



WEYL'S CLASSIFICATION OF SINGULAR IMPULSIVE HAHN–STURM–LIOUVILLE EQUATIONS

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Abstract. In this work, Weyl's famous classification was made for singular impulsive Hahn–Sturm–Liouville equations.

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1. INTRODUCTION

Sturm–Liouville problems are one of the important problems in the theory of differential equations. When solving partial differential equations with the Fourier method, encountering such equations increases the importance of Sturm–Liouville problems. Such problems under various conditions are intensively investigated. Impulsive boundary conditions are one of them. There are many studies in the literature under such conditions ([2, 5–9, 12–17]). In 1910 Weyl published his famous work on Sturm–Liouville problems in the singular case [18]. With this study, he made the famous limit circle/limit point classification.

In 2018, Annaby et al. investigated the Hahn–Sturm–Liouville problems in the regular case by taking the Hahn derivative instead of the classical derivative in the Sturm–Liouville problems [4]. The Hahn derivative operator gives the opportunity to gather two different operators under one roof. These are the q -difference and forward difference operators. Recently in [16], the author proved a spectral expansion theorem by constructing the spectral function of the Hahn–Sturm–Liouville equation in the singular case under impulsive conditions.

In this study, the singular Hahn–Sturm–Liouville problems defined as

$$\left[-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} + r(\zeta) \right] y(\zeta) = \lambda y(\zeta), \quad \zeta \in (\omega_0, d) \cup (d, \infty),$$

are considered under impulsive conditions. Weyl's classification has been obtained.

2. PRELIMINARIES

In this context, we provide a concise overview of the Hahn calculus [3, 4, 10, 11]. Let $q \in (0, 1)$, $\omega_0 := \omega / (1 - q)$, $\omega > 0$, and let $\Psi : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $\omega_0 \in J$.

Definition 1 ([10], [11]). We define the Hahn derivative $D_{\omega, q}\Psi$ by

$$D_{\omega, q}\Psi(\zeta) = \begin{cases} \frac{\Psi(\omega + q\zeta) - \Psi(\zeta)}{\omega + (q-1)\zeta}, & \zeta \neq \omega_0, \\ \Psi'(\omega_0), & \zeta = \omega_0. \end{cases}$$

Definition 2 ([3]). Let $a, b, \omega_0 \in J$. We define the Hahn integral (ω, q -integral) by

$$\int_a^b \Psi(\zeta) D_{\omega, q}\zeta := \int_{\omega_0}^b \Psi(\zeta) d_{\omega, q}\zeta - \int_{\omega_0}^a \Psi(\zeta) d_{\omega, q}\zeta,$$

where

$$\int_{\omega_0}^{\zeta} \Psi(t) d_{\omega, q}t := ((1 - q)\zeta - \omega) \sum_{n=0}^{\infty} q^n \Psi\left(\omega \frac{1 - q^n}{1 - q} + xq^n\right), \quad \zeta \in J$$

provided that the series converges at $\zeta = a$ and $\zeta = b$.

3. MAIN RESULTS

Consider now the following impulsive boundary-value problem (BVP)

$$\Gamma y := \left[-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} + r(\zeta) \right] y(\zeta) = \lambda y(\zeta), \quad (3.1)$$

$$y(\omega_0) \cos \beta + D_{\omega, q}y(\omega_0) \sin \beta = 0, \quad (3.2)$$

$$y(d-) - k_1 y(d+) = 0, \quad (3.3)$$

$$D_{-\frac{\omega}{q}, \frac{1}{q}}y(d-) - k_2 D_{-\frac{\omega}{q}, \frac{1}{q}}y(d+) = 0, \quad (3.4)$$

$$y\left(\frac{1}{q^\zeta}\right) \cos \delta + D_{\omega, q}y\left(\frac{1}{q^\zeta}\right) \sin \delta = 0, \quad (3.5)$$

where $\zeta \in (\omega_0, d) \cup (d, q^\zeta)$, $\beta, \delta \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\zeta \in \mathbb{N} := \{1, 2, \dots\}$, $q^{-\zeta} > d$, $k_1 k_2 = \eta > 0$; r is a real-valued, continuous function on $[\omega_0, d) \cup (d, \infty)$, one-sided limits $r(d^\pm)$ exist. The same problem has been studied by Allahverdiev and Tuna without impulsive boundary conditions [1].

