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(CO)LOCALIZATION IN (n-)COTILTING COMODULES

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Abstract. The purpose of this article is to study the localization and colocalization for cotilting comodules and n-cotilting comodules. In this paper, firstly we give the definition of (co)localization for preenvelopes and special preenvelopes. And secondly, we apply the theory of (co)localization to cotilting comodules. Following this result, we investigate the (co)localization for n-cotilting comodules.

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1. INTRODUCTION

The (co)localization is developed continuously from many aspects, which plays an important role in the theory of coalgebras. One of the most famous procedure is the localization in rings, which is a systematic method of adding multiplicative inverses to a ring. Following the localization for rings, Năstăsescu and Torrecillas [15] developed a theory of localization for coalgebras. Concretely, if C is a coalgebra over a field K, and $\mathcal{T}$ is a dense subcategory or a Serre class of the category $\mathcal{M}^C$ of right C-comodules, Năstăsescu and Torrecillas considered the quotient category $\mathcal{M}^C/\mathcal{T}$ and the canonical functor $T\colon \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$. What's more, Năstăsescu and Torrecillas considered the colocalizing subcategory $\mathcal{T}$, where the functor T has a left adjoint H, rather than considering the localizing subcategory $\mathcal{T}$, where $\mathcal{T}$ is closed under arbitrary direct sums (or equivalently, T has a right adjoint S). In addition, Gabriel described the localization abstractly in abelian and Grothendieck category in [9]. Later, Navarro developed the ideas of Gabriel in the categories of comodules (Grothendieck categories of finite type) in [10, 11]. The main reason for this is that the category $\mathcal{M}^C$ of right comodules over a coalgebra C is a locally finite Grothendieck category in which the theory of localization can be applied. The key-point of

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the theory lies in the fact that the quotient category becomes a comodule category. In other words, a quotient category $\mathcal{M}^C/\mathcal{T}$ is a category of comodules $\mathcal{M}^D$ for certain coalgebra D, where C is a coalgebra, and $\mathcal{T}$ is a localizing subcategory of a Grothendieck category. Indeed, it is better understood than in the case of modules over arbitrary algebras.

The notions of (pre)envelope and (n-)cotilting are well-established and of significant importance in general category theory, particularly within the context of Abelian categories. These concepts have proven to be fundamental in understanding the structural and homological properties of various mathematical objects. Similarly, (n-)cotilting theory in Abelian categories has led to a deeper understanding of the duality between projective and injective objects. Given the profound impact of these concepts in Abelian categories, it is natural to extend their application to other categorical settings where similar structural insights might be gained. One such setting is the category of comodules over a coalgebra, known as the comodule category.

The representation theory of coalgebra has been developed by some scholars, see [1–5, 7, 8, 10–25]. The fifth open question raised by Simson in [16] is to develop cotilting theory in the categories of comodules, which is an important part of the representation theory of coalgebras. Therefore it is rather natural to think about the following question:

Question 1. How to apply the theory of (co)localization in comodule categories to the (pre)envelope and n-cotilting comodule?

We present our main result as follows.

Theorem 1. *Let K be a field, C be a K-coalgebra and $e \in C^*$ be an idempotent, then we have:*

(1) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-preenvelope of eM for each idempotent $e \in C^*$.*
(2) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-envelope of eM for each idempotent $e \in C^*$.*
(3) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM, eX)$ is a special $e\mathcal{F}$-preenvelope of eM for each idempotent $e \in C^*$.*

Theorem 2. *Let K be a field, C be a K-coalgebra, and $e \in C^*$ be an idempotent defining a perfect localization. Suppose that $X \cong Ce$ is a quasi-finite injective cogenerator, then M is an n-cotilting eCe-comodule if and only if $H(M)$ is an n-cotilting C-comodule.*

2. PRELIMINARIES

Let (C,Δ,ε) be a coalgebra over a field K, where Δ is the comultiplication map $\Delta\colon C\to C\otimes C$ and $\varepsilon\colon C\to K$ is the counit map. We denote by $\mathcal{M}^C$ the category of all right C-comodules, where C is a coalgebra over a field K. Unless otherwise stated, all comodules are right comodules throughtout this paper. Give a comodule class,

$$^{\perp}\mathcal{F}=\{X\in \text{Comod-}C \mid \text{for any } F\in\mathcal{F}, \text{have } \operatorname{Ext}^1_C(X,F)=0\}.$$

Following [11], a full subcategory $\mathcal{T}$ of $\mathcal{M}^C$ is called dense if for every exact sequence $0\to M'\to M\to M''\to 0$ in $\mathcal{M}^C$, the comodule M lies in $\mathcal{T}$ if and only if each of the comodules M' and M'' lies in $\mathcal{T}$. For any dense subcategory $\mathcal{T}$ of $\mathcal{M}^C$, there exists an abelian category $\mathcal{M}^C/\mathcal{T}$ and an exact functor $T\colon \mathcal{M}^C\to\mathcal{M}^C/\mathcal{T}$ such that $T(M)=0$, for each $M\in\mathcal{T}$, satisfying the following universal property: for any exact functor $F\colon \mathcal{M}^C\to\mathcal{C}$ such that $F(M)=0$ for each $M\in\mathcal{T}$, there exists a unique functor $\overline{F}\colon \mathcal{M}^C/\mathcal{T}\to\mathcal{C}$ verifying that $F=\overline{F}T$, where $\mathcal{C}$ is an arbitrary abelian category. The category $\mathcal{M}^C/\mathcal{T}$ is called the quotient category of $\mathcal{M}^C$ with respect to $\mathcal{T}$, and T is known as the quotient functor.

