frac{\left|\psi'(\mu)\right|^q+2\left|\psi'(\upsilon)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|\psi'(\mu)\right|^q+\left|\psi'(\upsilon)\right|^q}{3}\right)^{\frac{1}{q}}\right].$$

Remark 16. If we choose $\alpha(\varkappa)=\beta$ (constant), then we have Theorem 4.5 in [16].

Theorem 7. *Let $\psi\times\mathbb{I}^\circ\to\mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ,\mu,\upsilon\in\mathbb{I}^\circ$, $\mu<\upsilon$, $\alpha(\mu)>0$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|\psi'|^q$ is convex on $[\mu,\upsilon]$ for $q>1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon-}^{\alpha(\mu)}\psi(\mu)\right|$$

$$\leq(\upsilon-\mu)\left[\mathbb{A}_{11}^{\frac{1}{p}}(\alpha(\mu),p)\left(\frac{(\alpha(\mu))^2}{2(\alpha(\mu)+1)^2}\left|\psi'(\mu)\right|^q+\frac{\alpha(\mu)^2+2\alpha(\mu)}{2(\alpha(\mu)+1)^2}\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\mathbb{A}_{12}^{\frac{1}{p}}(\alpha(\mu),p)\left(\frac{2\alpha(\mu)+1}{2(\alpha(\mu)+1)^2}\left|\psi'(\mu)\right|^q+\frac{1}{2(\alpha(\mu)+1)^2}\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\Bigg],$$

where

$$\mathbb{A}_{11}(\alpha(\mu),p)=\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)p}\,\mathrm{d}\tau,\quad \mathbb{A}_{12}(\alpha(\mu),p)=\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)^p\,\mathrm{d}\tau.$$

Proof. Using Lemma 5, Hölder inequality and the convexity of $|\psi'|^q$, we obtain

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon-}^{\alpha(\mu)}\psi(\mu)\right|\\
&\le(\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau+\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau\right]\\
&\le(\upsilon-\mu)\left[\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)p}\,\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)^p\,\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&\le(\upsilon-\mu)\left[\mathbb{A}_{11}^{\frac{1}{p}}(\alpha(\mu),p)\left(\frac{(\alpha(\mu))^2}{2(\alpha(\mu)+1)^2}\left|\psi'(\mu)\right|^q+\frac{(\alpha(\mu))^2+2\alpha(\mu)}{2(\alpha(\mu)+1)^2}\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\mathbb{A}_{12}^{\frac{1}{p}}(\alpha(\mu),p)\left(\frac{2\alpha(\mu)+1}{2(\alpha(\mu)+1)^2}\left|\psi'(\mu)\right|^q+\frac{1}{2(\alpha(\mu)+1)^2}\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Remark 17. In Theorem 7, when $\alpha(\varkappa)=1$ is chosen, we get Theorem 2.3 in [15].

Remark 18. If we choose $\alpha(\varkappa)=\beta$ (constant), then we have Theorem 4.6 in [16].

4. Numerical Examples

In this section, we represent some examples of variable order functions.

Example 1. We define a convex mapping $\psi(\varkappa)=\varkappa^2$. Then, from inequality (2.2) for $\alpha(\varkappa)=\sin\varkappa$, $\mu=1$ and $\upsilon=2$, we have

$$\psi\left(\frac{(\sin(1))2+1}{\sin(1)+1}\right)=2\frac{37087}{100000},\tag{4.1}$$

$$\frac{\Gamma(\sin(1)+1)}{(2-1)^{\sin(1)}}\mathbb{J}_{2-}^{\sin(1)}\psi(1)=2\frac{4201}{20000}\tag{4.2}$$

and

$$\frac{\sin(1)\psi(2)+\psi(1)}{\sin(1)+1}=2\frac{767}{6250}. \tag{4.3}$$

From (4.1), (4.2) and (4.3), we attain

$$2\frac{37807}{100000}<2\frac{4201}{20000}<2\frac{767}{6250}.$$

Which shows that the inequality (2.2) is valid for convex functions.