$H = L^2_{\omega, q}(\omega_0, d) + L^2_{\omega, q}(d, \infty)$ is a Hilbert space endowed with the following inner product

$$\langle y, z \rangle_H := \int_{\omega_0}^d y^{(1)} \overline{z^{(1)}} d_{\omega, q}\zeta + \eta \int_d^\infty y^{(2)} \overline{z^{(2)}} d_{\omega, q}\zeta,$$

where

$$y(\zeta) = \begin{cases} y^{(1)}(\zeta), & \zeta \in (\omega_0, d) \\ y^{(2)}(\zeta), & \zeta \in (d, \infty), \end{cases} \quad \text{and} \quad z(\zeta) = \begin{cases} z^{(1)}(\zeta), & \zeta \in (\omega_0, d) \\ z^{(2)}(\zeta), & \zeta \in (d, \infty). \end{cases}$$

Set

$$\mathcal{D}_{\max} = \left\{ y \in H : \begin{array}{l} y \text{ and } D_{-\frac{\omega}{q}, \frac{1}{q}} y \text{ are continuous at } \omega_0; \text{ one-sided} \\ \text{limits } y(d\pm) \text{ and } D_{-\frac{\omega}{q}, \frac{1}{q}} y(d\pm) \text{ exist and finite,} \\ y(d-) - k_1 y(d+) = 0, \\ D_{-\frac{\omega}{q}, \frac{1}{q}} y(d-) - k_2 D_{-\frac{\omega}{q}, \frac{1}{q}} \phi_1(d+, \lambda) = 0 \\ \text{and } \Gamma y \in H \end{array} \right\}.$$

Then the *maximal operator* \mathcal{L}_{\max} on \mathcal{D}_{\max} is defined by $\mathcal{L}_{\max} y = \Gamma y$, $y \in \mathcal{D}_{\max}$.

The ω, q -Green formula is given by

$$\begin{aligned} & \int_{\omega_0}^{\zeta} \left[(\Gamma y)(t) \overline{z(t)} - y(t) \overline{(\Gamma z)(t)} \right] d_{\omega, q} t \\ &= [y, z](\zeta) - [y, z](d+) + [y, z](d-) - [y, z](\omega_0), \end{aligned} \quad (3.6)$$

where $y, z \in \mathcal{D}_{\max}$, $\zeta \in (d, \infty)$ and

$$[y, z] := y \left(\overline{D_{-\frac{\omega}{q}, \frac{1}{q}} z} \right) - \left(D_{-\frac{\omega}{q}, \frac{1}{q}} y \right) \bar{z}.$$

Theorem 1. For any $\lambda \in \mathbb{C}$, Eq. (3.1) has a solution $\phi(\zeta, \lambda)$ satisfying conditions (3.2)-(3.4) which is an entire function of λ for every $\zeta \in [\omega_0, d) \cup (d, \infty)$.

Proof. According to a study by Annaby et al. [4], the following problem

$$\begin{aligned} \left[-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} + r(\zeta) \right] y(\zeta) &= \lambda y(\zeta), \quad \zeta \in [\omega_0, d), \\ y(\omega_0) &= -\sin \beta, \\ D_{-\frac{\omega}{q}, \frac{1}{q}} y(\omega_0) &= \cos \beta, \end{aligned}$$

has a unique solution $\phi^{(1)}(\zeta, \lambda)$ which is an entire function of λ .

Now let us consider the problem

$$\left[-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} + r(\zeta) \right] y(\zeta) = \lambda y(\zeta), \quad \zeta \in (d, \infty), \quad (3.7)$$

$$y(d+) = \frac{1}{k_1} \phi^{(1)}(d-, \lambda), \quad (3.8)$$

$$D_{-\frac{\omega}{q}, \frac{1}{q}} y(d+) = \frac{1}{k_2} D_{-\frac{\omega}{q}, \frac{1}{q}} \phi^{(1)}(d-, \lambda). \quad (3.9)$$

Let

$$u_n(\zeta, \lambda) = u_0(\zeta, \lambda) + q \int_d^{\zeta} \left(\frac{S_{\omega, q}(\sqrt{\lambda} \zeta; q)}{\sqrt{\lambda}} C_{\omega, q}(\sqrt{\lambda} q \gamma; q) \right) \quad (3.10)$$

$$-C_{\omega,q}(\sqrt{\lambda}\zeta;q) \frac{S_{\omega,q}(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \Big) \times r(h(\gamma)) u_{n-1}(h(\gamma), \lambda) d_{\omega,q}\gamma,$$

where $h(\gamma) = q\gamma + \omega$,

$$u_0(\zeta, \lambda) = \frac{1}{k_1} \phi^{(1)}(d-, \lambda) + \frac{1}{k_2} (\zeta - d) D_{-\frac{\omega}{q}, \frac{1}{q}} \phi^{(1)}(d-, \lambda), \quad \zeta \in (d, \infty),$$

and the functions $\frac{S_{\omega,q}(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}}$, $C_{\omega,q}(\sqrt{\lambda}q\zeta;q)$ are the fundamental solutions of ([4])

$$-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega,q} y(\zeta) = \lambda y(\zeta). \quad (3.11)$$

It is obvious that u_n is the entire function.

