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A note on two Diophantine equations $x^2 \pm 2^a p^b = y^4$

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A NOTE ON TWO DIOPHANTINE EQUATIONS $x^2 \pm 2^a p^b = y^4$

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Abstract. In this paper we gave some results of the two Diophantine equations $x^2 \pm 2^a p^b = y^4, x, y \in \mathbb{N}, \gcd(x,y) = 1, a, b \in \mathbb{Z}, a \geq 0, b \geq 0$, where p is an odd prime.

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1. INTRODUCTION

Let D denote an odd positive integer without any squared factor > 1. In 1964, W.Ljunggren [15] showed that the Diophantine equation $x^2 + 4D = y^q$ has no solutions in rational integers if $q \not\equiv 3 \pmod 8$ where the class number of $\mathbb{Q}\sqrt{-D})$ is indivisible by the odd prime and also he showed that this equation has only a finite number of solutions in rational integers x and y and primes q for given D.

Let $\mathbb{Z}, \mathbb{N}$ be the sets of integers and positive integers respectively. Let p be a fixed odd prime. Recently many authors are interested in the equation

$$x^2 + \delta 2^a p^b = y^n, \quad x, y, a, b \in \mathbb{Z}, x, y \geq 1, \gcd(x,y) = 1, a, b \geq 0, n \geq 3, \qquad (1.1)$$

where p is an odd prime and $\delta = 1$. In 2002, F.Luca [18] found all positive integer solutions (x, y, a, b, n) of $x^2 + 2^a 3^b = y^n$ with $n \geq 3$ and coprime x and y. In 2008, F.Luca and A.Togbé [19] solved $x^2 + 2^a 5^b = y^n, \gcd(x,y) = 1$. In 2009, F.Luca and A.Togbé [20] found all positive integer solutions of $x^2 + 2^a 13^b = y^n, \gcd(x,y) = 1$. I.N.Cangül, M.Demirci, F.Luca, Á.Pintér and G.Soydan [5] solved $x^2 + 2^a 11^b = y^n, \gcd(x,y) = 1$. Recently, in [23], G.Soydan, M.Ulas and H.Zhu found all positive integer solutions of $x^2 + 2^a 19^b = y^n, \gcd(x,y) = 1$. Obviously, the authors above researched the special cases of (1.1) and $p \in \{3, 5, 11, 13, 19\}$. In [4] A.Bérczes and I.Pink gave all the solutions of the Diophantine equation (1.1), when $\delta = 1$, $a = 0$ and b is even, where p is any prime in the interval $[2, 100]$ and $\gcd(x,y) = 1$. In [25] all the positive integer solutions (x, y, n) of the Diophantine equation $x^2 + a^2 = 2y^n$ with $a \in \{3, 4, \ldots, 501\}$ were found under the conditions that $n \geq 3$ and that

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$\gcd(x,y)=1$.
For general odd prime p and general n, (1.1) is difficult to solve completely, but for special n, maybe we can get some information. For example, when $n=4$, (1.1) is rewritten as

$$x^2+2^a p^b = y^4,\ \ x\geq 1,\ y>1,\ \gcd(x,y)=1,\ a\geq 0,\ b\geq 0. \tag{1.2}$$

where $\delta=1$. Obviously, y^4-x^2 can be factorized as $(y^2-x)(y^2+x)$. Our research is based on $(y^2-x)(y^2+x)=2^a p^b$ and $\gcd(x,y)=1$. Similarly, our research on another equation

$$x^2-2^a p^b = y^4,\ \ x\geq 1,\ y\geq 1,\ \gcd(x,y)=1,\ a\geq 0,\ b\geq 0 \tag{1.3}$$

is based on $(x-y^2)(x+y^2)=2^a p^b$ and $\gcd(x,y)=1$. When $b=0$, (1.1) is solved by J.H.E. Cohn [8, 9], S.A.Arif and F.S.Abu Muriefah [1, 2] and M.Le [14]. When $a=1$ and $b\geq 3$, we find (1.2) is concerned with the famous equation

$$x^2-2=y^n,\ \ x>1,\ y\geq 1,\ n\geq 3. \tag{1.4}$$

This is still unsolved and is one of the most exciting questions on "classical Diophantine equations". We do have good bounds for n, something like $n<1237$ or so, see the Appendix written by S.Siksek in the GTM book of [7]. And it has been solved for "half" the primes (namely those $n\equiv 1 \pmod 3$), by I.Chen [6]. Most people believe (1.4) has no solution, but this has not been proved up to day.

