Miskolc Mathematical Notes
Vol. 26 (2025), No. 2, pp. 897–906

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4831

OSCILLATION THEOREMS FOR PARTIAL DYNAMIC EQUATIONS WITH TIME-SCALE SOLUTIONS

FATIMA ZOHRA LADRANI, AMIN BENAISSA CHERIF,
ABDERRAHMANE BENIANI, AND KHALED ZENNIR

Received 31 October, 2023

Abstract. For the even-order nonlinear functional dynamic equations with unbounded neutral coefficients on time scales, we will establish some oscillation criteria in this paper.

$$a(t)\Delta_x u(x,t) = \frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial u}{\Delta t}(x,t)\right)\right) + q(t)u^{\Delta}(x,t) + p(x,t)f\left(t,x,u(x,t)\right),$$

for all $(x,t) \in \Omega \times \mathcal{J}_{t_0}$, utilizing either the Neumann boundary condition

$$\nabla_x u(x,t) = 0 \quad \text{for all } (x,t) \in \Gamma \times \mathcal{J}_{t_0},$$

or the Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{for all } (x,t) \in \Gamma \times \mathcal{J}_{t_0}.$$

We assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and $\mathcal{J}_{t_0} = [t_0,\infty) \cap \mathbb{T}$. Here, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\Gamma = \partial\Omega$, $\frac{\partial u}{\Delta t}$ is the partial dynamic operator with respect to t, Δ_x is laplacian operator on Ω, and N is the unit exterior normal vector to Γ.The purpose of this study is to introduce some fresh, necessary conditions for the oscillatory solution of partial dynamic equations on time scales.

2010 *Mathematics Subject Classification:* 35L10; 26E70; 34N05

Keywords: neutral differential equation, oscillation, partial dynamic equations, time scales

1. INTRODUCTION

Stefan Hilger [4] invented the time scale calculus in 1988. It integrates and advances both continuous and discrete analytic theories. A large portion of time scale calculus is summarized and organized in the works of Bohner and Peterson, [1, 2], on the subject of time scale, or measure chain. The study of oscillatory events in technology, the natural and social sciences, and other fields is connected to the theory of oscillations, a significant subfield of the applied theory of dynamic equations. Additionally, [3, 5, 6] have examined the oscillatory behavior of dynamic equations on time scales. We point the reader to the articles in the references because there has

been relatively little research on the oscillation of solutions to these kinds of partial dynamic equations on time scales.

We consider the partial dynamic equation.

$$a(t)\Delta_x u(x,t) = \frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial u}{\Delta t}(x,t)\right)\right) + q(t)\frac{\partial u}{\Delta t}(x,t) + p(x,t)f(t,x,u(x,t)), \quad (1.1)$$

for all $(x,t) \in \Omega \times \mathcal{J}_{t_0}$, with either the Neumann boundary condition

$$\nabla_x u(x,t) = 0 \quad \text{for all } (x,t) \in \Gamma \times \mathcal{J}_{t_0}, \quad (1.2)$$

or the Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{for all } (x,t) \in \Gamma \times \mathcal{J}_{t_0}. \quad (1.3)$$

We assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and $\mathcal{J}_{t_0} = [t_0,\infty) \cap \mathbb{T}$. Here, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\Gamma = \partial\Omega$, Δ_t is the partial dynamic operator with respect to t, Δ_x is a laplacian operator on, and N is the unit exterior normal vector to. Under the following presumptions, the equations (1.1)-(1.2) will be examined.

(C_1) The function $f\colon \mathcal{J}_{t_0} \times \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$f \in \mathcal{C}(\mathcal{J}_{t_0} \times \Omega \times \mathbb{R}, \mathbb{R}), \quad uf(t,x,u) > 0 \quad \text{for all } (t,x,u) \in \mathcal{J}_{t_0} \times \Omega \times \mathbb{R} - \{0\}$$

and there is $b \in \mathcal{C}(\mathcal{J}_{t_0}, [0,\infty))$ such that

$$\frac{f(t,x,u)}{u} \geq b(t) \quad \text{for all } (t,x,u) \in \mathcal{J}_{t_0} \times \Omega \times \mathbb{R} - \{0\}. \quad (1.4)$$