A dense subcategory $\mathcal{T}$ of $\mathcal{M}^C$ is said to be localizing if the quotient functor $T\colon \mathcal{M}^C\to\mathcal{M}^C/\mathcal{T}$ has a right adjoint functor $S\colon \mathcal{M}^C/\mathcal{T}\to\mathcal{M}^C$, called the section functor. If the section functor S is exact, a localizing subcategory $\mathcal{T}$ is called perfect localizing. $\mathcal{T}$ is said to be colocalizing if T has a left adjoint functor $H\colon \mathcal{M}^C/\mathcal{T}\to\mathcal{M}^C$, called the colocalizing functor. $\mathcal{T}$ is called a perfect colocalizing subcategory if the colocalizing functor H is exact.

Consider the dual algebra $C^*=\operatorname{Hom}_K(C,K)$ with the convolution product $f*g(C)=\sum f(C_1)g(C_2)$, where $f,g\in C^*$. The counit $\varepsilon\colon C\to K$ of C is the identity element of the algebra C^*. In [6, 11, 24], a localizing subcategory of $\mathcal{M}^C$ was described by means of idempotent $e\in C^*$. Furthermore, it was proved that the quotient category is the category of right comodules over the coalgebra eCe, where e is an idempotent associated to the localizing subcategory. The coalgebra structure of eCe is given by

$$\triangle_{eCe}(ece)=\sum_{(c)} ec_{(1)}e\otimes ec_{(2)}e \quad\text{and}\quad \varepsilon_{eCe}(ece)=e(c),$$

for any $c\in C$, where $\triangle_C(c)=\sum_{(c)} c_{(1)}\otimes c_{(2)}$ is the Sweedler's notation in [21]. If M is a right C-comodule, then eM has a natural structure of right eCe-comodule given by

$$\rho(em)=\sum_{(m)} em_{(0)}\otimes em_{(1)}e, \quad\text{where}\quad \rho_M(m)=\sum_{(m)} m_{(0)}\otimes m_{(1)},$$

for any $m\in M$ by the Sweedler's notation in [21]. Throughout we denote by $\mathcal{T}_e$ the localizing subcategory associated to e for an idempotent $e\in C^*$ for any coalgebra C over a field K.

The following two lemmas are the properties of the (co)localizing functor which are from (cf.[11],[15] and [9]).

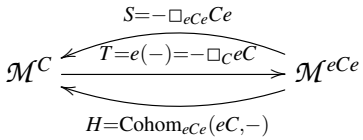
Lemma 1. *Let $\mathcal{T}$ be a dense subcategory of the category of right comodules $\mathcal{M}^C$ over a coalgebra C. Then the following statements hold.*

(1) *T is exact.*
(2) *If $\mathcal{T}$ is localizing, then the section functor S is left exact and the equivalence $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ holds.*
(3) *If $\mathcal{T}$ is colocalizing, then the colocalizing functor H is right exact and the equivalence $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ holds.*

Lemma 2. *Let C be a coalgebra, and e be an idempotent in C^*. Then the following statements hold.*

(1) *The quotient functor $T\colon \mathcal{M}^C \to \mathcal{M}^{eCe}$ is naturally equivalent to the functor $e(-)$. The quotient functor T is also equivalent to the cotensor functor $-\square_C eC$.*
(2) *The section functor $S\colon \mathcal{M}^{eCe} \to \mathcal{M}^C$ is naturally equivalent to the cotensor functor $-\square_{eCe}Ce$.*
(3) *If $\mathcal{T}$ is a colocalizing subcategory of $\mathcal{M}^C$, then the colocalizing functor $H\colon \mathcal{M}^{eCe} \to \mathcal{M}^C$ is naturally equivalent to the functor* $\mathrm{Cohom}_{eCe}(eC,-)$.

For the convenience of the reader, the functors can be presented as shown in the following diagram:



For the convenience of writing, we use $T\cdot$ instead of $T(\cdot)$.

3. (Co)localization in (pre)envelopes

In this section, we give the concepts of (pre)envelopes and special preenvelopes for comodules and apply the theory of (co)localization to them.

Definition 1. Assume that $\mathcal{F}$ is a class of comodules in $\mathcal{M}^C$ and $M \in \mathcal{M}^C$. We call $\phi \in \mathrm{Com}_C(M,X)$ with $X \in \mathcal{F}$ an $\mathcal{F}$-preenvelope of M if it satisfies that $\mathrm{Com}_C(\phi,F)\colon \mathrm{Com}_C(X,F) \to \mathrm{Com}_C(M,F)$ is surjective for every $F \in \mathcal{F}$.

Remark 1. Let $\phi \in \mathrm{Com}_C(M,X)$ be an $\mathcal{F}$-preenvelope of M

(i) ϕ is called an $\mathcal{F}$-envelope of M, if $g\phi = \phi$ and $g \in \mathrm{End}(X)$, then g is an automorphism of X;
(ii) ϕ is said to be special, if $\phi \in \mathrm{Com}_C(M,X)$ is injective and $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$.

A comodule class $\mathcal{F} \subseteq \mathcal{M}^C$ is called a preenvelope (envelope) class if every comodule in $\mathcal{M}^C$ has an $\mathcal{F}$-preenvelope ($\mathcal{F}$-envelope).

Proposition 1. *Let C be a K-coalgebra, $e \in C^*$ be an idempotent, $\phi \in \mathrm{Com}_C(M,X)$ be a right C-comodule homomorphism.*

(1) *If $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M, then right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-preenvelope of eM.*

(2) *If $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M, then the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-envelope of eM.*

(3) *If $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M, then the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is a special $e\mathcal{F}$-preenvelope of eM.*

Proof.