Now we introduce some notations and symbols. For any positive integer k, let

$$u_k=\frac{1}{2}(\rho^k+\bar{\rho}^k),\ \ v_k=\frac{1}{2\sqrt{2}}(\rho^k-\bar{\rho}^k), 2\nmid k, \tag{1.5}$$

$$U_k=\frac{1}{2}(\rho'^k+\overline{\rho'}^k),\ \ V_k=\frac{1}{2\sqrt{2}}(\rho'^k-\overline{\rho'}^k), \tag{1.6}$$

where

$$\rho=1+\sqrt{2}, \bar{\rho}=1-\sqrt{2}, \rho'=3+2\sqrt{2}, \bar{\rho'}=3-2\sqrt{2}. \tag{1.7}$$

By basic properties of Pell equations [26], $(u,v)=(u_k,v_k)(k=1,3,5,\cdots)$, and $(U,V)=(U_k,V_k)(k=1,2,3,\cdots)$ are all solutions of equations

$$u^2-2v^2=-1, u,v\in\mathbb{N}, \tag{1.8}$$

and

$$U^2-2V^2=1, U,V\in\mathbb{N}, \tag{1.9}$$

respectively.

2. Statement of the results

Theorem 1. *If* (1.4) *has no solution, then all solutions of the equation*

$$x^2 + \delta 2^a p^b = y^4, \;\; x \ge 1, \;\; y > 1, \; \gcd(x,y) = 1, \; a \ge 0, \; b \ge 0, \delta \in \{1, -1\}$$

are given as follows:

a	b	p	x	y	δ	restrictions on f, k, r, s, t
0	1	$2f^2-1$	f^2-1	f	1	$f \in \mathbb{N}, f \ge 2$
		$2f^2+1$	f^2+1	f	-1	$f \in \mathbb{N}$
0	2	u_k	$\frac{u_k^2-1}{2}$	v_k	1	$k \in \mathbb{N}, 2 \nmid k$
		U_k	V_k^2+1	V_k	-1	$k \in \mathbb{N}$
0	3	3	122	11	-1	-
0	5	23	6083	78	1	-
3	0	p	3	1	-1	-
3	1	$\frac{f^2+1}{2}$	f^2+2	f	-1	$f \in \mathbb{N}, f \ge 3$
3	2	V_k	u_k^2+2	u_k	-1	$k \in \mathbb{N}, 2 \nmid k$
3	3	3	29	5	-1	-
3	4	13	5713	239	-1	-
4	3	5	129	11	-1	-
5	0	p	7	3	1	-
7	2	3	287	17	1	-
7	4	3	113	7	-1	-
9	3	17	4785	71	1	-
11	3	7	855	13	-1	-
$s+2$	1	$2^s - f^2$	$2^{s+1} - f^2$	f	-1	$f, s \in \mathbb{N}, 2 \nmid f$
	1	$f^2 + 2^s$	$f^2 + 2^{s+1}$	f	-1	$f, s \in \mathbb{N}, 2 \nmid f$
$s+4$	1	$2^s - 1$	$2^{2s+2} - 2^{s+2} - 1$	$2^{s+1} - 1$	1	$s \in \mathbb{N}, s \ge 2$
	2	$2^s - 1$	$\lvert 2^{2s} - 2^{s+3} + 2^{s+1} + 1 \rvert$	$2^s + 1$	1	$s \in \mathbb{N}, s \ge 2$
$t+2$	1	$\lvert f^2 - 2^t \rvert$	$\lvert f^2 - 2^{t+1} \rvert$	f	1	$f, t \in \mathbb{N}, 2 \nmid f$
2^r+4	1	$2^{2^r}+1$	$2^{2^{r+1}+2} + 2^{2^r+2} - 1$	$2^{2^r+1}+1$	1	$r \in \mathbb{N}$
	2	$2^{2^r}+1$	$2^{2^{r+1}} + 2^{2^r+3} - 2^{2^r+1} + 1$	$2^{2^r} - 1$	-1	$r \in \mathbb{N}$

3. Proof of the theorem

We consider the equation

$$x^2 + \delta 2^a p^b = y^4, x \ge 1, y > 1, a \ge 0, b \ge 0, \gcd(x,y) = 1, \delta \in \{1, -1\} \tag{3.1}$$

where p is an odd prime. For the case $\delta = 1$ and $b = 0$, from [1, 2, 8, 9] and [14], we get $(x,y,a) = (7,3,5)$. For the case $\delta = -1$ and $b = 0$, we get $x^2 - 2^a = y^4$. From W.Ivorra [13] and S.Siksek [22], we know when $a \ge 2$, this equation has the solution $(x,y,a) = (3,1,3)$. When $a = 1$, from $\gcd(x,y) = 1$ we have $2 \nmid xy$ and $x^2 \equiv y^4 \equiv 1 (\bmod 8)$, so $a \ge 3$, it is a contradiction. When $a = 0$, from P. Mihăilescu [21], this equation has no solutions.