The basic estimate

$$|u_{n+1}(\zeta, \lambda) - u_n(\zeta, \lambda)| \leq q^{n+1} \widetilde{\sigma}(\lambda) \frac{(A\sigma(\lambda) [\zeta(1-q) - \omega])^n}{(q; q)_n} \quad (3.12)$$

is proved by induction. For $n = 1$,

$$\begin{aligned} & |u_1(\zeta, \lambda) - u_0(\zeta, \lambda)| \\ & \leq \left| q \int_d^\zeta \begin{pmatrix} \frac{S_{\omega,q}(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}} C_{\omega,q}(\sqrt{\lambda}q\gamma;q) \\ -C_{\omega,q}(\sqrt{\lambda}\zeta;q) \frac{S_{\omega,q}(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \end{pmatrix} r(h(\gamma)) u_0(h(\gamma), \lambda) d_{\omega,q}\gamma \right| \\ & \leq q\sigma(\lambda) A \widetilde{\sigma}(\lambda) \left| \int_{\omega_0}^\zeta d_{\omega,q}\gamma \right| = q\sigma(\lambda) A \widetilde{\sigma}(\lambda) \frac{[\zeta(1-q) - \omega]}{(1-q)}, \end{aligned}$$

where $\lambda \in \mathbb{C}$; $\sigma(\lambda)$, $\widetilde{\sigma}(\lambda)$ and A are positive numbers such that

$$\left| \begin{pmatrix} \frac{S_{\omega,q}(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}} C_{\omega,q}(\sqrt{\lambda}q\gamma;q) \\ -C_{\omega,q}(\sqrt{\lambda}\zeta;q) \frac{S_{\omega,q}(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \end{pmatrix} \right| \leq \sigma(\lambda),$$

and $\max_{\zeta \in (d, a]} |r(\zeta)| = A$, $|u_0(\zeta, \lambda)| \leq \widetilde{\sigma}(\lambda)$, $\zeta \in (d, a]$.

Based on the induction hypothesis

$$\begin{aligned} & |u_n(\zeta, \lambda) - u_{n-1}(\zeta, \lambda)| \\ & \leq \left| q \int_d^\zeta \begin{pmatrix} \frac{S_{\omega,q}(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}} C_{\omega,q}(\sqrt{\lambda}q\gamma;q) \\ -C_{\omega,q}(\sqrt{\lambda}\zeta;q) \frac{S_{\omega,q}(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \end{pmatrix} \right. \\ & \quad \left. \times r(h(\gamma)) [u_{n-1}(h(\gamma), \lambda) - u_{n-2}(h(\gamma), \lambda)] d_{\omega,q}\gamma \right| \end{aligned}$$

$$\leq q^n \widetilde{\sigma}(\lambda) \frac{(A\sigma(\lambda) [\zeta(1-q) - \omega])^{n-1}}{(q; q)_{n-1}}$$

The remainder of the argument is straightforward. Hence, the series

$$u_1(\zeta, \lambda) + \sum_{n=1}^{\infty} \{u_{n+1}(\zeta, \lambda) - u_n(\zeta, \lambda)\} \quad (3.13)$$

is uniformly convergent with respect to the variable ζ on $(d, a]$, due to the series

$$\sum_{n=1}^{\infty} q^{n+1} \widetilde{\sigma}(\lambda) \frac{(A\sigma(\lambda) [\zeta(1-q) - \omega])^n}{(q; q)_n}$$

is convergent.

Let

$$\phi^{(2)}(\zeta, \lambda) = u_1(\zeta, \lambda) + \sum_{n=1}^{\infty} \{u_{n+1}(\zeta, \lambda) - u_n(\zeta, \lambda)\}.$$

Then we find

$$\lim_{n \rightarrow \infty} u_n(\zeta, \lambda) = \phi^{(2)}(\zeta, \lambda).$$