(C_2) The functions $r,q,a\colon \mathcal{J}_{t_0} \to [0,\infty)$ and $p\colon \mathcal{J}_{t_0} \times \Omega \to \mathbb{R}$, such as

$$r,q,a \in \mathcal{C}(\mathcal{J}_{t_0}, [0,\infty)), \quad p \in \mathcal{C}(\mathcal{J}_{t_0} \times \Omega, \mathbb{R}), \quad r(t) - \mu(t)q(t) \neq 0 \quad \text{for all } t \in \mathcal{J}_{t_0}.$$

In order to satisfy (1.1) on $\mathcal{J}_{t_0}$, a nontrivial real-valued function $u \in \mathcal{C}(\mathcal{J}_{t_0} \times \Omega, \mathbb{R})$ must satisfy the following conditions: $\frac{\partial u}{\Delta t}$, $\left(r\left(\frac{\partial u}{\Delta t}\right)\right)^{\Delta}$, $\partial_{x_i} u$, $\partial_{x_i,y_i} u \in \mathcal{C}(\mathcal{J}_{t_0} \times \Omega, \mathbb{R})$, for $i,j \in \{1,2,\ldots,n\}$. We shall not investigate any solutions that disappear in some neighborhood of infinity.

If there is a $t_1 \in \mathcal{J}_{t_0}$ such that $u > 0$ and $\int_\Omega u dx > 0$, (resp. $u < 0$ and $\int_\Omega u dx < 0$) for all $t \in \mathcal{J}_{t_1}$, then the function $u\colon \mathcal{J}_{t_0} \times \Omega \to \mathbb{R}$ is said to be ultimately positive (eventually negative). If a solution u to equation (1.1) is neither ultimately positive nor ultimately negative, it is said to be oscillatory; otherwise, it is said to be nonoscillatory. If all of the solutions to equation (1.1) are oscillatory, the equation is said to be oscillatory. There haven't been any oscillatory findings for (1.1) to date. This paper's goal is to provide some oscillation criteria for this.

2. PRELIMINARIES

An arbitrary nonempty closed subset of real numbers is referred to as a temporal scale. The ideas of differential equations and difference equations are combined in the theory of dynamic equations. We assume that the reader is already aware with the fundamental ideas behind dynamic equations' time scales math. Otherwise, the majority of the information required to understand this article may be found in the books [1] and [2].

If a function $u\colon \mathbb{T}\to\mathbb{R}$ is continuous at each right-dense point and has a left-sided limit at each point, it is said to be rd-continuous, and we write $u\in C_{rd}(\mathbb{T})=C_{rd}(\mathbb{T},\mathbb{R})$.

Let $\mathbb{T}$ be a time scale, and let $u\colon \mathbb{T}\to\mathbb{R}$ be a real valued function. Then, given $t\in\mathbb{T}^k$, we define $u^{\Delta}(t)$, if one exists, as the integer such that for all $\varepsilon>0$, a neighborhood $\mathcal{V}$ of t such that for all $s\in\mathcal{V}$,

$$\left|u^{\sigma}(t)-u(s)-u^{\Delta}(t)\left(\sigma(t)-s\right)\right|\leq\varepsilon\left|\sigma(t)-s\right|.$$

where the definition of the forward jump operator $\sigma\colon \mathbb{T}\to\mathbb{T}$is defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}\colon s>t\}\quad \text{for any } t\in\mathbb{T}.$$

If $u^{\Delta}(t)$ exists for every $t\in\mathbb{T}^k=\mathbb{T}-\{\sup\mathbb{T}\}$, then we may claim that u is delta differentiable on $\mathbb{T}$.

The Cauchy integral of u is defined as for $a,b\in\mathbb{T}$ and a differentiable function u, then

$$\int_a^b u^{\Delta}(s)\,\Delta s=u(b)-u(a).$$

Regressive function $p\colon \mathbb{T}\to\mathbb{R}$ is defined as follows

$$1+\mu(t)\,p(t)\neq 0\quad \text{for all } t\in\mathbb{T}^{\kappa},$$

where the graininess function $\mu\colon \mathbb{T}\to[0,+\infty)$ is defined by $\mu(t)=\sigma(t)-t$. The set of all regressive and rd-continuous functions $p\colon \mathbb{T}\to\mathbb{R}$ will be denoted by $\mathcal{R}=\mathcal{R}(\mathbb{T},\mathbb{R})$.

