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ON THE REPRESENTATION OF SOLUTION FOR A CLASS OF PERTURBED CONTROLLED NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATION WITH THE CONTINUOUS INITIAL CONDITION

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Abstract. The analytic relation between of solutions of the original Cauchy problem and a corresponding perturbed problem is established for the controlled neutral functional-differential equation with the continuous initial condition, whose right-hand side is linear with respect to the prehistory of the phase velocity. In the representation formula of solution the effects of perturbations of the delay parameter containing in the phase coordinates, of the initial and control functions are revealed. Continuity at the initial moment means that at the initial moment values of the initial function and trajectory always coincide. The representation formula of solution plays an important role in proving the necessary conditions of optimality in neutral optimization problems, allows one to get an approximate solution of the perturbed equation and to carry out a sensitivity analysis of mathematical models.

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1. Introduction

The neutral functional-differential equation is a mathematical model of such system whose behavior at a given moment depends on the velocity of the system in the past. Many real processes are described by neutral functional-differential equations [3, 6, 8, 14, 23]. Many works are dedicated to the investigation of neutral functional-differential equations and neutral optimization problems, including [2, 8, 11, 14, 18, 20, 23, 24]. In the paper the quasi-linear neutral functional-differential equation

$$\dot{x}(t) = A(t,x(t))\dot{x}(t-\sigma) + f(t,x(t),x(t-\tau_0),u_0(t)), t \in I = [t_0,t_1] \tag{1.1}$$

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with the continuous initial condition

$$x(t) = \varphi_0(t), t \le t_0 \tag{1.2}$$

is considered. The condition (1.1) is called the continuous initial condition because always $x(t_0) = \varphi_0(t_0)$. Let $x_0(t)$ be solution of the original Cauchy problem (1.1) - (1.2) and let $x(t)$ be solution of the corresponding perturbed (with respect to delay τ_0, initial function $\varphi_0(t)$ and control function $u_0(t)$) problem. In the paper, for the first time the analytic relation between solutions $x_0(t)$ and $x(t)$ is proved on the interval I in the case when the coefficient $A(\cdot)$ depends on the phase coordinate and perturbations do not depends on a small parameter $\varepsilon > 0$. Moreover, the essential novelty here is the effects of the continuous initial condition (1.2) and perturbation of the delay τ_0 in the representation formula. We note that such analytic relation plays an important role in proving the necessary conditions of optimality [1, 4, 5, 7, 9, 10, 12, 13, 16, 17, 21, 23]. Besides, such relation allows one to get an approximate solution of the perturbed equation and to carry out a sensitivity analysis of mathematical models. The case when $A(t,x(t)) \equiv A(t)$ and perturbations depends on a parameter ε is considered in [11, 18, 24]. The case when $A(t,x(t)) \equiv 0$ is considered in [13, 15, 19, 21, 22]. The paper is organized as follows. In Section 2, the main theorem is formulated and some comments are given. In Section 3 the auxiliary lemmas are given and proved. In Section 4 the main theorem is proved.

2. FORMULATION OF THE MAIN RESULT

Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x = (x^1,\dots,x^n)^T$ and let $O \subset \mathbb{R}^n$, $U \subset \mathbb{R}^r$ be convex open sets; let $\sigma > 0$ and $\tau_2 > \tau_1 > 0$ be given numbers, with

$$t_0 + \max\{\sigma, \tau_2\} < t_1. \tag{2.1}$$

Suppose that the $n \times n$ -dimensional matrix function $A(t,x)$ is continuous on the set $I \times O$ and continuously differentiable with respect to $x^i, i = 1,2,\dots,n$; moreover, there exists $M_1 > 0$ such that

$$|A(t,x)| + \sum_{i=1}^{n} \left| \frac{\partial}{\partial x^i} A(t,x) \right| \le M_1, \forall (t,x) \in I \times O. \tag{2.2}$$

Let the n-dimensional function $f(t,x,y,u)$ be continuous on the set $I \times O^2 \times U$ and continuously differentiable with respect to x,y,u; moreover, there exists $M_2 > 0$ such that

$$|f(t,x,y,u)| + |f_x(\cdot)| + |f_y(\cdot)| + |f_u(\cdot)| \le M_2, \forall (t,x,y,u) \in I \times O^2 \times U. \tag{2.3}$$

Further, denote by Φ and Ω the sets of continuous differentiable functions $\varphi(t) \in O, t \in I_1 = [\hat{\tau}, t_0]$, where $\hat{\tau} = t_0 - \max\{\sigma, \tau_2\}$ and measurable functions $u(t) \in U, t \in I$, respectively, with the set $clu(I)$ is compact and $clu(I) \subset U$.