(1) Since the right C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M, based on Definition 1, we can deduce that

$$\phi^* = \mathrm{Com}_C(\phi,F)\colon \mathrm{Com}_C(X,F) \to \mathrm{Com}_C(M,F)$$

is surjective for each $F \in \mathcal{F}$, i.e., $\mathrm{Com}_C(X,F) \xrightarrow{\phi^*} \mathrm{Com}_C(M,F) \to 0$ in $\mathcal{M}^C$ is exact. Now, for any $\alpha \in \mathrm{Com}_{eCe}(eM, eF)$, where $F \in \mathcal{F}$, there exists an $f \in \mathrm{Com}_C(M,F)$ such that $e(f) = \alpha$ since $e(-)$ is full. Consequently, there is a $g \in \mathrm{Com}_C(X,F)$ such that $f = g\phi$ since ϕ is an $\mathcal{F}$-preenvelope of M. Thereby, we can determine that $e(f) = e(g\phi) = e(g).e(\phi)$, i.e., $\alpha = e(g).e(\phi)$. Hence,$(e\phi)^*$ is surjective, i.e. $\mathrm{Com}_{eCe}(eX,eF) \xrightarrow{(e\phi)^*} \mathrm{Com}_{eCe}(eM,eF) \to 0$ in $\mathcal{M}^{eCe}$ is exact, where $eF \in e\mathcal{F}$. This shows that the eCe-comodule homomorphism $e\phi$ is an $e\mathcal{F}$-preenvelope of eM.

(2) Because the right C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M, according to Remark 1(i), it can be inferred that $\phi \in \mathrm{Com}_C(M,X)$ is primarily an $\mathcal{F}$-preenvelope. Furthermore, if $g\phi = \phi$ and $g \in \mathrm{End}(X)$, then g is an automorphism of X. From Proposition 1(1), we know that $e\phi$ is an $e\mathcal{F}$-preenvelope of eM. Furthermore, since the C-comodule endomorphism g is an automorphism of X, we can deduce from [12, Lemma 3.7(3)] that for any eCe-comodule endomorphism $eg\colon eX \to eX$, eg is an automorphism of eX. Based on the above two aspects, therefore, it can be concluded that the eCe-comodule homomorphism $e\phi$ is an $e\mathcal{F}$-envelope of eM.

(3) Because the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M, according to Remark 1(ii), it can be inferred that $\phi \in \mathrm{Com}_C(M,X)$ is primarily an $\mathcal{F}$-preenvelope. Furthermore, we infer that $\phi \in \mathrm{Com}_C(M,X)$ is injective and $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$. From Proposition 1(1), we know that $e\phi$ is an $e\mathcal{F}$-preenvelope of eM. Let $N = \mathrm{Coker}\phi$, then we get the exact

sequence $0 \to M \xrightarrow{\phi} X \to N \to 0$ in $\mathcal{M}^C$. Since T is an exact functor, there is the short exact sequence $0 \to eM \xrightarrow{e\phi} eX \to eN \to 0$ in $\mathcal{M}^{eCe}$. As Coker$\phi \in {}^{\perp}\mathcal{F}$, it holds that $\mathrm{Ext}^1_C(N,F) = 0$ for any $F \in \mathcal{F}$. From [12, Lemma 3.5], it follows that $e\mathrm{Ext}^1_C(N,F) = 0$, i.e., $\mathrm{Ext}^1_{eCe}(eN,eF) = 0$. This shows that $eN = \mathrm{Coker}(e\phi) \in {}^{\perp}(e\mathcal{F})$. Consequently, the eCe-comodule homomorphism $e\phi$ is a special $e\mathcal{F}$-preenvelope.

□

Corollary 1. *Suppose that C is a coalgebra and $\mathcal{F} \subseteq \mathcal{M}^C$ is a (pre)envelope class, then $e\mathcal{F} \subseteq \mathcal{M}^{eCe}$ is also a (pre)envelope class.*

Proof. It can be proved by Proposition 1. □

Proposition 2. *Let k be a field, C be a K-coalgebra, $\phi \in \mathrm{Com}_{eCe}(M,X)$ be a right eCe-comodule homomorphism and $e \in C^*$ be an idempotent which defines a perfect localization.*

(1) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M, then the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M),S(X))$ is an $S(\mathcal{F})$-preenvelope of $S(M)$.*

(2) *If $\phi \in \mathrm{Com}_{eCe}(M,\ X)$ is an $\mathcal{F}$-envelope of M, then the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M),\ S(X))$ is an $S(\mathcal{F})$-envelope of $S(M)$.*

(3) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M, then the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M),S(X))$ is a special $S(\mathcal{F})$-preenvelope of $S(M)$.*

Proof.