For $p\in\mathcal{R}(\mathbb{T},\mathbb{R})$ and $t_0,t\in\mathbb{T}$, the generalized exponential function is defined by

$$e_p(t,t_0)=\exp\left(\int_{t_0}^{t}\xi_{\mu(s)}\left(p(s)\right)\Delta s\right),\quad \xi_h(z)=\begin{cases}\frac{1}{z}\ln(1+hz), & h\neq 0;\\ z, & h=0.\end{cases}$$

Then $e_p(t,t_0)$ is a nonzero real valued function, and is the unique solution of the IVP

$$\begin{cases}u^{\Delta}(t)=p(t)\,v(t) & \text{for all } t\in\mathbb{T}^k,\\ v(t_0)=1.\end{cases}$$

If $p,q\in\mathcal{R}(\mathbb{T},\mathbb{R})$, then the function $p\oplus q$ defined by

$$(p\oplus q)(t)=p(t)+q(t)+\mu(t)\,p(t)\,q(t)\quad \text{for all } t\in\mathbb{T}^k$$

and the function $\ominus p$ defined by

$$\ominus p(t) = \frac{-p(t)}{1+\mu(t)\,p(t)} \quad \text{for all } t \in \mathbb{T}^k.$$

The following result is used frequently in the remainder of this paper, let $p,q \in \mathcal{R}(\mathbb{T},\mathbb{R})$, and $t,t_0 \in \mathbb{T}^k$, we have

(i) $e_{p\oplus q}(t,t_0) = e_p(t,t_0)\,e_q(t,t_0)$,
(ii) $e_p^{\sigma}(t,t_0) = e_p(\sigma(t),t_0) = (1+\mu(t)\,p(t))\,e_p(t,t_0)$,
(iii) $\frac{1}{e_p(t,t_0)} = e_{\ominus p}(t,t_0)$.

3. Oscillation with Neumann boundary condition

We create certain necessary criteria in this section that ensure any solution u of (1.1)-(1.2) oscillates on $\mathcal{J}_{t_0}$. We present the key findings and note them for simplicity.

$$P(t) := \min\{p(t,x) : x \in \Omega\} > 0 \quad \text{for all } t \in \mathcal{J}_{t_0}. \tag{3.1}$$

Theorem 1. *Assume that (C_1) to (C_2) are true. If $\delta \in C^1_{rd}(\mathcal{J}_{t_0},\mathbb{R})$ is a positive function that exists, then*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{e_{\ominus \frac{q(t)}{r(t)}}(s,t)}{r(s)} \Delta s = \infty, \tag{3.2}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Lambda_\delta(s,t_0)\,\Delta s = \infty, \tag{3.3}$$

where

$$\Lambda_\delta(t,t_0) := \delta(t)\,e_{\ominus\left[-\frac{q}{r}\right]}(\sigma(t),t_0)\,b(t)\,P(t) - e_{\ominus\left[-\frac{q}{r}\right]}(t,t_0)\,\frac{r(t)\left[\delta^{\Delta}(t)\right]^2}{4\delta(t)}.$$

Any (1.1)-(1.2) solution is then oscillatory.

Proof. If (1.1)-(1.2) has a non-oscillatory solution u on $\mathcal{J}_{t_0}$, then this is assumed. Without sacrificing generality, we can assume that there is $t_1 \in \mathcal{J}_{t_0}$ such that $u(t,x) \geq 0$ and $v(t) = \int_\Omega u(t,x)\,\mathrm{d}x > 0$, for $t \in \mathcal{J}_{t_0}$. Since identical justifications can be provided for the cases $u(t,x) \geq 0$ and $v(t) = \int_\Omega u(t,x)\,\mathrm{d}x < 0$, for $t \in \mathcal{J}_{t_0}$. Let u be an eventual solution of (1.1)-(1.2) that is positive. Green's theorem and (1.2) provide the following results:

$$\int_\Omega \Delta_x u(x,t)\mathrm{d}x = \int_\Gamma \nabla_x u(x,t).\mathrm{d}S = 0.$$