To each element

$$\mu = (\tau, \varphi(t), u(t)) \in \Lambda = (\tau_1, \tau_2) \times \Phi \times \Omega$$

we assign the quasi-linear neutral functional-differential equation

$$\dot{x}(t) = A(t, x(t))\dot{x}(t-\sigma) + f(t, x(t), x(t-\tau), u(t)), t \in I \tag{2.4}$$

with the continuous initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0]. \tag{2.5}$$

Definition 1. Let $\mu \in \Lambda$, a function $x(t) = x(t;\mu) \in O, t \in [\hat{\tau}, t_1]$ is called a solution of equation (2.4) with the condition (2.5) or a solution corresponding to the element μ and defined on the interval I_1 if it satisfies the condition (2.5) and is absolutely continuous on the interval I and satisfies equation (2.4) almost everywhere (a.e.) on I.

Let us introduce the notations:

$$|\mu| = |\tau| + \|\varphi\|_1 + \|u\|, \quad \Lambda_\varepsilon(\mu_0) = \Big\{\mu \in \Lambda : |\mu - \mu_0| \le \varepsilon\Big\},$$

where

$$\|\varphi\|_1 = \sup\Big\{|\varphi(t)| + |\dot{\varphi}(t)| : t \in I_1\Big\}, \quad \|u\| = \sup\Big\{|u(t)| : t \in I\Big\},$$

$\varepsilon > 0$ is a fixed number and $\mu_0 = (\tau_0, \varphi_0(t), u_0(t)) \in \Lambda$ is a fixed element; furthermore,

$$\delta\tau = \tau - \tau_0, \quad \delta\varphi(t) = \varphi(t) - \varphi_0(t), \quad \delta u(t) = u(t) - u_0(t),$$
$$\delta\mu = \mu - \mu_0 = (\delta\tau, \delta\varphi(t), \delta u(t)),$$

$$\begin{cases} |\delta\mu| = |\delta\tau| + \|\delta\varphi\|_1 + \|\delta u\|, \\ \|\delta\varphi\|_1 = \sup\{|\delta\varphi(t)| + |\dot{\delta\varphi}(t)| : t \in I_1\}. \end{cases} \tag{2.6}$$

Let $x(t;\mu_0)$ be solution corresponding to the element $\mu_0 \in \Lambda$ and defined on the interval I_1. Then there exists a number $\varepsilon_1 > 0$ such that to each element $\mu = \mu_0 + \delta\mu \in \Lambda_{\varepsilon_1}(\mu_0)$ corresponds solution $x(t;\mu)$ i. e. Cauchy's perturbed problem has solution, defined on the interval I_1 (see Lemma 1 in the Section 3).

Theorem 1. *Let $x_0(t) = x(t;\mu_0)$ be solution corresponding to the element $\mu_0 = (\tau_0, \varphi_0(t), u_0(t)) \in \Lambda$ and defined on the interval I_1. Then there exist number $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for arbitrary $\mu \in \Lambda_{\varepsilon_2}(\mu_0)$ on the interval I the following representation holds:*

$$x(t;\mu) = x_0(t) + \delta x(t;\delta\mu) + o(t;\delta\mu), \tag{2.7}$$

where

$$\begin{aligned} \delta x(t;\delta\mu) = {} & \Psi(t_0;t)\delta\varphi(t_0) + \int_{t_0-\sigma}^{t_0} Y(\xi+\sigma;t)A[\xi+\sigma]\dot{\delta\varphi}(\xi)d\xi \\ & + \int_{t_0-\tau_0}^{t_0} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi + \int_{t_0}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi \\ & - \Big\{\int_{t_0}^{t} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big\}\delta\tau \end{aligned} \tag{2.8}$$

and

$$\lim_{|\delta\mu|\to 0} o(t;\delta\mu)/|\delta\mu| = 0 \text{ uniformly for } t\in I.$$