(1) Since the right eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M, we infer that

$$\phi^* = \mathrm{Com}_{eCe}(\phi,F)\colon \mathrm{Com}_{eCe}(X,F) \xrightarrow{\phi^*} \mathrm{Com}_{eCe}(M,F)$$

is surjective for each $F \in \mathcal{F}$, i.e., $\mathrm{Com}_{eCe}(X,F) \xrightarrow{\phi^*} \mathrm{Com}_{eCe}(M,\ F) \to 0$ in $\mathcal{M}^{eCe}$ is exact. Now, for any $\alpha \in \mathrm{Com}_C(S(M),S(F))$, where $F \in \mathcal{F}$, there exists an $f \in \mathrm{Com}_{eCe}(M,F)$ such that $S(f) = \alpha$ since S is full. In consequence, there is a $g \in \mathrm{Com}_{eCe}(X,F)$ such that $f = g\phi$ since ϕ is an $\mathcal{F}$-preenvelope of M.Consequently, we have $S(f) = S(g)S(\phi)$, i.e., $\alpha = S(g)S(\phi)$. Thus, $(S(\phi))^*$ is surjective, that is, $\mathrm{Com}_C(S(X),S(F)) \xrightarrow{(S(\phi))^*} \mathrm{Com}_C(S(M),S(F)) \to 0$ in $\mathcal{M}^C$ is exact, where $S(F) \in S(\mathcal{F})$. This shows that the C-comodule homomorphism $S(\phi)$ is an $S(\mathcal{F})$-preenvelope of $S(M)$.

(2) Since $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M, according to Remark 1(i), it can be inferred that $\phi \in \mathrm{Com}_{eCe}(M,X)$ is mainly an $\mathcal{F}$-preenvelope. Additionally, since $g\phi = \phi$ and $g \in \mathrm{End}_{eCe}(X)$, we derive that the eCe-comodule endomorphism g is an automorphism of X. From Proposition 2(1), we know that $S(\phi)$ is an $S(\mathcal{F})$-preenvelope of $S(M)$. And also since the eCe-comodule

endomorphism $g \in \mathrm{Com}_{eCe}(X,X)$ is an automorphism of X, by [12, Lemma 3.8(4)], it follows that the C-comodule endomorphism $S(g) = \mathrm{Com}_C(S(X), S(X))$ is also an automorphism of $S(X)$. Taking into account the above two aspects, we can conclude that C-comodule homomorphism $S(\phi)$ is an $S(\mathcal{F})$-envelope of $S(M)$.

(3) Since $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M, it follows that $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope, and furthermore, $\phi \in \mathrm{Com}_{eCe}(M,X)$ is injective and $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$. From Proposition 2(1), it can be concluded that $S(\phi)$ is an $S(\mathcal{F})$-preenvelope of $S(M)$. By [12, Lemma 3.8(2)], the C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M),S(X))$ is injective. Let $N = \mathrm{Coker}\phi$, then we have the exact sequence $0 \to M \xrightarrow{\phi} X \to N \to 0$ in $\mathcal{M}^{eCe}$. Since $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$, it follows that $\mathrm{Ext}^1_{eCe}(N,F) = 0$ for any $F \in \mathcal{F}$. Since S is an exact functor, there is the short exact sequence $0 \to S(M) \xrightarrow{S(\phi)} S(X) \to S(N) \to 0$ in $\mathcal{M}^C$. We obtain the following commutative diagram (denote $S(\cdot)$ by $S\cdot$ for convenience)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Com}_{eCe}(N,F) & \longrightarrow & \mathrm{Com}_{eCe}(X,F) & \longrightarrow & \mathrm{Com}_{eCe}(M,F) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Com}_C(SN,SF) & \longrightarrow & \mathrm{Com}_C(SX,SF) & \longrightarrow & \mathrm{Com}_C(SM,SF) & \longrightarrow & \mathrm{Ext}^1_C(SN,SF)
\end{array}$$

where $S(F) \in S(\mathcal{F})$, because S is a fully faithful functor. This shows that $\mathrm{Ext}^1_C(S(N), S(F)) = 0$ and $S(N) \in {}^{\perp}S(\mathcal{F})$. Consequently, $S(N) = \mathrm{Coker}(S(\phi)) \in {}^{\perp}S(\mathcal{F})$. Based on the above two aspects, we can conclude that the C-comodule homomorphism $S(\phi)$ is a special $S(\mathcal{F})$-preenvelope.

□

Proposition 3. *Let C be a coalgebra, $\phi \in \mathrm{Com}_{eCe}(M,X)$ be a right eCe-comodule homomorphism and $e \in C^*$ be an idempotent which defines a perfect colocalization.*

(1) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M, then the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M),H(X))$ is an $H(\mathcal{F})$-preenvelope of $H(M)$.*

(2) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M, then the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-envelope of $H(M)$.*

(3) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M, then the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M),H(X))$ is a special $H(\mathcal{F})$-preenvelope of $H(M)$.*

Proof.

(1) Since the right eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M, we infer that

$$\phi^* = \mathrm{Com}_{eCe}(\phi,F)\colon \mathrm{Com}_{eCe}(X,F) \xrightarrow{\phi^*} \mathrm{Com}_{eCe}(M,F)$$

is surjective for each $F \in \mathcal{F}$, i.e., $\mathrm{Com}_{eCe}(X,F) \xrightarrow{\phi^*} \mathrm{Com}_{eCe}(M,\ F) \to 0$ in $\mathcal{M}^{eCe}$ is exact. Now, for any $\alpha \in \mathrm{Com}_C(H(M),H(F))$, where $F \in \mathcal{F}$, there exists an $f \in \mathrm{Com}_{eCe}(M,F)$ such that $H(f) = \alpha$ since H is full. Consequently, there exists a $g \in \mathrm{Com}_{eCe}(X,F)$ such that $f = g\phi$ since ϕ is an $\mathcal{F}$-preenvelope of M. Thus, we can conclude that $H(f) = H(g)H(\phi)$, i.e., $\alpha = H(g) \cdot H(\phi)$. As a result, it follows that $(H(\phi))^*$ is surjective, i.e.

$$\mathrm{Com}_C(H(X),H(F)) \xrightarrow{(H(\phi))^*} \mathrm{Com}_C(H(M),H(F)) \to 0$$

in $\mathcal{M}^C$ is exact, where $H(F) \in H(\mathcal{F})$. This shows that the C-comodule homomorphism $H(\phi)$ is an $H(\mathcal{F})$-preenvelope of $H(M)$.