(3.10) implies

$$\begin{aligned} & D_{\omega, q} u_{n+1}(\zeta, \lambda) - D_{\omega, q} u_n(\zeta, \lambda) \\ &= q \int_d^\zeta \left(\begin{array}{c} D_{\omega, q} \frac{S_{\omega, q}(\sqrt{\lambda}\zeta; q)}{\sqrt{\lambda}} C_{\omega, q}(\sqrt{\lambda}q\gamma; q) \\ - D_{\omega, q} C_{\omega, q}(\sqrt{\lambda}\zeta; q) \frac{S_{\omega, q}(\sqrt{\lambda}q\gamma; q)}{\sqrt{\lambda}} \end{array} \right) \\ & \quad \times r(h(\gamma)) \begin{bmatrix} u_n(h(\gamma), \lambda) \\ -u_{n-1}(h(\gamma), \lambda) \end{bmatrix} d_{\omega, q}\gamma, \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_{n+1}(\zeta, \lambda) + \frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_n(\zeta, \lambda) \\ &= q \int_d^\zeta \left(\begin{array}{c} -\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} \frac{S_{\omega, q}(\sqrt{\lambda}\zeta; q)}{\sqrt{\lambda}} C_{\omega, q}(\sqrt{\lambda}q\gamma; q) \\ + \frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} C_{\omega, q}(\sqrt{\lambda}\zeta; q) \frac{S_{\omega, q}(\sqrt{\lambda}q\gamma; q)}{\sqrt{\lambda}} \end{array} \right) \\ & \quad \times r(h(\gamma)) \begin{bmatrix} u_n(h(\gamma), \lambda) \\ -u_{n-1}(h(\gamma), \lambda) \end{bmatrix} d_{\omega, q}\gamma - r(\zeta) [u_n(\zeta, \lambda) - u_{n-1}(\zeta, \lambda)]. \end{aligned}$$

From (3.12), we conclude that the series

$$\sum_{n=1}^{\infty} (D_{\omega, q} u_{n+1}(\zeta, \lambda) - D_{\omega, q} u_n(\zeta, \lambda))$$

and

$$\sum_{n=1}^{\infty} \left(-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_{n+1}(\zeta, \lambda) + \frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_n(\zeta, \lambda) \right)$$

are uniformly convergent on (d, ∞) with respect to the variable ζ for every $\lambda \in \mathbb{C}$. From (3.11), we find

$$\begin{aligned} & -\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} \phi^{(2)}(\zeta, \lambda) \\ &= \sum_{n=1}^{\infty} \left(-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_{n+1}(\zeta, \lambda) + \frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_n(\zeta, \lambda) \right) \\ &= (\lambda - v(\zeta)) \sum_{n=1}^{\infty} (u_n(\zeta, \lambda) - u_{n-1}(\zeta, \lambda)) = (\lambda - v(\zeta)) \phi^{(2)}(\zeta, \lambda). \end{aligned}$$

It follows easily that $\phi^{(2)}$ satisfies (3.8)-(3.9). Thus, we see that the function

$$\phi(\zeta, \lambda) = \begin{cases} \phi^{(1)}(\zeta, \lambda), & \zeta \in [0, d] \\ \phi^{(2)}(\zeta, \lambda), & \zeta \in (d, \infty) \end{cases} \quad (3.14)$$

satisfies the BVP (3.1)-(3.4). \square

A similar theorem can be proved.

Theorem 2. For any $\lambda \in \mathbb{C}$, Eq. (3.1) has a solution

$$\psi(\zeta, \lambda) = \begin{cases} \psi^{(1)}(\zeta, \lambda), & \zeta \in [\omega_0, d] \\ \psi^{(2)}(\zeta, \lambda), & \zeta \in (d, \infty) \end{cases} \quad (3.15)$$

satisfying conditions (3.3)-(3.5) which is an entire function of λ for every $\zeta \in [\omega_0, d] \cup (d, \infty)$.

Consider

$$\phi(\zeta, \lambda) = \begin{cases} \phi^{(1)}(\zeta), & \zeta \in [\omega_0, d] \\ \phi^{(2)}(\zeta), & \zeta \in (d, \infty) \end{cases}$$

and

$$\psi(\zeta, \lambda) = \begin{cases} \psi^{(1)}(\zeta), & \zeta \in [\omega_0, d] \\ \psi^{(2)}(\zeta), & \zeta \in (d, \infty) \end{cases}$$

as solutions of the BVP (3.1)-(3.3) satisfying

$$\begin{aligned} \phi^{(1)}(\omega_0, \lambda) &= \sin \beta, \quad \psi^{(1)}(\omega_0, \lambda) = \cos \beta, \\ D_{\omega, q} \phi^{(1)}(\omega_0, \lambda) &= -\cos \beta, \quad D_{\omega, q} \psi^{(1)}(\omega_0, \lambda) = \sin \beta, \end{aligned} \quad (3.16)$$

where $0 \leq \beta < \pi$.