Here,

$$A[\xi]=A(\xi,x_0(\xi)), f_y[\xi]=f_y(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi));$$

for the fixed $t\in(t_0,t_1)$, $n\times n$ matrix functions $\Psi(\xi;t)$ and $Y(\xi;t)$ satisfy the linear system

$$\begin{cases}\Psi_\xi(\xi;t) & =-Y(\xi;t)\Big\{\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]+f_x[\xi]\Big\}\\ & \quad -Y(\xi+\tau_0;t)f_y[\xi+\tau_0],\\ Y(\xi;t) & =\Psi(\xi;t)+Y(\xi+\sigma;t)A[\xi+\sigma],\xi\in(t_0,t)\end{cases} \tag{2.9}$$

and the condition

$$\Psi(\xi;t)=Y(\xi;t)=\begin{cases}E, \ \xi=t,\\ \Theta, \ \xi>t,\end{cases} \tag{2.10}$$

where E is the identity matrix and Θ is the zero matrix and

$$\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]=\frac{\partial}{\partial x}\Big[A(\xi,x)\dot{x}_0(\xi-\sigma)\Big]_{x=x_0(\xi)}.$$

Some Comments. The function $\delta x(t;\delta\mu)$ in (2.7) is called the first variation of solution $x_0(t)$. The expression (2.8) is called the variation formula of solution. The term "variation formula of solution" has been introduced by R. V. Gamkrelidze and proved for the ordinary differential equation in [5].

The expression

$$Y(\xi;t_0)\delta\varphi(t_0)+\int_{t_0-\sigma}^{t_0}Y(\xi+\sigma;t)A[\xi+\sigma]\delta\dot{\varphi}(\xi)d\xi$$
$$+\int_{t_0-\tau_0}^{t_0}Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

in formula (2.8) is the effect of perturbation $\varphi_0(t)$, where $Y(\xi;t_0)\delta\varphi(t_0)$ is the effect of the continuous initial condition.

The addend

$$\int_{t_0}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi$$

in formula (2.8) is the effect of perturbation $u_0(t)$.

The expression

$$\Big\{\int_{t_0}^{t}Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big\}\delta\tau$$

in formula (2.8) is the effect of perturbation τ_0.

Formula (2.7) allows us to obtain an approximate solution of the perturbed equation in the analytical form on the interval I. In fact, for a small $|\delta\mu|$ from (2.7) it follows

$$x(t;\delta\mu)\approx x_0(t)+\delta x(t;\delta\mu),$$

where $\delta x(t;\delta\mu)$ has the form (2.8). We note that in order to construct $\delta x(t;\delta\mu)$ it is sufficient to find a solution to the linear problem (2.9)- (2.10).

3. AUXILIARY ASSERTIONS

Lemma 1. *[23] Let $x_0(t) = x(t;\mu_0)$ be the solution corresponding to the element $\mu_0 = (\tau_0, \varphi_0(t), u_0(t)) \in \Lambda$, defined on the interval I_1. Then there exists a number $\varepsilon_1 > 0$ such that to each element $\mu = (\tau, \varphi(t), u(t)) \in \Lambda_{\varepsilon_1}(\mu_0)$ there corresponds solution $x(t) = x(t;\mu) \in O$ defined on the interval I_1.*

Lemma 2. *There exist numbers $L_i > 0, i = 1,2$ such that the following inequalities hold*

$$|A(t,x) - A(t,y)| \le L_1|x-y|, \forall (t,x,y) \in I \times O^2;$$
$$|f(t,x_1,y_1,u_1) - f(t,x_2,y_2,u_2)| \le L_2\left[|x_1-x_2| + |y_1-y_2| + |u_1-u_2|\right],$$
$$\forall t \in I,\ \forall (x_i,y_i,u_i) \in O^2 \times U, i = 1,2.$$

On the basis of (2.2) and (2.3) the Lemma 2 can be proved analogously to Lemma 2.2 (see [21]).