(2) Since $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M, firstly, $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope, and secondly, with $g\phi = \phi$ and $g \in \mathrm{End}_{\mathrm{eCe}}(X)$, we deduced that the eCe-comodule endomorphism g is an automorphism of X. As a result of Proposition 3(1), we can assert that the C-comodule homomorphism $H(\phi)$ is an $H(\mathcal{F})$-preenvelope of $H(M)$. Additionally, since the eCe-comodule endomorphism $g \in \mathrm{Com}_{eCe}(X,X)$ is an automorphism of X, [12, Lemmma 3.10(4)] allows us to infer that the C-comodule endomorphism $H(g) = \mathrm{Com}_C(H(X),H(X))$ is also an automorphism of $H(X)$. With the two aspects in mind, we can assert that the C-comodule homomorphism $H(\phi)$ is an $H(\mathcal{F})$-envelope of $H(M)$.

(3) Considering $\phi \in \mathrm{Com}_{eCe}(M,X)$ as a special $\mathcal{F}$-preenvelope of M, it is clear that $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope at first, and then it is observed that $\phi \in \mathrm{Com}_{eCe}(M,X)$ is injective and $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$. From the perspective of Proposition 3(1), it is evident that the C-comodule homomorphism $H(\phi)$ is an $H(\mathcal{F})$-preenvelope of $H(M)$. From [12, Lemma 3.10(2)], it follows that the C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M),H(X))$ is injective. Then there is an exact sequence $0 \to M \xrightarrow{\phi} X \to N \to 0$ in $\mathcal{M}^{eCe}$, where $N = \mathrm{Coker}\phi$. It follows that $\mathrm{Ext}^1_{eCe}(N,F) = 0$ for any $F \in \mathcal{F}$, since $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$. Since H is an exact functor, there is the following exact sequence $0 \to H(M) \xrightarrow{H(\phi)} H(X) \to H(N) \to 0$ in $\mathcal{M}^C$. Then there is the following commutative diagram (denote $H(\cdot)$ by $H\cdot$ for convenience)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Com}_{eCe}(N,F) & \longrightarrow & \mathrm{Com}_{eCe}(X,F) & \longrightarrow & \mathrm{Com}_{eCe}(M,F) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{Com}_C(HN,HF) & \longrightarrow & \mathrm{Com}_C(HX,HF) & \longrightarrow & \mathrm{Com}_C(HM,HF) & \longrightarrow & \mathrm{Ext}^1_C(HN,HF)
\end{array}$$

where $HF \in H\mathcal{F}$, because H is a fully faithful functor. This shows that $\mathrm{Ext}^1_C(H(N),H(F)) = 0$ and $H(N) \in {}^{\perp}H(\mathcal{F})$. As a consequence, $H(N) =$

$\mathrm{Coker}(H(\phi)) \in {}^{\perp}H(\mathcal{F})$. Given the information from the two aspects, we conclude that the C-comodule homomorphism $H(\phi)$ is a special $H(\mathcal{F})$-preenvelope.

□

Theorem 3. *Let K be a field, C be a K-coalgebra and $e \in C^*$ be an idempotent, then we have:*

(1) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-preenvelope of eM for each idempotent $e \in C^*$.*

(2) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-envelope of eM for each idempotent $e \in C^*$.*

(3) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM, eX)$ is a special $e\mathcal{F}$-preenvelope of eM for each idempotent $e \in C^*$.*

Proof.

(1) The necessity is obtained from Proposition 1(1).

Sufficiency. If the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM, eX)$ is an $e\mathcal{F}$-preenvelope of eM, then

$$\mathrm{Com}_{eCe}(eX,eF) \xrightarrow{(e\phi)^*} \mathrm{Com}_{eCe}(eM,eF) \to 0$$

in $\mathcal{M}^{eCe}$ is exact. Consequently, for any $\alpha\colon eM \to eF$, there exists a $\beta\colon eX \to eF$ such that $\alpha = \beta \cdot e\phi$. Since $e(-)$ is full, there must be $f\colon M \to F$ and $g\colon X \to F$ such that $e(f) = \alpha$, $e(g) = \beta$. Therefore, it can be inferred that $e(f) = e(g)e(\phi)$, which means $e(f) = e(g\phi)$. Since $e(-)$ is full, it can be obtained that $f = g\phi$. Therefore, $\mathrm{Com}_C(X,F) \xrightarrow{\phi^*} \mathrm{Com}_C(M,F) \to 0$ in $\mathcal{M}^C$ is exact. This indicates that the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M.

(2) According to Proposition 1(2), the necessity is obvious.

Sufficiency. Suppose that the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-envelope of eM. First of all, the eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-preenvelope of eM. From (1), it follows that the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M. Furthermore, with $(eg)(e\phi) = (e\phi)$ and $eg \in \mathrm{End}_{eCe}(eX, eX)$, we deduce that the eCe-comodule endomorphism eg is an automorphism of eX. By [12, Lemma 3.7(3)], we can derive that the C-comodule endomorphism g is an automorphism of X from $\phi g = \phi$ and $g \in \mathrm{End}_C(X)$. Consequently, the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M.

(3) The necessity follows from Proposition 1(3).