When $a = 0$ and $b > 0$, from (3.1) we have $\delta p^b = (y^2 - x)(y^2 + x)$. From $\gcd(x,y) = 1$, we have $\gcd(\delta(y^2 - x), y^2 + x) = 1$. Otherwise $p | (y^2 - x)$ and $p | (y^2 + x)$, $p | 2y^2$ and $p | 2x$, $p | y$ and $p | x$. It is a contradiction with $\gcd(x,y) = 1$.

So we have

$$y^2 - x = \delta,\ y^2 + x = p^b$$

and it leads to

$$2y^2 = p^b + 1. \tag{3.2}$$

and

$$p^b = 2y^2 + 1. \tag{3.3}$$

where $\delta = 1$ and $\delta = -1$, respectively. From Theorem 1.1 of M.A.Bennet [3], we know that when $b \geq 4$, (3.2) has no solutions. When $b = 3$, we get $(4y)^2 = (2p)^3 + 8$ and from J.Gebel [11] and J.London [17], we know that $p = 23, y = 78, x = 6083$. When $b = 1$, we obtain $p = 2f^2 - 1, x = f^2 - 1$, where $f \in \mathbb{N}$, $f \geq 2$. When $b = 2$, we obtain $p^2 - 2y^2 = -1$ and $p = u_k, y = v_k, x = \frac{u_k^2 - 1}{2}$ where $k \in \mathbb{N}, 2 \nmid k$.

From J.H.E.Cohn [10] and E.Herrmann [12], we know that when $b \geq 3$, (3.3) has the only solution $(p, b, y) = (3, 5, 11)$. When $b = 1$, we obtain $p = 2f^2 + 1, x = f^2 + 1$, where $f \in \mathbb{N}$. When $b = 2$, we obtain $p^2 - 2y^2 = 1$ and $p = U_k, y = V_k, x = V_k^2 + 1$, where $k \in \mathbb{N}$.

When $a > 0$ and $b > 0$, from (3.1) we have the factorization

$$\delta 2^a p^b = (y^2 - x)(y^2 + x),\ \ 2 \nmid xy. \tag{3.4}$$

Because $\gcd(x, y) = 1$, we have $\gcd(\delta \frac{y^2 - x}{2}, \frac{y^2 + x}{2}) = 1$ and $\frac{y^2 - x}{2} \cdot \frac{y^2 + x}{2} = \delta 2^{a-2} p^b$. Therefore, we get

$$\frac{y^2 - x}{2} = \delta,\ \ \frac{y^2 + x}{2} = 2^{a-2} p^b, \tag{3.5}$$

$$\frac{y^2 - x}{2} = \delta 2^{a-2},\ \ \frac{y^2 + x}{2} = p^b \tag{3.6}$$

and

$$\frac{y^2 - x}{2} = \delta p^b,\ \ \frac{y^2 + x}{2} = 2^{a-2}. \tag{3.7}$$

In the following, we discuss the three cases:

Case 1: From (3.5), we get

$$y^2 - \delta = 2^{a-2} p^b, \tag{3.8}$$

For the case $\delta = 1$, we obtain

$$\frac{y-1}{2} \cdot \frac{y+1}{2} = 2^{a-4} p^b. \tag{3.9}$$

So from (3.9) we have

$$\frac{y-1}{2} = 1,\ \ \frac{y+1}{2} = 2^{a-4} p^b \tag{3.10}$$

or

$$\frac{y-1}{2} = 2^{a-4}, \quad \frac{y+1}{2} = p^b \tag{3.11}$$

or

$$\frac{y-1}{2} = p^b, \quad \frac{y+1}{2} = 2^{a-4}. \tag{3.12}$$

(3.10) leads to $y = 3, a = 5, b = 0$. (3.11) leads to

$$p^b - 2^{a-4} = 1. \tag{3.13}$$

When $b \geq 2$, we have $a-4 \geq 2$ and from P.Mihăilescu [21] we have $(p,a,b) = (3,7,2)$, the responding $(x,y) = (287,17)$. When $b = 1$, $p = 2^{a-4}+1$ is a Fermat prime. We have $p = 2^{2^r}+1, a = 2^r+4, y = 2^{2^r+1}+1, x = 2^{2^{r+1}+2}+2^{2^r+2}-1$, where $r \in \mathbb{Z}$. (3.12) leads to