Integrating (1.1) with respect to x over Ω, we obtain

$$\int_\Omega \frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial u}{\Delta t}(x,t)\right)\right)\mathrm{d}x + q(t)\int_\Omega \frac{\partial u}{\Delta t}(x,t)\mathrm{d}x + \int_\Omega p(x,t) f(t,x,u(x,t))\,\mathrm{d}x = 0.$$

From (1.4), we have

$$\int_\Omega \frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial u}{\Delta t}(x,t)\right)\right)\mathrm{d}x + q(t)\int_\Omega \frac{\partial u}{\Delta t}(x,t)\mathrm{d}x + P(t)b(t)\int_\Omega u(x,t)\,\mathrm{d}x \le 0.$$

Using Theorem [1, Theorem 1.117], we obtain

$$\frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial}{\Delta t}\int_\Omega u(x,t)\,\mathrm{d}x\right)\right) + q(t)\frac{\partial}{\Delta t}\int_\Omega u(x,t)\mathrm{d}x + b(t)P(t)\int_\Omega u(x,t)\mathrm{d}x \le 0.$$

Let $v(t) = \int_\Omega u(x,t)\,\mathrm{d}x$, for all $t \in \mathcal{J}_{t_0}$, It implies that.

$$\left(r(t)v^\Delta(t)\right)^\Delta + q(t)v^\Delta(t) + b(t)P(t)v(t) \le 0 \quad \text{for all } t \in \mathcal{J}_{t_1}. \tag{3.4}$$

Let $\alpha(t) = \ominus - \frac{q(t)}{r(t)}$, for $t \in \mathcal{J}_{t_0}$, but, we also have

$$\begin{aligned}
\left(e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t)\right)^\Delta &= e^\sigma_{\alpha(t)}(t,t_0)\left(r(t)v^\Delta(t)\right)^\Delta + \alpha(t)e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t)\\
&= e^\sigma_{\alpha(t)}(t,t_0)\left[\left(r(t)v^\Delta(t)\right)^\Delta + \oplus\alpha(t)r(t)v^\Delta(t)\right]\\
&= e^\sigma_{\alpha(t)}(t,t_0)\left[\left(r(t)v^\Delta(t)\right)^\Delta + q(t)v^\Delta(t)\right]\\
&\le -e^\sigma_{\alpha(t)}(t,t_0)b(t)P(t)v(t) \quad \text{for all } t \in \mathcal{J}_{t_1}.
\end{aligned}$$

Thus, $e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t)$ is a nonincreasing function on $\mathcal{J}_{t_0}$. We now claim that $e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t) > 0$, for $t \in \mathcal{J}_{t_1}$. If not, then there exists $t_2 \in \mathcal{J}_{t_1}$, such that

$$e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t) \le -e_{\alpha(t)}(t_2,t_0)b(t_2)v^\Delta(t_2) = -c < 0 \quad \text{for all } t \in \mathcal{J}_{t_2}.$$

It means that

$$v^\Delta(t) \le -c\frac{e_{\ominus\alpha(t)}(t,t_0)}{r(t)} \quad \text{for all } t \in \mathcal{J}_{t_2}.$$

Integrating from t_2 to t, we get

$$v(t) \le v(t_2) - c\int_{t_2}^t \frac{e_{\ominus\alpha(t)}(s,t_0)}{r(s)}\Delta s \to -\infty \quad \text{as } t \to \infty,$$

which defies the assertion tha $v(t) > 0$ for $t \in \mathcal{J}_{t_2}$ exists.