Sufficiency. If the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM, eX)$ is a special $e\mathcal{F}$-preenvelope of eM, then $e\phi$ is injective and $\mathrm{Coker}(e\phi) \in {}^{\perp}(e\mathcal{F})$, where $\phi \in \mathrm{Com}_C(M,X)$. From [12, Lemma 3.7(1)], it follows that ϕ is injective. Let $\mathrm{Coker}(e\phi) = eN$, then there is the exact sequence $0 \to eM \to eX \to eN \to 0$ in $\mathcal{M}^{eCe}$. Take $eF \in e\mathcal{F}$, we get $\mathrm{Ext}^1_{\mathrm{eCe}}(eN, eF) = e\mathrm{Ext}^1_{\mathrm{C}}(N,F) = 0$, since $eN \in {}^{\perp}(e\mathcal{F})$. It follows from [12, Lemma 3.5] that $\mathrm{Ext}^1_{\mathrm{C}}(N,F) = 0$, where $F \in \mathcal{F}$. This shows that $N = \mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$, where $\phi \in \mathrm{Com}_C(M,X)$. As a consequence, the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M.

□

Theorem 4. *Let C be a coalgebra and $e \in C^*$ be an idempotent defining a localization, then we have:*

(1) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M if and only if the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is an $S(\mathcal{F})$-preenvelope of $S(M)$.*
(2) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M if and only if the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is an $S(\mathcal{F})$-envelope of $S(M)$.*
(3) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M if and only if the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is a special $S(\mathcal{F})$-preenvelope of $S(M)$.*

Proof.

(1) The necessity follows from Proposition 2(1).

Sufficiency. If the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is an $S(\mathcal{F})$-preenvelope of $S(M)$, then the eCe-comodule homomorphism $T(S(\phi))$ is a $TS(\mathcal{F})$-preenvelope of $T(S(M))$ by Proposition 1(1). By Lemma 1(2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, we infer that the eCe-comodule homomorphism $T(S(\phi)) \cong \phi$ is an $\mathcal{F}$-preenvelope of M.

(2) The necessity is obtained from Proposition 2(2).

Sufficiency. If the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is an $S(\mathcal{F})$-envelope of $S(M)$, then the eCe-comodule homomorphism $T(S(\phi))$ is a $TS(\mathcal{F})$-envelope of $T(S(M))$ by Proposition 1(2). By Lemma 1(2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, the eCe-comodule homomorphism $T(S(\phi)) \cong \phi$ is an $\mathcal{F}$-envelope of M.

(3) By Proposition 2(3), the necessity is clear.

Sufficiency. If the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is a special $S(\mathcal{F})$-preenvelope of $S(M)$, then the eCe-comodule homomorphism $T(S(\phi))$ is a special $TS(\mathcal{F})$-preenvelope of $T(S(M))$ by Proposition 1(3). By Lemma 1(2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, we infer that the eCe-comodule homomorphism $T(S(\phi)) \cong \phi$ is a special $\mathcal{F}$-preenvelope of M.

□

Theorem 5. *Let C be a coalgebra and $e \in C^*$ be an idempotent which defines a colocalization, then we have:*

(1) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M if and only if the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-preenvelope of $H(M)$.*

(2) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M if and only if the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-envelope of $H(M)$.*

(3) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M if and only if the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is a special $H(\mathcal{F})$-preenvelope of $H(M)$.*

Proof.

(1) The necessity is obtained from Proposition 3(1).
Sufficiency. If the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-preenvelope of $H(M)$, then the eCe-comodule homomorphism $T(H(\phi))$ is a $TH(\mathcal{F})$-preenvelope of $T(H(M))$ by Proposition 1(1). By Lemma 1(3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, the eCe-comodule homomorphism $T(H(\phi)) \cong \phi$ is an $\mathcal{F}$-preenvelope of M.

(2) The necessity follows from Proposition 3(2).
Sufficiency. If the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-envelope of $H(M)$, then the eCe-comodule homomorphism $T(H(\phi))$ is a $TH(\mathcal{F})$-envelope of $T(H(M))$ by Proposition 1(2). By Lemma 1(3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, the eCe-comodule homomorphism $T(H(\phi)) \cong \phi$ is an $\mathcal{F}$-envelope of M.

(3) The necessity is obtained from Proposition 3(3).
Sufficiency. If the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is a special $H(\mathcal{F})$-preenvelope of $H(M)$, then the eCe-comodule homomorphism $T(H(\phi))$ is a special $TH(\mathcal{F})$-preenvelope of $T(H(M))$ by Proposition 1(3). From Lemma 1(3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, it follows that the eCe-comodule homomorphism $T(H(\phi)) \cong \phi$ is a special $\mathcal{F}$-preenvelope of M.

□

4. (Co)localization in cotilting comodules

This section is devoted to applying the theory of (co)localization to cotilting comodules.

Definition 2. A comodule M is called a cotilting comodule if the following three conditions hold:

(1) $\mathrm{proj.dim}(M) \leq 1$;
(2) $\mathrm{Ext}^1_C(M, M^{(X)}) = 0$ for any cardinal X;
(3) there exists an exact sequence $0 \to W \to M_0 \to M_1 \to 0$, where W is a projective generator in $\mathcal{M}^C$, and $M_i \in \mathrm{Add}\, T$ for $i = 0, 1$.