Example 2. We define a convex mapping $\psi(\varkappa)=\varkappa^2$. Then, from inequality (2.2) for $\alpha(\varkappa)=\cos\varkappa, \mu=0$ and $\upsilon=1$, we have

$$\psi\left(\frac{(\cos(0))1+0}{\cos(0)+1}\right)=\frac{1}{4}, \tag{4.4}$$

$$\frac{\Gamma(\cos(0)+1)}{(1-0)^{\cos(0)}}\mathbb{J}_{1^-}^{\cos(0)}\psi(0)=\frac{1}{3} \tag{4.5}$$

and

$$\frac{\cos(0)\psi(1)+\psi(0)}{\cos(0)+1}=\frac{1}{2}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we achieve

$$\frac{1}{4}<\frac{1}{3}<\frac{1}{2}.$$

Which shows that the inequality (2.2) is valid for convex functions.

Remark 19. Both Examples 1 and 2 demonstrate the validity of the inequalities stated in Theorem 1. This conclusion is further emphasized by a comparative study, as shown in Figures 1 and 2.

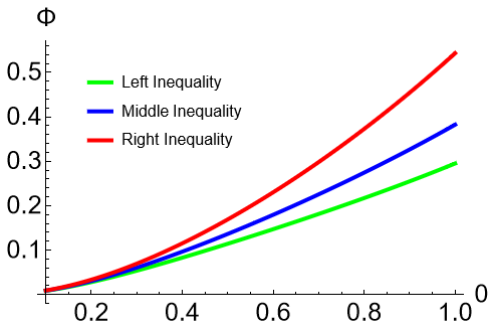

FIGURE 1. In 2D-Plot when μ lies between 1 and 2 and υ is fixed.

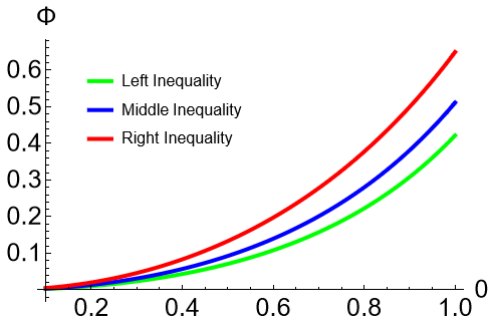

FIGURE 2. In 2D-Plot when μ lies between 0 and 1 and υ is fixed.

Remark 20. When we see 2-D plot as 'μ' and 'υ' lies between 1 and 2 we can clearly see from Figure 1, inequality stated in Theorem 1 is satisfied.

Remark 21. When we see 2-D plot as 'μ' and 'υ' lies between 0 and 1, we can clearly see from Figure 2, inequality stated in Theorem 1 is satisfied.

CONCLUSION

This research has contributed novel findings to the realm of H-H inequalities, particularly in their application to convex functions and fractional integrals with variable order. We have established new H-H type inequalities through an examination of convex functions and fractional integrals, as well as identified innovative fractional trapezoid and midpoint-type inequalities. Furthermore, we have provided numerical examples and graphical analysis of these newly established inequalities. Future research endeavors may focus on expanding the scope of these inequalities and exploring their implications in diverse mathematical disciplines. Additionally, researchers may apply the Riemann-Liouville fractional integral with variable order for different types of inequalities and explore these results in a real-world scenario.

REFERENCES

[1] S. Abbas, M. Ali, A. Hanif, and H. Budak, "Hermite-Hadamard like inequalities for exponentially subadditive functions via fractional integrals," *Malaysian Journal of Mathematical Sciences*, vol. 17, no. 1, 2023, doi: 10.47836/mjms.17.1.04.

[2] M. A. Ali, C. S. Goodrich, and H. Budak, "Some new parameterized Newton-type inequalities for differentiable functions via fractional integrals," *Journal of Inequalities and Applications*, vol. 2023, no. 1, pp. 1–17, 2023, doi: 10.1186/s13660-023-02953-x.

[3] M. A. Ali, J. Soontharanon, H. Budak, T. Sitthiwirattham, and M. Fečkan, "Fractional Hermite-Hadamard inequality and error estimates for Simpson's formula through convexity with respect to a pair of functions," *Miskolc Mathematical Notes*, vol. 24, no. 2, pp. 553–568, 2023, doi: 10.18514/MMN.2023.4214.

[4] B. Bayraktar, "Some new generalizaitons of Hadamard-type midpoint inequalities involving fractional integrals," *Problemy Analiza– Issues of Analysis*, vol. 9, no. 27-3, pp. 66–82, 2020, doi: 10.15393/j3.art.2020.8270.

[5] H. Budak, M. A. Ali, N. Alp, and H. M. Awais, "Some new q-Hermite-Hadamard type inequalities for the product of convex functions," *Journal of Interdisciplinary Mathematics*, vol. 25, no. 8, pp. 2141–2166, 2022, doi: 10.1080/09720502.2021.1925452.