By using the generalized Riccati substitution, define the function w,

$$w(t) = \delta(t)e_{\alpha(t)}(t,t_0)r(t)\frac{v^\Delta(t)}{v(t)} \quad \text{for all } t \in \mathcal{J}_{t_2}. \tag{3.5}$$

Then $w(t) > 0$ for $t \in \mathcal{J}_{t_2}$. Differentiating (3.5) and using (3.4), we have

$$\begin{aligned}
w^\Delta(t) &= \frac{\delta^\Delta(t)}{\delta(t)}w(t) + \frac{\delta(t)}{v(t)}\left(e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t)\right)^\Delta - \delta(t)e_{\alpha(t)}(t,t_0)r(t)\frac{\left(v^\Delta(t)\right)^2}{v(t)v^\sigma(t)}\\
&\le -\delta(t)e^\sigma_{\alpha(t)}(t,t_0)b(t)P(t) + \frac{\delta^\Delta(t)}{\delta(t)}w(t) - \frac{w(t)}{\delta(t)e_{\alpha(t)}(t,t_0)r(t)}.
\end{aligned}$$

Using the inequality, $Bx - Ax^2 \leq \frac{B^2}{4A}$, we have

$$w^{\Delta}(t) \leq -\delta(t) e^{\sigma}_{\alpha(t)}(t,t_0) b(t) P(t) + \frac{1}{4} e_{\alpha(t)}(t,t_0) r(t) \frac{\left[\delta^{\Delta}(t)\right]^2}{\delta(t)}.$$

Integrating the last inequality from t_2 to t and taking the limit supremum on both sides, yields

$$\limsup_{t\to\infty} \int_{t_2}^{t} \delta(s) e^{\sigma}_{\alpha(t)}(s,t_0) b(s) P(s) - \frac{1}{4} e_{\alpha(t)}(s,t_0) r(s) \frac{\left[\delta^{\Delta}(s)\right]^2}{\delta(s)} \Delta s \leq w(t_2),$$

which is in opposition to equation (3.3). The theorem has been fully proven. □

Theorem 2. *Assume that conditions* (C_1) *through* (C_2) *are met, in which case, for all suitably large* $t_1 \in \mathcal{I}_{t_0}$,

$$\int_{t}^{\infty} \frac{1}{\pi(s,t_1)} \left(\int_{t_1}^{s} b(\tau) P(\tau) \Delta\tau \right) \Delta s > 1 \quad \text{for all } t \in \mathcal{I}_{t_1}, \tag{3.6}$$

where

$$\pi(t,t_1) = r(t) + e_{\ominus\left[-\frac{q}{r}\right]}(t,t_0) r(t) \int_{t_1}^{t} e_{-\frac{q}{r}}(s,s_0) \frac{q(s)}{r(s)} \Delta s \quad \text{for all } t \in \mathcal{I}_{t_1}. \tag{3.7}$$

If there exists a positive function $\delta \in \mathcal{C}^1_{rd}(\mathcal{I}_{t_0}, \mathbb{R})$, *such that* (3.3) *holds. Then any solution of* (1.1)-(1.2) *is oscillatory.*

Proof. If (1.1)-(1.2) has a non-oscillatory solution u on $\mathcal{I}_{t_0}$, then this is assumed. Without sacrificing generality, we can assume that there is $t_1 \in \mathcal{I}_{t_0}$ such that $u(t,x) \geq 0$ and $v(t) = \int_{\Omega} u(t,x)\,dx > 0$, for $t \in \mathcal{I}_{t_1}$. Since similar justifications can be provided for the cases where $u(t,x) \geq 0$ and $v(t) = \int_{\Omega} u(t,x)\,dx < 0$, for $t \in \mathcal{I}_{t_1}$. Let u be an eventual solution of (1.1)-(1.2) that is positive. Inequality (3.4) must be satisfied for all $t \in \mathcal{I}_{t_1}$ such that only one of the two following cases occurs, just like in the proof of Theorem 1.

Case 1: $v^{\Delta}(t) < 0$ and $\left(e_{\alpha(t)}(t,t_0) r(t) v^{\Delta}(t)\right)^{\Delta} < 0$, for $t \in \mathcal{I}_{t_1}$. By integrating the inequality (3.4) from time t_1 to t, we arrive at

$$r(t)v^{\Delta}(t) + \int_{t_1}^{t} q(s) v^{\Delta}(s) \Delta s + \int_{t_1}^{t} b(s) P(s) v(s) \Delta s \leq r(t_1) v^{\Delta}(t_1) \quad \text{for } t \in \mathcal{I}_{t_1}.$$