Proposition 4. *Let C be a semiperfect coalgebra and $e \in C^*$ be an idempotent. Assume that M is a cotilting C-comodule and $X \cong Ce$ is a quasi-finite injective cogenerator, then $T(M)$ is a cotilting eCe-comodule.*

Proof. The proof is similar to that of Proposition 6 in Section 5. □

Proposition 5. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent defining a perfect colocalization. If M is a cotilting eCe-comodule, then $H(M)$ is a cotilting C-comodule.*

Proof. The proof is similar to that of Proposition 7 in Section 5. □

Theorem 6. *Assume that C is a K-coalgebra, idempotent $e \in C^*$ defines a perfect localization and $X \cong Ce$ is a quasi-finite injective cogenerator, then M is a cotilting eCe-comodule if and only if $H(M)$ is a cotilting C-comodule.*

Proof. The proof is similar to that of Theorem 7 in Section 5. □

5. (Co)localization in n-cotilting comodules

This section is devoted to studying the theory of the (co)localization for n-cotilting comodules.

Definition 3. A comodule M is called an n-cotilting comodule if the following three conditions hold:

(1) $\mathrm{proj.dim}(M) \leq n$;
(2) $\mathrm{Ext}^i_C(M, M^{(X)}) = 0$ for each $i > 0$ and for any cardinal X;
(3) there exists a long exact sequence $0 \to W \to M_0 \to M_1 \to \ldots \to M_n \to 0$, where W is a projective generator in $\mathcal{M}^C$ and $M_i \in \mathrm{Add}\, M$, for every $0 \leq i \leq n$.

Proposition 6. *Let C be a semi-perfect coalgebra and $e \in C^*$ be an idempotent. If M is an n-cotilting C-comodule and $X \cong Ce$ is a quasi-finite injective cogenerator, then $T(M)$ is an n-cotilting eCe-comodule.*

Proof. By [24, Theorem 1.13], we infer that $X \cong Ce$ is a quasi-finite injective cogenerator if and only if the functor T is an equivalence.

(1) By the hypothesis, we have proj.dim$(M) \leq n$, i.e., there is an exact sequence $0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$ in $\mathcal{M}^C$, where P_i's, for $0 \leq i \leq n$, are projective C-comodules. Since T is an equivalence, it follows that $0 \to T(P_n) \to \ldots \to T(P_1) \to T(P_0) \to T(M) \to 0$ is exact, where $T(P_i)'$s, for $0 \leq i \leq n$, are projective right eCe-comodules. It implies that proj.dim$(T(M)) \leq n$.

(2) Take an exact sequence $0 \to K \to P \xrightarrow{\phi} M \to 0$ in $\mathcal{M}^C$, where P is a projective C-comodule and $K = \text{Ker}\phi$. By the hypothesis, we infer that $\text{Ext}^i{}_C(M, M^{(X)}) = 0$, for each $i > 0$. Since T is an equivalence, it follows that $T(\text{Ext}^i_C(M, M^{(X)})) = \text{Ext}^i_{eCe}(TM, (TM)^{(X)}) = 0$, for every $i \geq 0$. As a consequence, we obtain $\text{Ext}^i_{eCe}(TM, (TM)^{(X)}) = 0$.

(3) From the assumption, we get the exact sequence $0 \to W \to M_0 \to M_1 \to \ldots \to M_n \to 0$ in $\mathcal{M}^C$, where W is a projective generator in $\mathcal{M}^C$ and C-comodules $M_i \in \text{Add } M$. Since T is an equivalence, there is the short exact sequence $0 \to T(W) \to T(M_0) \to T(M_1) \to \ldots \to T(M_n) \to 0$ in $\mathcal{M}^{eCe}$, where eCe-comodules $T(M_i) \in \text{Add } T(M)$.

□

Proposition 7. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent defining a perfect colocalization. If M is an n-cotilting eCe-comodule, then $H(M)$ is an n-cotilting C-comodule.*

Proof.

(1) By the hypothesis, we infer that proj.dim$(M) \leq n$, i.e., there is an exact sequence $0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$ in $\mathcal{M}^{eCe}$, where $P_i's$ are projective eCe-comodules. Since T is an exact functor, H preserves projective comodules. It follows that there is the short exact sequence $0 \to H(P_n) \to \ldots \to H(P_1) \to H(P_0) \to H(M) \to 0$ in $\mathcal{M}^C$, where $H(P_i)'s$ are projective right C-comodules, for $0 \leq i \leq n$. It implies that proj.dim$(H(M)) \leq n$.

(2) Take an exact sequence $0 \to K \to P \xrightarrow{\delta} M \to 0$ in $\mathcal{M}^{eCe}$, where P is an projective eCe-comodule and $K = \text{Ker}\delta$. By the hypothesis, we infer that $\text{Ext}^i_{eCe}(M, M^{(X)}) = 0$. Since H is a fully faithful functor, there is the following commutative diagram, see Diagram 1 (denote $H(\cdot)$ by $H\cdot$ for convenience) where $H(M^{(X)}) = (HM)^{(X)}$ because H preserves coproducts of comodules. Since H is a fully faithful functor, it follows that $H(\text{Ext}^i_{eCe}(M, M^{(X)})) = \text{Ext}^i_C(HM, H(M^{(X)})) = 0$, for each $i \geq 0$. As a consequence, we get $\text{Ext}^i_C(HM, (HM)^{(X)}) = 0$.

(3) According to the assumption, it infers that $0 \to W \to M_0 \to M_1 \to \ldots \to M_n \to 0$, where W is a projective generator in $\mathcal{M}^{eCe}$ and eCe-comodules $M_i \in \text{Add } M$. Since H is an exact functor and preserves coproducts of comodules,

there is the short exact sequence $0 \to H(W) \to H(M_0) \to H(M_1) \to \ldots \to H(M_n) \to 0$ in $\mathcal{M}^C$, where C-comodules $H(M_i) \in \text{Add}\, H(M)$.