Lemma 3. *There exist a number $N > 0$ such that the following inequality holds*

$$|\dot{x}_0(t)| \le N,\ a.e.\ t \in I_1. \tag{3.1}$$

Proof. Let $t \in [\hat{\tau}, t_0)$ then

$$|\dot{x}_0(t)| = |\dot{\varphi}_0(t)| < \|\varphi_0\|_1 = N_0.$$

Let $t \in [t_0, t_0+\sigma]$ then

$$|\dot{x}_0(t)| = |A[t]\dot{x}_0(t-\sigma) + f[t]| \le M_1N_0 + M_2 = N_1,$$

where

$$A[t] = A(t,x_0(t)),\ f[t] = f(t,x_0(t),x_0(t-\tau_0),u_0(t))$$

(see (2.2),(2.3)). If $t \in [t_0+\sigma, t_0+2\sigma]$ we get

$$|\dot{x}_0(t)| \le M_1N_1 + M_2 = N_2.$$

Continuing this process to t_1 we get the finite quantity numbers $N_0, N_1, \ldots, N_k$, where

$$k = \begin{cases} m & \text{if } t_0 + m\sigma < t_1 < t_0 + (m+1)\sigma, \\ m-1 & \text{if } t_0 + m\sigma = t_1 \end{cases}$$

(see (2.1)). Thus, $N = \max\{N_0, N_1, \ldots, N_k\}$. □

Lemma (1) allows one to introduce the increment of the solution $x_0(t)$:

$$\Delta x(t) = \Delta x(t;\delta\mu) = x(t;\mu) - x_0(t), \quad t \in I_1, \tag{3.2}$$

where $\mu = \mu_0 + \delta\mu \in \Lambda_{\varepsilon_1}(\mu_0)$, i.e. $\delta\mu \in \Lambda_{\varepsilon_1}(\mu_0) - \mu_0$.

Lemma 4. *For arbitrary* $\delta\mu \in \Lambda_{\varepsilon_1}(\mu_0) - \mu_0$ *the following inequality holds*

$$\sup_{t\in I_1} |\Delta x(t)| \leq O(\delta\mu), \tag{3.3}$$

where

$$\lim_{|\delta\mu|\to 0} O(\delta\mu)|\delta\mu| < \infty.$$

Proof. Let $t \in [\hat{\tau}, t_0]$ then

$$|\Delta x(t)| = |\delta\varphi(t)| \leq |\delta\mu| = O(\delta\mu) \tag{3.4}$$

(see (2.6)). It is not difficult to see that the function $\Delta x(t) = x(t) - x_0(t)$ satisfies the equation

$$\dot{\Delta} x(t) = A(t, x_0(t) + \Delta x(t))\dot{\Delta} x(t-\sigma) + \alpha(t;\delta\mu) + \beta(t;\delta\mu), \; a.e.\; t \in I \tag{3.5}$$

and the initial condition

$$\Delta x(t) = \delta\varphi(t), t \in [\hat{\tau}, t_0], \tag{3.6}$$

where

$$\alpha(t;\delta\mu) = \Big[A(t, x_0(t) + \Delta x(t)) - A[t]\Big]\dot{x}_0(t-\sigma)$$
$$\beta(t;\delta\mu) = f(t, x_0(t) + \Delta x(t), x_0(t-\tau) + \Delta x(t-\tau), u_0(t) + \delta u(t)) - f[t].$$

The solution $\Delta x(t)$ of the problem (3.5)-(3.6) can be represented on the interval I in the following form

$$\Delta x(t) = \delta\varphi(t_0) + \int_{t_0-\sigma}^{t_0} Y_0(\xi;t)A(\xi+\sigma, x_0(\xi+\sigma) + \Delta x(\xi+\sigma))\dot{\delta\varphi}(\xi)d\xi$$
$$+ \int_{t_0}^{t} Y_0(\xi;t)[\alpha(\xi;\delta\mu) + \beta(\xi;\delta\mu)]d\xi$$

(see Theorem 1.7 in [23]), where $Y_0(\xi;t) = Y_0(\xi;t,\delta\mu)$ is $n\times n$ matrix function satisfying the difference equation

$$Y_0(\xi;t) = E + Y_0(\xi+\sigma;t)A(\xi+\sigma, x_0(\xi+\sigma) + \Delta x(\xi+\sigma)), \xi \in [t_0, t]$$

and the condition

$$Y_0(\xi;t) = \begin{cases} E, \; \xi = t, \\ \Theta, \; \xi > t, \end{cases}$$

where E is the identity matrix and Θ is the zero matrix. It is clear that

$$|Y_0(\xi;t,\delta\mu)| < const, \forall(\xi,t,\delta\mu) \in I^2 \times (\Lambda_{\varepsilon_1}(\mu_0) - \mu_0)$$