Since $v < 0$ and $e_{\alpha(t)}(.,t_0) r v^{\Delta}$are decreasing on $\mathcal{I}_{t_1}$, we obtain

$$\left[r(t) + e_{\alpha(t)}(t,t_0) r(t) \int_{t_1}^{t} \frac{q(s)}{e_{\alpha(t)}(s,s_0) r(s)} \Delta s \right] v^{\Delta}(t) \leq -v(t) \int_{t_1}^{t} b(s) P(s) \Delta s.$$

It means that

$$v^{\Delta}(t) \leq -\frac{v(t)}{\pi(t,t_1)} \int_{t_1}^{t} b(s) P(s) \Delta s \quad \text{for all } t \in \mathcal{I}_{t_1}.$$

Integrating the above inequality over $[t,\infty)$, we obtain

$$\lim_{t\to\infty} v(t) \le v(t) - \int_t^\infty \frac{v(s)}{\pi(s,t_1)} \left(\int_{t_1}^s b(\tau) P(\tau) \Delta\tau \right) \Delta s.$$

Since v is positive and nonincreasing on $\mathcal{I}_{t_1}$, then $\lim_{t\to\infty} v(t)$ exists and is positive, we have

$$0 \le v(t) \left(1 - \int_t^\infty \frac{1}{\pi(s,t_1)} \left(\int_{t_1}^s b(\tau) P(\tau) \Delta\tau \right) \Delta s \right) \quad \text{for all } t \in \mathcal{I}_{t_1}.$$

Since $v(t) > 0$, for $t \in \mathcal{I}_{t_1}$, what implies

$$\int_t^\infty \frac{1}{\pi(s,t_1)} \left(\int_{t_1}^s b(\tau) P(\tau) \Delta\tau \right) \Delta s \le 1 \quad \text{for all } t \in \mathcal{I}_{t_1},$$

which is in conflict (3.6).

Case 2: $v^\Delta(t) > 0$ and $\left(e_{\alpha(t)}(t,t_0) r(t) v^\Delta(t)\right)^\Delta < 0$, for $t \in \mathcal{I}_{t_1}$. The proof in this instance is identical to that of Theorem 1, so it is omitted. The evidence is now complete.

□

4. Oscillation with Dirichlet boundary condition

In this section, we establish some sufficient conditions which guarantee that every solution u of (1.1)-(1.3) oscillates on $\mathcal{I}_{t_0}$.

Theorem 3. *Assume that conditions* $(C_1)-(C_2)$ *and* (3.2) *holds. If there exists a positive function* $\theta \in C^1_{rd}(\mathcal{I}_{t_0},\mathbb{R})$ *and a constant* $\lambda > 0$*, such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \gamma_\theta(s,t_0) \Delta s = \infty. \tag{4.1}$$

where

$$\gamma_\theta(t,t_0) := \theta(t) e_{\ominus\left[-\frac{q}{r}\right]}(\sigma(t),t_0) \left[b(t) P(t) + \lambda a(t) \right] - e_{\ominus\left[-\frac{q}{r}\right]}(t,t_0) \frac{r(t)\left[\theta^\Delta(t)\right]^2}{4\theta(t)},$$

and P *is defined in* (3.1).

Then any solution of (1.1)-(1.3) *is oscillatory.*

Proof. Suppose that (1.1)-(1.3) has a nonoscillatory solution u on $\mathcal{I}_{t_0}$. We may assume without loss of generality that there exists $t_1 \in \mathcal{I}_{t_0}$ such that $u(t,x) \ge 0$ and $v(t) = \int_\Omega u(t,x)\,dx > 0$, for $t \in \mathcal{I}_{t_0}$. Since similar arguments can be made, for the case $u(t,x) \ge 0$ and $v(t) = \int_\Omega u(t,x)\,dx > 0$, for $t \in \mathcal{I}_{t_0}$. Let u be an eventually positive solution of (1.1)-(1.3). Let $\psi\colon \Omega \to \mathbb{R}$ be a function defined by

$$\psi(x) := \sum_{i=1}^{i=n} e^{\sqrt{\lambda} x_i} \quad \text{for all } x \in \Omega.$$

By Green's theorem and (1.3), we have

$$\int_\Omega \psi(x)\,\Delta_x u(x,t)\,dx = -\int_\Omega u(x,t)\,\Delta\psi(x)\,dx + \int_\Gamma u(x,t)\,\nabla\psi(x)\,ds$$
$$= -\int_\Omega u(x,t)\,\Delta\psi(x)\,dx = -\lambda\int_\Omega u(x,t)\,\psi(x)\,dx.$$