Write

$$W_{\omega, q}^{(i)} := W_{\omega, q}(\psi^{(i)}, \phi^{(i)}) \quad (i = 1, 2),$$

where

$$W_{\omega, q}(y, z) := y D_{\omega, q} z - z D_{\omega, q} y. \quad (3.17)$$

An explicit calculation shows that $W_{\omega,q}^{(1)} = \eta W_{\omega,q}^{(2)}$. Let $W_{\omega,q} := W_{\omega,q}^{(1)} = \eta W_{\omega,q}^{(2)}$. It follows from (3.6) that

$$\int_{\omega_0}^{\zeta} \Gamma y \bar{z} d_{\omega,q} t - \int_{\omega_0}^{\zeta} y \bar{\Gamma z} d_{\omega,q} t = W_{\omega,q}(y, \bar{z})(h^{-1}(\zeta)) - W_{\omega,q}(y, \bar{z})(\omega_0), \quad (3.18)$$

where $h(\zeta) = q\zeta + \omega$ and $\zeta \in (\omega_0, d)$.

Theorem 3. *Let ϕ and ψ be two solutions of the BVP (3.1), (3.3), (3.4). Then, $W_{\omega,q}(\phi, \psi)$ is independent of ζ .*

Proof. Let $\lambda \in \mathbb{C}$ and $\zeta \in (\omega_0, d)$. By (3.6), we find

$$\begin{aligned} \int_{\omega_0}^{\zeta} \Gamma \phi \psi d_{q,t} - \int_{\omega_0}^{\zeta} \phi \Gamma \psi d_{q,t} &= (\lambda - \lambda) \int_{\omega_0}^{\zeta} \phi(t, \lambda) \psi(t, \lambda) d_{\omega,q} t \\ &= W_{\omega,q}(\phi, \psi)(h^{-1}(\zeta), \lambda) - W_{\omega,q}(\phi, \psi)(\omega_0, \lambda). \end{aligned} \quad (3.19)$$

since $\Gamma \phi = \lambda \phi$ and $\Gamma \psi = \lambda \psi$. Hence

$$W_{\omega,q}(\phi, \psi)(h^{-1}(\zeta), \lambda) = W_{\omega,q}(\phi, \psi)(\omega_0, \lambda).$$

Likewise, for $\zeta \in (d, \infty)$, we obtain

$$W_{\omega,q}(\phi, \psi)(h^{-1}(\zeta), \lambda) = \frac{1}{\eta} W_q(\phi, \psi)(\omega_0, \lambda).$$

□

Lemma 1. *Let $\phi(\zeta, \lambda)$ be a solution of the BVP (3.1), (3.3), (3.4) and let $\sigma = \text{Im } \lambda$, where $\lambda \in \mathbb{C}$. Then the following relations hold:*

1)

$$2i\sigma \int_{\omega_0}^b |\phi(\zeta, \lambda)|^2 d_{\omega,q} \zeta = W_{\omega,q}(\phi, \bar{\phi})(h^{-1}(b), \lambda) - W_{\omega,q}(\phi, \bar{\phi})(\omega_0, \lambda), \quad (3.20)$$

where $b \in (\omega_0, d)$.

2)

$$\begin{aligned} 2i\sigma \left[\int_{\omega_0}^d |\phi(\zeta, \lambda)|^2 d_{\omega,q} \zeta + \eta \int_d^b |\phi(\zeta, \lambda)|^2 d_{\omega,q} \zeta \right] \\ = W_{\omega,q}(\phi, \bar{\phi})(h^{-1}(b), \lambda) - W_{\omega,q}(\phi, \bar{\phi})(\omega_0, \lambda), \end{aligned} \quad (3.21)$$

where $b \in (d, \infty)$.

Proof. The proof is straightforward. □

The solution of (3.1) is given by

$$\vartheta(\zeta, \lambda) := \phi(\zeta, \lambda) + m\psi(\zeta, \lambda),$$

where $\zeta \in [\omega_0, d) \cup (d, \infty)$ and m is a constant. Then ϑ satisfies (3.5) if and only if

$$m = -\frac{\Xi \cos \delta + \Omega \sin \delta}{\Phi \cos \delta + \Upsilon \sin \delta}, \quad (3.22)$$

where

$$\begin{aligned} \Omega &= D_{\omega, q} \phi \left(\frac{1}{q^\zeta}, \lambda \right), \quad \Upsilon = D_{\omega, q} \psi \left(\frac{1}{q^\zeta}, \lambda \right), \\ \Xi &= \phi \left(\frac{1}{q^\zeta}, \lambda \right), \quad \Phi = \psi \left(\frac{1}{q^\zeta}, \lambda \right). \end{aligned}$$