(see (2.2)). Further,

$$|\Delta x(t)| \leq |\delta\varphi(t_0)| + \sigma M_1 \parallel Y_0 \parallel ||\delta\varphi||_1 + ||Y_0|| \int_{t_0}^{t} \Big[L_1 N|\Delta x(\xi)| + L_2|\Delta x(\xi)|$$
$$+ L_2|\Delta x(\xi-\tau)| + L_2[x_0(\xi-\tau) - x_0(\xi-\tau_0)] + L_2|\delta u(\xi)|\Big]d\xi$$

$$\leq (1+||Y_0||M_1\sigma+||Y_0||L_2(t_1-t_0))|\delta\mu|+||Y_0||(L_1N+L_2)\int_{t_0}^{t}|\Delta x(\xi)|d\xi$$
$$+||Y_0||L_2\int_{t_0}^{t}|\Delta x(\xi-\tau)|d\xi+||Y_0||L_2\int_{t_0}^{t_1}|x_0(\xi-\tau)-x_0(\xi-\tau_0)|d\xi \tag{3.7}$$

(see Lemmas 2 and 3), where

$$||Y_0||=\sup\{|Y_0(\xi;t,\delta\mu)|:(\xi,t)\in I^2,\delta\mu\in(\Lambda_{\varepsilon_1}(\mu_0)-\mu_0)\}.$$

Now we transform the two last addends of (3.7). We have,

$$\int_{t_0}^{t}|\Delta x(\xi-\tau)|d\xi=\int_{t_0-\tau}^{t-\tau}|\Delta x(\xi)|d\xi.$$

Let $t_0+\tau\geq t_0+\sigma$ then if $t\in[t_0,t_0+\sigma]$ we have $t-\tau\leq t_0+\sigma-\tau\leq t_0+\tau-\tau=t_0$, i. e.

$$\int_{t_0-\tau}^{t-\tau}|\Delta x(\xi)|d\xi\leq\int_{t_0-\tau}^{t_0}|\delta\varphi(\xi)|d\xi\leq\tau|\delta\mu|\leq\tau_2|\delta\mu|=O(\delta\mu). \tag{3.8}$$

Let $t_0+\tau<t_0+\sigma$ then if $t\in[t_0,t_0+\tau]$ we have $t-\tau\leq t_0$; if $t\in[t_0+\tau,t_0+\sigma]$ then we have $t-\tau\geq t_0+\tau-\tau=t_0$. Consequently, if $t\in[t_0,t_0+\tau]$ we have (3.8) and if $t\in[t_0+\tau,t_0+\sigma]$ we have

$$\int_{t_0-\tau}^{t-\tau}|\Delta x(\xi)|d\xi=\int_{t_0-\tau}^{t_0}|\Delta x(\xi)|d\xi+\int_{t_0}^{t-\tau}|\Delta x(\xi)|d\xi\leq O(\delta\mu)$$
$$+\int_{t_0}^{t}|\Delta x(\xi)|d\xi.$$

(see (3.4)). Thus,

$$\int_{t_0}^{t}|\Delta x(\xi-\tau)|d\xi\leq O(\delta\mu)+\int_{t_0}^{t}|\Delta x(\xi)|d\xi. \tag{3.9}$$

Further,

$$\int_{t_0}^{t_1}|x_0(\xi-\tau)-x_0(\xi-\tau_0)|d\xi=\int_{t_0}^{t_1}\Big|\int_{\xi-\tau}^{\xi-\tau_0}|\dot{x}_0(\varsigma)|d\varsigma\Big|d\xi$$
$$\leq(t_1-t_0)N|\delta\mu|=O(\delta\mu). \tag{3.10}$$

Taking into account (3.9), (3.10) from (3.7) it follows

$$|\Delta x(t)|\leq O(\delta\mu)+||Y_0||(L_1N+2L_2)\int_{t_0}^{t}|\Delta x(\xi)|d\xi,\ t\in I.$$