We multiply (1.1) by ψ and integrate with respect to x over Ω and get

$$\left(r(t)\left(v^\Delta(t)\right)\right)^\Delta + q(t)v^\Delta(t) + \left(b(t)P(t) + \lambda a(t)\right)v(t) \le 0 \quad \text{for all } t \in \mathcal{J}_{t_1}. \tag{4.2}$$

The rest of proof is similar to that of Theorem 1. □

Theorem 4. *Assume that conditions $(C_1)-(C_2)$ holds, such that for all sufficiently large $t_1 \in \mathcal{J}_{t_0}$. If there exists a constant $\beta > 0$, such that*

$$\int_t^\infty \frac{1}{\pi(s,t_1)}\left(\int_{t_1}^s b(\tau)P(\tau) + \beta a(\tau)\Delta\tau\right)\Delta s > 1 \quad \text{for all } t \in \mathcal{J}_{t_1},$$

where $\pi(.,t_1)$ is defined in (3.7) and P is defined in (3.1).

If there exists a positive function $\theta \in C^1_{rd}(\mathcal{J}_{t_0},\mathbb{R})$ and a constant $\lambda > 0$, such that (4.1) holds. Then any solution of (1.1)-(1.2) is oscillatory.

Proof. Suppose that (1.1)-(1.3) has a nonoscillatory solution u on $\mathcal{J}_{t_0}$. We may assume without loss of generality that there exists $t_1 \in \mathcal{J}_{t_0}$ such that $u(t,x) \ge 0$ and $v(t) = \int_\Omega u(t,x)\,dx > 0$, for $t \in \mathcal{J}_{t_0}$. Since similar arguments can be made, for the case $u(t,x) \ge 0$ and $v(t) = \int_\Omega u(t,x)\,dx > 0$, for $t \in \mathcal{J}_{t_0}$. Let u be an eventually positive solution of (1.1)-(1.3). As in the proof of Theorem 3, inequality (4.2) is satisfied for all $t \in \mathcal{J}_{t_1}$, such that only one of the following two cases happens. Case 1. $v^\Delta(t) > 0$ and $\left(e_{\alpha(t)}(t,t_0)\,r(t)\,v^\Delta(t)\right)^\Delta < 0$, for $t \in \mathcal{J}_{t_1}$. The proof, in this case, is the same as that of Theorem 3. Case 2. $v^\Delta(t) < 0$ and $\left(e_{\alpha(t)}(t,t_0)\,r(t)\,v^\Delta(t)\right)^\Delta < 0$, for $t \in \mathcal{J}_{t_1}$. Integrating the inequality (4.2) over $[t_1,t)$, we obtain

$$r(t)v^\Delta(t) + \int_{t_1}^t q(s)v^\Delta(s)\Delta s + \int_1^t \left(b(s)P(s) + \lambda a(s)\right)v(s)\,\Delta s \le 0 \quad \text{for all } t \in \mathcal{J}_{t_1}.$$

The rest of proof is similar to that of Theorem 2. This completes the proof. □

5. Examples

We provide an example to demonstrate our major finding in this section.

Example 1. Consider the Neumann boundary problem

$$\begin{cases} \partial_t^2 u(x,t) + \partial_t u(x,t) + (\|x\| + t)\,u(x,t) = t\Delta_x u(x,t) & \text{for } (x,t) \in B_\rho(0,d) \times \mathbb{R}^+, \\ \nabla_x u(x,t) = 0 & \text{for all } (x,t) \in S(0,d) \times \mathbb{R}^+, \end{cases}$$

with $d > 0$ and $B_\rho(0,d) = \{x \in \mathbb{R}^n : \|x\| < d\}$. Here, $\Omega = B_\rho(0,d)$, $\Gamma = S(0,\rho)$, $\mathbb{T} = \mathbb{R}$, $r(t) = q(t) = 1$, $p(t,x) = \|x\| + t$, $a(t) = t$, and $f(t,x,u) = u$. Then $b(t) =$

t, $P(t)=t$, $e_{\ominus-\frac{q(t)}{r(t)}}(s,t)=e^{t-s}$ and the hypotheses (C_1)-(C_2), (3.2) are true. Let $\delta(t)=1$, for $t\in\mathbb{R}^+$. Therefore, (1) is valid. Equation (1) is oscillatory according to Theorem1.