$$2^{a-4} - p^b = 1. \tag{3.14}$$

When $b \geq 2$, we have $a-4 \geq 3$ and from P.Mihăilescu [21] we know (3.14) has no solutions. When $b = 1$, $p = 2^{a-4}-1$ is a Mersenne prime. We have $p = 2^s-1, a = s+4, y = 2^{s+1}-1, x = 2^{2s+2}-2^{s+2}-1$, where $s \in \mathbb{N}, s \geq 2$.

For the case $\delta = -1$, from (3.8) we get

$$y^2 + 1 = 2^{a-2} p^b. \tag{3.15}$$

Because $2 \nmid y$, we have $y^2+1 \equiv 2 \pmod 8$. So $a = 3$ and

$$y^2 + 1 = 2p^b. \tag{3.16}$$

By using [16,24], we know (3.16) with $b \geq 3$ has the solution $(y,p,b) = (239,13,4)$, the responding $x = 57123$. When $b = 1$, $p = \dfrac{f^2+1}{2}, x = f^2+2$, where $f \in \mathbb{N}, f \geq 3$. When $b = 2$, we have $y^2 - 2p^2 = -1$ and $y = u_k, p = v_k, x = u_k^2+2$, where $k \in \mathbb{N}, 2 \nmid k$.

Case 2: For the case $\delta = 1$, from (3.6) we get

$$y^2 - 2^{a-2} = p^b. \tag{3.17}$$

When $b = 1$, from (3.17) we have $p = |f^2 - 2^t|$, where $f,t \in \mathbb{N}, 2 \nmid f$. When $2|b$, from (3.17) we have $\left(\frac{y-p^{\frac{b}{2}}}{2}\right)\left(\frac{y+p^{\frac{b}{2}}}{2}\right) = 2^{a-4}$ and $p^{\frac{b}{2}} = 2^{a-4}-1$. If $b = 2$, then $p = 2^{a-4}-1$, which is a Mersenne prime. We have $p = 2^s-1, a = s+4, y = 2^s+1, x = |2^{2s}-2^{s+3}+2^{s+1}+1|$, where $s \in \mathbb{N}, s \geq 2$. If $2|b$ and $b > 2$, then $a-4 > 3$. From [21], we know it has no solution. So $2 \nmid b$ and $b \geq 3$. From W.Ivorra [13] and S.Siksek [22] we know (3.17) with $a-2 \geq 2$ has the only solution $(y,p,a,b) = (71,17,9,3)$, the responding $x = 4785$. There remainders the following unsolved equation

$$y^2 - 2 = p^b, \ \ 2 \nmid b, \ b \geq 3. \tag{3.18}$$

For the case $\delta = -1$, from (3.6) we get

$$y^2 + 2^{a-2} = p^b. \tag{3.19}$$

When $b = 1$, from (3.19) we have $p = f^2 + 2^s, a = s+2, x = f^2 + 2^{s+1}$, where $f, s \in \mathbb{N}, 2 \nmid f$. When $b = 2$, from (3.19) we have $\left(\frac{p-y}{2}\right)\left(\frac{p+y}{2}\right) = 2^{a-4}$ and $p = 2^{a-4} + 1$ is a Fermat prime. So we have $p = 2^{2^r} + 1, a = 2^r + 4, y = 2^{2^r} - 1, x = 2^{2^{r+1}} + 2^{2^r+3} - 2^{2^r+1} + 1$, where $r \in \mathbb{Z}$. If $b > 2$, from [1, 2, 8, 9] and [14], we know that (3.19) has the solutions $(y, p, a, b) = (5,3,3,3), (7,3,7,4), (11,5,4,3)$, the responding $x = 29, 113, 129$.

Case 3: For the case $\delta = 1$ from (3.7), we also get (3.17) and discuss it similarly with Case 2.

For the case $\delta = -1$ from (3.7) we get

$$y^2 + p^b = 2^{a-2}. \tag{3.20}$$

When $b = 1$, we have $p = 2^s - f^2, a = s+2, x = 2^{s+1} - f^2$, where $f, s \in \mathbb{N}, 2 \nmid f$. When $b \geq 2$, from Theorem 8.4 of M.A.Bennet [3], we see (3.20) has the solution $(y, p, a, b) = (13, 7, 11, 3)$, the responding $x = 855$. We complete the proof of the theorem.

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