By the Gronwall-Bellman inequality, from the last inequality it follows

$$|\Delta x(t)|\leq O(\delta\mu)\exp[||Y_0||(L_1N+2L_2)(t-t_0)]\leq O(\delta\mu),t\in I. \tag{3.11}$$

On the basis of (3.4) and (3.11) we get (3.3). □

Lemma 5. *The following inequality holds*

$$|\Delta\dot{x}(t)|\leq O(\delta\mu),\ a.e.\ t\in I_1.$$

Using Lemma 2 and equation (3.5) Lemma 5 without principle difficulties can be proved by the step method with respect to σ (see proof of Lemma 3).

4. PROOF OF THEOREM 1

The function $\Delta x(t)$ satisfies the equation (see the previous section)

$$\begin{aligned}\dot{\Delta}x(t) = A[t]\dot{\Delta}x(t-\sigma) + \Big\{\frac{\partial}{\partial x}\Big[A[t]\dot{x}_0(t-\sigma)\Big] + f_x[t]\Big\}\Delta x(t)\\ + f_y[t]\Delta x(t-\tau_0) + f_u[t]\delta u(t) + a(t;\delta\mu) + b(t;\delta\mu) \qquad (4.1)\end{aligned}$$

with the initial condition

$$\Delta x(t) = \delta\varphi(t), t \in [\hat{\tau}, t_0],$$

where

$$\begin{aligned} a(t;\delta\mu) &= A(t, x_0(t)+\Delta x(t))\dot{x}_0(t-\sigma) - A[t]\dot{x}_0(t-\sigma)\\ &\quad - \frac{\partial}{\partial x}\Big[A[t]\dot{x}_0(t-\sigma)\Big]\Delta x(t) + \Big(A(t,x_0(t)+\Delta x(t)) - A[t]\Big)\dot{\Delta}x(t-\sigma);\\ b(t;\delta\mu) &= b_0(t;\delta\mu) - f_x[t]\Delta x(t) - f_y[t]\Delta x(t-\tau_0) - f_u[t]\delta u(t),\\ b_0(t;\delta\mu) &= f(t, x_0(t)+\Delta x(t), x_0(t-\tau)+\Delta x(t-\tau), u_0(t)+\delta u(t)) - f[t]. \end{aligned}$$

By using the Cauchy formula (see Theorem 1.5, [23]) one can represent the solution of equation (4.1) in the form

$$\begin{aligned}\Delta x(t) = \Psi(t_0;\xi)\delta\varphi(t_0) + \int_{t_0-\sigma}^{t_0} Y(\xi+\sigma;t)A[\xi+\sigma]\dot{\delta\varphi}(\xi)d\xi\\ + \int_{t_0-\tau_0}^{t_0} Y(\xi+\tau_0;\xi)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi + a_1(t;\delta\mu) + b_1(t;\delta\mu), \qquad (4.2)\end{aligned}$$

where

$$a_1(t;\delta\mu) = \int_{t_0}^{t} Y(\xi;t)a(\xi;\delta\mu)d\xi,\ \ b_1(t;\delta\mu) = \int_{t_0}^{t} Y(\xi;t)b(\xi;\delta\mu)d\xi;$$

$\Psi(\xi;t)$ and $Y(\xi;t)$ are matrix functions satisfying equation (2.9) and condition (2.10). Now we estimate $|a_1(t;\delta\mu)|$, we have

$$\begin{aligned}|a_1(t;\delta\mu)| = \Big|\int_{t_0}^{t} Y(\xi;t)\Big[\int_0^1 \frac{d}{ds}A(\xi, x_0(\xi)+s\Delta x(\xi))\dot{x}_0(\xi-\sigma)ds\Big]d\xi\\ - \int_{t_0}^{t} Y(\xi;t)\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]\Delta x(\xi)d\xi\\ + \int_{t_0}^{t} Y(\xi;t)\Big(A(\xi,x_0(\xi)+\Delta x(\xi)) - A[\xi]\Big)\dot{\Delta}x(\xi-\sigma)\Big|\\ \le \Big|\int_{t_0}^{t} Y(\xi;t)\Big\{\int_0^1 \Big(\frac{\partial}{\partial x}\Big[A(\xi,x_0(\xi)+s\Delta x(\xi))\dot{x}_0(\xi-\sigma)\Big]\end{aligned}$$