□

Theorem 7. *Let K be a field, C be a K-coalgebra, and $e \in C^*$ be an idempotent defining a perfect localization. Suppose that $X \cong Ce$ is a quasi-finite injective cogenerator, then M is an n-cotilting eCe-comodule if and only if $H(M)$ is an n-cotilting C-comodule.*

Proof. Necessity. It follows from Proposition 7.

Sufficiency. It follows from [24, Theorem 1.13] that the functor T is an equivalence if and only if $X \cong Ce$ is a quasi-finite injective cogenerator.

(1) According to the assumption, we obtain $\text{proj.dim} H(M) \leq n$, i.e., there is an exact sequence $0 \to Q_n \to \ldots \to Q_1 \to Q_0 \to H(M) \to 0$ in $\mathcal{M}^C$, where Q_i's are projective C-comodules, for each $0 \leq i \leq n$. Since T is an equivalence, it follows that $0 \to T(Q_n) \to \ldots \to T(Q_1) \to T(Q_0) \to TH(M) \to 0$ in $\mathcal{M}^{eCe}$ is exact, where $T(P_i)'$s, for every $0 \leq i \leq n$, are projective right eCe-comodules. By Lemma 1(3) $TH \simeq 1_{\mathcal{M}/\mathcal{T}}$, we obtain the short exact sequence $0 \to T(Q_n) \to \ldots \to T(Q_1) \to T(Q_0) \to M \to 0$ in $\mathcal{M}^{eCe}$, where $T(Q_i)'$s are projective eCe-comodule. Consequently, we obtain $\text{proj.dim}(M) \leq n$.

(2) Take an exact sequence $0 \to H(K) \to H(P) \xrightarrow{\lambda} H(M) \to 0$ in $\mathcal{M}^C$, where $H(P)$ is an projective C-comodule and $H(K) = \text{Ker}\lambda$. According to the given conditions, we get $\text{Ext}^1_C(H(M), (HM)^{(X)}) = 0$. Since T is an equivalence, there is the following commutative diagram, see Diagram 2 (denote $H(\cdot)$ by $H\cdot$ for convenience) where $T((HM)^{(X)}) = TH(M^{(X)})$ because H preserves coproducts. Due to the equivalent functor T, we infer that

$$T(\text{Ext}^i_C(HM, (HM)^{(X)})) = \text{Ext}^i_{eCe}(THM, T((HM)^{(X)})) = 0.$$

As a consequence, $\text{Ext}^i_{eCe}(M, M^{(X)}) = 0$.

(3) According to the assumption, there is the short exact sequence $0 \to H(W) \to H(M_0) \to H(M_1) \to \ldots \to H(M_n) \to 0$ in $\mathcal{M}^C$, where $H(W)$ is a projective generator in $\mathcal{M}^C$ and C-comodules $H(M_i) \in \text{Add}\, H(M)$. Since T is an equivalence, we obtain the short exact sequence $0 \to T(H(W)) \to T(H(M_0)) \to T(H(M_1)) \to \ldots \to T(H(M_n)) \to 0$ in $\mathcal{M}^{eCe}$, where $TH(M_i) \in \text{Add}\, TH(M)$. By Lemma 1(3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, it infers that there is the short exact sequence $0 \to W \to M_0 \to M_1 \to \ldots \to M_n \to 0$ in $\mathcal{M}^{eCe}$, where W is a projective generator in $\mathcal{M}^{eCe}$ and eCe-comodules $M_i \in \text{Add}\, M$.

□

Remark 2. When n in Definition 3 takes the value of 1, the result is consistent with Definition 2 in Section 4. Thus, it can be seen that Section 4 is a special case

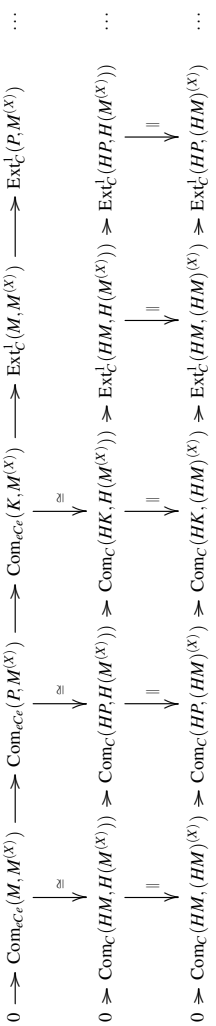


Diagram 1

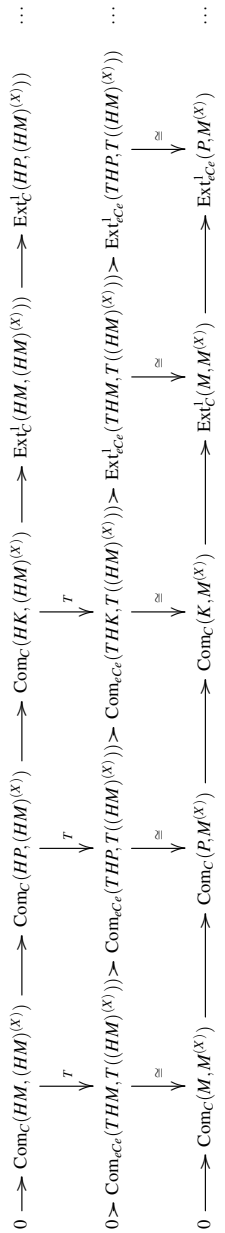


Diagram 2

of Section 5 when $n = 1$. In view of this, we omit the proofs of Proposition 4, Proposition 5, and Theorem 6 in Section 4.

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