From (3.22), we obtain

$$m = -\frac{\Xi z + \Omega}{\Phi z + \Upsilon}, \quad (3.23)$$

where $\cot \delta = z$. Thus, m describes a circle $C_{\frac{1}{q^\zeta}}$ ($\zeta \in \mathbb{N}$, $q^{-\zeta} > d$). Now let's calculate the radius and center of this circle

Theorem 4. *Let $\zeta \in \mathbb{N}$, $q^{-\zeta} > d$ and $\lambda \in \mathbb{C}$, where $\sigma = \text{Im} \lambda \neq 0$. Then, we have*

$$\Theta_\zeta(\lambda) = -\frac{W_{\omega, q}(\phi, \bar{\psi}) \left(\frac{1}{q^\zeta}, \lambda \right)}{W_{\omega, q}(\psi, \bar{\psi}) \left(\frac{1}{q^\zeta}, \lambda \right)}, \quad (3.24)$$

$$r_\zeta(\lambda) = \left(2\sigma \int_{\omega_0}^d |\psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + 2\sigma \eta \int_d^{\frac{1}{q^\zeta}} |\psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta \right)^{-1}, \quad (3.25)$$

where Θ_ζ is the center of the circle $C_{\frac{1}{q^\zeta}}$ and r_ζ is its radius.

Proof. Clearly, $\Theta_\zeta(\lambda)$ is the symmetric point at ∞ . Set $m(\lambda, z') = \infty$, $m(\lambda, z'') = \Theta_\zeta(\lambda)$. Then we conclude that $z' = \bar{z}''$. On the other hand,

$$m(\lambda, z') = \infty \Leftrightarrow z' = -\frac{D_{\omega, q} \Psi \left(\frac{1}{q^\zeta}, \lambda \right)}{\Psi \left(\frac{1}{q^\zeta}, \lambda \right)}.$$

Hence we find

$$\begin{aligned} \Theta_\zeta(\lambda) &= m \left(\lambda, -\frac{D_{\omega, q} \Psi \left(\frac{1}{q^\zeta}, \lambda \right)}{\Psi \left(\frac{1}{q^\zeta}, \lambda \right)} \right) \\ &= \frac{\phi \left(\frac{1}{q^\zeta}, \lambda \right) \left(-\frac{D_{\omega, q} \Psi \left(\frac{1}{q^\zeta}, \lambda \right)}{\Psi \left(\frac{1}{q^\zeta}, \lambda \right)} \right) + D_{\omega, q} \phi \left(\frac{1}{q^\zeta}, \lambda \right)}{\Psi \left(\frac{1}{q^\zeta}, \lambda \right) \left(-\frac{D_{\omega, q} \Psi \left(\frac{1}{q^\zeta}, \lambda \right)}{\Psi \left(\frac{1}{q^\zeta}, \lambda \right)} \right) + D_{\omega, q} \Psi \left(\frac{1}{q^\zeta}, \lambda \right)} \end{aligned}$$

$$\begin{aligned}
&= -\frac{\phi\left(\frac{1}{q^\zeta}, \lambda\right) D_{\omega, q} \Psi\left(\frac{1}{q^\zeta}, \bar{\lambda}\right) - \Psi\left(\frac{1}{q^\zeta}, \bar{\lambda}\right) D_{\omega, q} \phi\left(\frac{1}{q^\zeta}, \lambda\right)}{\Psi\left(\frac{1}{q^\zeta}, \lambda\right) D_{\omega, q} \Psi\left(\frac{1}{q^\zeta}, \bar{\lambda}\right) - \Psi\left(\frac{1}{q^\zeta}, \bar{\lambda}\right) D_{\omega, q} \Psi\left(\frac{1}{q^\zeta}, \lambda\right)} \\
&= -\frac{W_{\omega, q}(\phi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right)}{W_{\omega, q}(\Psi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right)},
\end{aligned}$$

where $\zeta \in \mathbb{N}$. Since $W_{\omega, q}(\phi, \Psi)\left(\frac{1}{q^\zeta}, \lambda\right) = \frac{1}{\eta}$ ($\zeta \in \mathbb{N}$, $q^{-\zeta} > d$) we see that

$$\begin{aligned}
r_n(\lambda) &= \left| \frac{D_{\omega, q} \phi\left(\frac{1}{q^\zeta}, \lambda\right)}{D_{\omega, q} \Psi\left(\frac{1}{q^\zeta}, \lambda\right)} - \frac{W_{\omega, q}(\phi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right)}{W_{\omega, q}(\Psi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right)} \right| \\
&= \left| \frac{D_{\omega, q} \Psi\left(\frac{1}{q^\zeta}, \bar{\lambda}\right) W_{\omega, q}(\phi, \Psi)\left(\frac{1}{q^\zeta}, \lambda\right)}{D_{\omega, q} \Psi\left(\frac{1}{q^\zeta}, \lambda\right) W_{\omega, q}(\Psi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right)} \right| \\
&= \left| \frac{W_{\omega, q}(\phi, \Psi)\left(\frac{1}{q^\zeta}, \lambda\right)}{W_{\omega, q}(\Psi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right)} \right| = \frac{1}{\eta \left| W_{\omega, q}(\Psi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right) \right|}.
\end{aligned}$$