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]\Big)\Delta x(\xi)ds\Big\}d\xi \\
&+\int_{t_0}^{t} Y(\xi;t)\Big(A(\xi,x_0(\xi)+\Delta x(\xi))-A[\xi]\Big)\dot{\Delta x}(\xi-\sigma)d\xi\Big| \\
\le\ &\|Y\|O(\delta\mu)\Big\{\int_{t_0}^{t_1}\rho(\xi;\delta\mu)d\xi+L_1(t_1-t_0)O(\delta\mu)\Big\}
\end{aligned}
$$

(see Lemma 5), where

$$
\begin{aligned}
&\|Y\|=\sup\{|Y(\xi;t)|:\xi,t\in I\},\\
&\rho(\xi;\delta\mu)=\int_0^1\Big|\frac{\partial}{\partial x}\Big[A(\xi,x_0(\xi)+s\Delta x(\xi))\dot{x}_0(\xi-\sigma)\Big]-\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]\Big|ds,
\end{aligned}
$$

By the Lebesgue theorem on the passage to the limit under the integral sign, we have

$$
\lim_{|\delta\mu|\to 0}\int_{t_0}^{t_1}\rho(\xi;\delta\mu)d\xi=0.
$$

Consequently,

$$
a_1(t;\delta\mu)=o(t;\delta\mu). \tag{4.3}
$$

We introduce the notations:

$$
\begin{aligned}
f[t;\theta,\delta\mu]&=f(t,x_0(t)+\theta\Delta x(t),x_0(t-\tau_0)+\theta[x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau)],\\
&\qquad u_0(t)+\theta\delta u(t))\\
\alpha_x(t;\theta,\delta\mu)&=f_x[t;\theta,\delta\mu]-f_x[t].
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
b_0(t;\delta\mu)&=\int_0^1\frac{d}{d\theta}f[t;\theta,\delta\mu]d\theta\\
&=\int_0^1\{f_x[t;\theta,\delta\mu]\Delta x(t)+f_y[t;\theta,\delta\mu][x_0(t-\tau)-x_0(t-\tau_0)\\
&\qquad+\Delta x(t-\tau)]+f_u[t;\theta,\delta\mu]\delta u(t)\}\,d\theta=\left[\int_0^1\alpha_x(t;\theta,\delta\mu)d\theta\right]\Delta x(t)\\
&\qquad+\left[\int_0^1\alpha_y(t;\theta,\delta\mu)d\theta\right](x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau))\\
&\qquad+\left[\int_0^1\alpha_u(t;\theta,\delta\mu)d\theta\right]\delta u(t)+f_x[t]\Delta x(t)+f_y[t][x_0(t-\tau)\\
&\qquad-x_0(t-\tau_0)+\Delta x(t-\tau)]+f_u[t]\delta u(t)\\
&=\alpha_x(t;\delta\mu)\Delta x(t)+\alpha_y(t;\delta\mu)\,(x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau))\\
&\qquad+\alpha_u(t;\delta\mu)\delta u(t)+f_x[t]\Delta x(t)+f_y[t][x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau)]\\
&\qquad+f_u[t]\delta u(t),
\end{aligned}
$$

where

$$\alpha_x(t;\delta\mu) = \int_0^1 \alpha_x(t;\theta,\delta\mu)d\theta.$$

Taking into account the last relations we have

$$b_1(t;\delta\mu) = \sum_{i=1}^{5} b_{2i}(t;\delta\mu),$$

where

$$b_{21}(t;\delta\mu) = \int_{t_0}^{t} Y(s;t)\alpha_x(s;\delta\mu)\Delta x(s)ds,$$

$$b_{22}(t;\delta\mu) = \int_{t_0}^{t} Y(s;t)\alpha_y(s;\delta\mu)[x_0(s-\tau)-x_0(s-\tau_0)+\Delta x(s-\tau)]ds,$$

$$b_{23}(t;\delta\mu) = \int_{t_0}^{t} Y(s;t)f_y[s][x_0(s-\tau)-x_0(s-\tau_0)]ds,$$

$$b_{24}(t;\delta\mu) = \int_{t_0}^{t} Y(s;t)f_y[s][\Delta x(s-\tau)-\Delta x(s-\tau_0)]ds,$$

$$b_{25}(t;\delta\mu) = \int_{t_0}^{t} Y(s;t)\alpha_u(s;\delta\mu)\delta u(s)ds.$$