Example 2. Consider the Dirichlet boundary problem

$$\begin{cases}\Delta_t\left(t.\Delta_t u(x,t)\right)+\frac{t}{2}\Delta_t u(x,t)+u(x,t)=t\Delta_x u(x,t) & \text{for } (x,t)\in\Omega\times\mathbb{Z}^+,\\ u(x,t)=0 & \text{for } (x,t)\in\Gamma\times\mathbb{Z}^+.\end{cases}$$

In this case, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\Gamma=\partial\Omega$. Here, $\mathbb{T}=\mathbb{R}$, $r(t)=t$, $q(t)=\frac{t}{2}$, $p(t,x)=1$, $a(t)=t$, and $f(t,x,u)=u$. Then $b(t)=t$, $P(t)=1$, and

$$e_{\ominus-\frac{q(t)}{r(t)}}(s,t)=e_1(s,t)=2^{t-s}\quad\text{for all } t,s\in\mathbb{Z}^+.$$

Then, (3.2) is maintained. By substituting in (4.1) and setting $\theta(t)=1$, for $t\in\mathbb{Z}^+$, we get

$$\limsup_{t\to\infty}\int_{t_0}^{t}2^{s+1}\left[s^2+\lambda s\right]\Delta s=+\infty\quad\text{for all }\lambda>0.$$

Therefore, (4.1) is valid. Equation (2) is oscillatory according to Theorem 3.

REFERENCES

[1] M. Bohner and A. Peterson, *Dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2001, an introduction with applications, doi: 10.1007/978-1-4612-0201-1.

[2] M. Bohner and A. Peterson, *Dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2001, an introduction with applications, doi: 10.1007/978-1-4612-0201-1.

[3] S. R. Grace, J. Džurina, I. Jadlovská, and T. Li, "On the oscillation of fourth-order delay differential equations," *Adv. Difference Equ.*, pp. Paper No. 118, 15, 2019, doi: 10.1186/s13662-019-2060-1.

[4] S. Hilger, *Ein Maßkettenakül mit Anwendung auf Zentrumsannigfaltigkeiten*. Universität Würzburg, 1988.

[5] T. Kusano and J. V. Manojlović, "Precise asymptotic behavior of regularly varying solutions of second order half-linear differential equations," *Electron. J. Qual. Theory Differ. Equ.*, pp. Paper No. 62, 24, 2016, doi: 10.14232/ejqtde.2016.1.62.

[6] A. B. Trajković and J. V. Manojlović, "Asymptotic behavior of intermediate solutions of fourth-order nonlinear differential equations with regularly varying coefficients," *Electron. J. Differential Equations*, pp. Paper No. 129, 32, 2016.

Authors' addresses

Fatima Zohra Ladrani

Department of Exact Sciences, Higher Training Teacher's School of Oran Ammour Ahmed(ENSO), M'Naouer, P. O. Box 1523, Oran 31000, Algeria

E-mail address: f.z.ladrani@gmail.com

Amin Benaissa Cherif

(**Corresponding author**) Department of Mathematics, Faculty of Mathematics and Informatics, University of Science and Technology of Oran Mohamed-Boudiaf (USTOMB), El Mnaouar, P. O. Box 1505, Bir El Djir, Oran 31000, Algeria

E-mail address: amin.benaissacherif@univ-usto.dz

Abderrahmane Beniani

Faculty of Science and Technology, Engineering and Sustainable Development Laboratory, University of Belhadj Bouchaib, P. O. Box 284 RP, Ain Temouchent 46000, Algeria,

E-mail address: abderrahmane.beniani@univ-temouchent.edu.dz

Khaled Zennir

Department of Mathematics, College of Sciences and Arts, Al-Ras, Qassim University, King Abdulaziz Road, Buraydah 51452, Kingdom of Saudi Arabia

E-mail address: k.Zennir@qu.edu.sa