By Lemma 1, we deduce that

$$W_{\omega, q}(\Psi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right) = 2i\sigma \int_{\omega_0}^d |\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + 2i\sigma\eta \int_d^{\frac{1}{q^\zeta}} |\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta.$$

implies

$$\left| W_{\omega, q}(\Psi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right) \right| = 2|\sigma| \int_{\omega_0}^d |\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + 2|\sigma|\eta \int_d^{\frac{1}{q^\zeta}} |\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta,$$

which proves the theorem. \square

Theorem 5. Let $\lambda = \rho + i\sigma$, where $\text{Im } \lambda = \sigma > 0$. Then the exterior of the circle $C_{\frac{1}{q^\zeta}}$ ($\zeta \in \mathbb{N}$) is mapped onto the upper half-plane of the z -plane.

Proof. By (3.21), we conclude that

$$\begin{aligned}
\text{Im} \left\{ \frac{D_{\omega, q} \Psi\left(\frac{1}{q^\zeta}, \lambda\right)}{\Psi\left(\frac{1}{q^\zeta}, \lambda\right)} \right\} &= \frac{1}{2} i \left\{ -\frac{D_{\omega, q} \Psi\left(\frac{1}{q^\zeta}, \lambda\right)}{\Psi\left(\frac{1}{q^\zeta}, \lambda\right)} + \frac{D_{\omega, q} \Psi\left(\frac{1}{q^\zeta}, \bar{\lambda}\right)}{\Psi\left(\frac{1}{q^\zeta}, \bar{\lambda}\right)} \right\} \\
&= \frac{1}{2} i \frac{W_{\omega, q}(\bar{\Psi}, \Psi)\left(\frac{1}{q^\zeta}, \lambda\right)}{\left| \Psi\left(\frac{1}{q^\zeta}, \lambda\right) \right|^2} = -\frac{1}{2} i \frac{W_{\omega, q}(\Psi, \bar{\Psi})\left(\frac{1}{q^\zeta}, \lambda\right)}{\left| \Psi\left(\frac{1}{q^\zeta}, \lambda\right) \right|^2}
\end{aligned}$$

$$= \frac{\sigma}{\left| \Psi\left(\frac{1}{q^\varsigma}, \lambda\right) \right|^2} \left(\int_{\omega_0}^d |\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + \eta \int_d^{\frac{1}{q^\varsigma}} |\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta \right) > 0$$

where $\varsigma \in \mathbb{N}$. □

Theorem 6. *Let $\varsigma \in \mathbb{N}$, $q^{-\varsigma} > d$ and $\text{Im } \lambda = \sigma > 0$. Then m lies on the circles $C_{\frac{1}{q^\varsigma}}$ if and only if*

$$\int_{\omega_0}^d |\Phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + \eta \int_d^{\frac{1}{q^\varsigma}} |\Phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta = \frac{\text{Im } m}{\sigma}, \quad (3.26)$$

and m is inside $C_{\frac{1}{q^\varsigma}} \Leftrightarrow$

$$\int_{\omega_0}^d |\Phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + \eta \int_d^{\frac{1}{q^\varsigma}} |\Phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta < \frac{\text{Im } m}{\sigma}.$$

Proof. Let $m \in \mathbb{C}$. Then, we obtain

$$\begin{aligned} W_{\omega, q}(\Phi + m\Psi, \overline{\Phi + m\Psi})(\omega_0, \lambda) &= W_{\omega, q}(\Phi, \overline{\Phi})(\omega_0, \lambda) + mW_q(\Psi, \overline{\Phi})(\omega_0, \lambda) \\ &\quad + \overline{m}W_{\omega, q}(\Phi, \overline{\Psi})(\omega_0, \lambda) + |m|^2 W_{\omega, q}(\Psi, \overline{\Psi})(\omega_0, \lambda) \\ &= -m + \overline{m} = -2i \text{Im } m. \end{aligned}$$