[6] H. Budak, P. Kösem, and H. Kara, "On new Milne-type inequalities for fractional integrals," *Journal of Inequalities and Applications*, vol. 2023, no. 1, pp. 1–15, 2023, doi: 10.1186/s13660-023-02921-5.

[7] M. L. Cartwright, *Jacques Hadamard, 1865-1963*. The Royal Society London, 1965. doi: 10.1098/rsbm.1965.0005.

[8] S. Dragomir and R. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998, doi: 10.1016/S0893-9659(98)00086-X.

[9] R. J. Dwilewicz, "A short history of convexity," *Differential Geometry–Dynamical Systems*, 2009.

[10] R. Garrappa, A. Giusti, and F. Mainardi, "Variable-order fractional calculus: A change of perspective," *Communications in Nonlinear Science and Numerical Simulation*, vol. 102, p. 105904, 2021, doi: 10.1016/j.cnsns.2021.105904.

[11] J. Green, "Recent applications of convex functions," *The American Mathematical Monthly*, vol. 61, no. 7P1, pp. 449–454, 1954, doi: 10.1137/050641867.

[12] K. Hasan, H. Budak, M. A. Ali, and F. Hezenci, "On inequalities of Simpson's type for convex functions via generalized fractional integrals," *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 71, no. 3, pp. 806–825, 2022, doi: 10.31801/cfsuasmas.1004300.

[13] I. Işcan, "Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals," *arXiv preprint arXiv:1404.7722*, 2014, doi: 10.48550/arXiv.1404.7722.

[14] M. Jleli and B. Samet, "On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function," *J. Nonlinear Sci. Appl*, vol. 9, no. 3, pp. 1252–1260, 2016, doi: 10.22436/jnsa.009.03.50.

[15] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004, doi: 10.1016/S0096-3003(02)00657-4.

[16] M. Kunt, D. Karapınar, S. Turhan, and I. Iscan, "The right Rieaman-Liouville fractional Hermite-Hadamard type inequalities for convex functions," *J. Inequal. Spec. Funct*, vol. 9, no. 1, pp. 45–57, 2018.

[17] S. Mandelbrojt and L. Schwartz, "Jacques Hadamard (1865-1963)," *Bull. Amer. Math. Soc.*, vol. 71, no. 1, pp. 107–129, 1965.

[18] C. E. Pearce and J. Pečarić, "Inequalities for differentiable mappings with application to special means and quadrature formulae," *Applied Mathematics Letters*, vol. 13, no. 2, pp. 51–55, 2000, doi: 10.1016/S0893-9659(99)00164-0.

[19] B. Ross, "The development of fractional calculus 1695-1900," *Historia Mathematica*, vol. 4, no. 1, pp. 75–89, 1977, doi: 10.1016/0315-0860(77)90039-8.

[20] S. G. Samko and B. Ross, "Integration and differentiation to a variable fractional order," *Integral Transforms and Special Functions*, vol. 1, no. 4, pp. 277–300, 1993, doi: 10.1080/10652469308819027.

[21] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013, doi: 10.1016/j.mcm.2011.12.048.

[22] P. K. Sarpong, A. Owusu-Hemeng, and J. Ackora-Prah, "Applications of convex function and concave functions," *Dama International Journal of Researchers (DIJR)*, vol. 3, no. 5, pp. 1–14, 2018.

[23] E. Set, M. Özdemir, and S. Dragomir, "On the Hermite-Hadamard inequality and other integral inequalities involving two functions," *Journal of Inequalities and Applications*, vol. 2010, pp. 1–9, 2010, doi: 10.1155/2010/148102.

[24] J. Xie, M. A. Ali, H. Budak, M. Fečkan, and T. Sitthiwirattham, "Fractional Hermite-Hadamard inequality, Simpson's and Ostrowski's type inequalities for convex functions with respect to a pair of functions," *Rocky Mountain Journal of Mathematics*, vol. 53, no. 2, pp. 611–628, 2023, doi: 10.1216/rmj.2023.53.611.

[25] X. You, M. A. Ali, H. Budak, J. Reunsumrit, and T. Sitthiwirattham, "Hermite-Hadamard-Mercer-type inequalities for harmonically convex mappings," *Mathematics*, vol. 9, no. 20, p. 2556, 2021, doi: 10.3390/math9202556.

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