By Lemma 1, we infer that

$$\begin{aligned} 2\sigma \int_{\omega_0}^d |\Phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + 2\sigma\eta \int_d^{\frac{1}{q^\varsigma}} |\Phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta \\ = \frac{1}{i} \left(W_{\omega, q}(\Phi + m\Psi, \Phi + m\Psi) \left(\frac{1}{q^\varsigma}, \lambda \right) + 2i \text{Im } m \right), \end{aligned} \quad (3.27)$$

where $\varsigma \in \mathbb{N}$, $q^{-\varsigma} > d$. From Theorem 5, for $\sigma > 0$, m is inside $C_{\frac{1}{q^\varsigma}}$ if $\text{Im } z < 0$. (3.23) implies

$$m = -\frac{\Xi z + \Omega}{\Phi z + \Upsilon} \Leftrightarrow z = -\frac{\Omega + m\Upsilon}{\Xi + m\Phi}.$$

Hence

$$i(z - \bar{z}) = i \left\{ -\frac{\Omega + m\Upsilon}{\Xi + m\Phi} + \frac{\overline{\Omega + m\Upsilon}}{\overline{\Xi + m\Phi}} \right\} = i \frac{W_{\omega, q}(\Phi + m\Psi, \overline{\Phi + m\Psi}) \left(\frac{1}{q^\varsigma}, \lambda \right)}{|\Xi + m\Phi|^2},$$

where $\varsigma \in \mathbb{N}$, $q^{-\varsigma} > d$. Therefore

$$\text{Im } z < 0 \Leftrightarrow iW_q(\Phi + m\Psi, \overline{\Phi + m\Psi}) \left(\frac{1}{q^\varsigma}, \lambda \right) > 0, \quad (3.28)$$

where $\varsigma \in \mathbb{N}, q^{-\varsigma} > d$. By combining (3.27) and (3.28), we see that

$$\int_{\omega_0}^d |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta + \eta \int_d^{\frac{1}{q^\varsigma}} |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta < \frac{\text{Im} m}{\sigma}, \quad (3.29)$$

where $\varsigma \in \mathbb{N}, q^{-\varsigma} > d$. Since m is on $C_{\frac{1}{q^\varsigma}} \Leftrightarrow \text{Im} z = 0$, we infer that

$$W_{\omega, q}(\phi + m\psi, \overline{\phi + m\psi})\left(\frac{1}{q^\varsigma}, \lambda\right) = 0, \quad (3.30)$$

where $\varsigma \in \mathbb{N}, q^{-\varsigma} > d$. By virtue of (3.30) and (3.27), we get the desired result. \square

Theorem 7. Let $\text{Im} \lambda = \sigma > 0$ and let k be a point such that $d < \frac{1}{q^k} < \frac{1}{q^\varsigma}$. Then $C_{\frac{1}{q^\varsigma}} \subset C_{\frac{1}{q^k}}$.

Proof. By (3.27), we see that

$$\begin{aligned} & \int_{\omega_0}^d |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta + \eta \int_d^{\frac{1}{q^k}} |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta \\ & < \int_{\omega_0}^d |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta + \eta \int_d^{\frac{1}{q^\varsigma}} |\phi(\zeta, \lambda) + m\psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta < \frac{\text{Im} m}{\sigma}, \end{aligned}$$

which proves the theorem. \square

Corollary 1. As $\varsigma \rightarrow \infty$, the circles $C_{\frac{1}{q^\varsigma}}$ contract either to a circle or a point. These represent limit-circle and limit-point cases respectively.

Theorem 8. Let m be a point lying on or inside the limiting circle C_∞ , and $\Psi(\zeta, \lambda) := \phi(\zeta, \lambda) + m\psi(\zeta, \lambda)$, $\text{Im} \lambda = \sigma > 0$, be the solution of (3.1). Then we have

$$\int_{\omega_0}^d |\Psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta + \eta \int_d^\infty |\Psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta < \infty,$$

i.e., $\Psi(\cdot, \lambda) \in H$.

Proof. Let $\varsigma \in \mathbb{N}(q^{-\varsigma} > d)$. Since

$$\int_{\omega_0}^d |\Psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta + \eta \int_d^{\frac{1}{q^\varsigma}} |\Psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta < \frac{\text{Im} m}{\sigma},$$

we arrive at

$$\int_{\omega_0}^d |\Psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta + \eta \int_d^\infty |\Psi(\zeta, \lambda)|^2 d_{\omega, q}\zeta < \frac{\text{Im} m}{\sigma},$$

i.e., $\Psi(\cdot, \lambda) \in H$. \square

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