It is not difficult to see that

$$|b_{21}(t;\delta\mu)| \le \|Y\|O(\delta\mu)\alpha_x(\delta\mu),\ \alpha_x(\delta\mu) = \int_{t_0}^{t_1} |\alpha_x(s;\delta\mu)|ds$$

(see Lemma 5);

$$|b_{22}(t;\delta\mu)| \le \|Y\|\alpha_y(\delta\mu)\int_{t_0}^{t_1}\Big\{\Big|\int_{s-\tau_0}^{s-\tau}|\dot x_0(\xi)|d\xi\Big| + O(\delta\mu)\Big\}d\xi$$
$$\le \|Y\|O(\delta\mu)\alpha_y(\delta\mu)$$

(see Lemma 4);

$$|b_{25}(t;\delta\mu)|\|Y\|O(\delta\mu)\alpha_u(\delta\mu).$$

Further,

$$b_{23}(t;\delta\mu) = \int_{t_0}^{t} Y(s;t)f_y[s]\Big\{\int_{s-\tau_0}^{s-\tau}\dot x_0(\xi)d\xi\Big\}ds.$$

The function $x_0(\xi), \xi\in I_1$, is absolutely continuous, therefore for each fixed Lebesgue point $s\in(t_0,t_1)$ of the function $\dot x(\varsigma-\tau_0),\quad \varsigma\in(t_0,t_1)$, we get

$$\int_{s-\tau_0}^{s-\tau}\dot x_0(\xi)d\xi = \int_{s}^{s-\delta\tau}\dot x_0(\varsigma-\tau_0)d\varsigma = -\dot x_0(s-\tau_0)\delta\tau$$
$$+\rho(s;\delta\tau), \tag{4.4}$$

where

$$\lim_{|\delta\tau|\to 0}\rho(s;\delta\tau)/|\delta\tau| = 0$$

From boundedness of the function $\dot{x}_0(s), s \in I_1$ and (4.4) it follows

$$|\rho(s;\delta\tau)|/|\delta\tau| \leq const$$

a. e. on I. Thus,

$$b_{23}(t;\delta\mu) = -\Big\{\int_{t_0}^{t} Y(s;t)f_y[s]\dot{x}_0(s-\tau_0)ds\Big\}\delta\tau + \rho_1(t;\delta\tau),$$

where

$$\rho_1(t;\delta\tau) = \int_{t_0}^{t} Y(s;t)f_y[s]\rho(s;\delta\tau)ds.$$

It is clear that

$$|\Delta x(s-\tau) - \Delta x(s-\tau)| \leq \Big|\int_{s-\tau_0}^{s-\tau} |\dot{\Delta x}(\xi)|d\xi\Big| \leq O(\delta\mu)|\delta\mu|$$

(see Lemma 5). On the basis above obtained estimates and the Lebesgue theorem can be concluded that

$$b_{21}(t;\delta\mu) = o(t;\delta\mu),\ b_{22}(t;\delta\mu) = o(t;\delta\mu),$$
$$b_{23}(t;\delta\mu) = -\Big\{\int_{t_0}^{t} Y(s;t)f_y[s]\dot{x}_0(s-\tau_0)ds\Big\}\delta\tau + o(t;\delta\mu),$$
$$b_{21}(t;\delta\mu) = o(t;\delta\mu),\ b_{22}(t;\delta\mu) = o(t;\delta\mu).$$

Consequently, we get

$$b_1(t;\delta\mu) = -\Big\{\int_{t_0}^{t} Y(s;t)f_y[s]\dot{x}_0(s-\tau_0)ds\Big\}\delta\tau + o(t;\delta\mu). \tag{4.5}$$

From (4.2) by virtue of (4.3) and (4.5), we obtain (2.7), where $\delta x(t;\delta\mu)$ has the form (2.8).

5. CONCLUSION

The formula (2.7) plays an important role in proving the necessary conditions of optimality in the optimization problems. Besides, this formula allows one to get an approximate solution of the perturbed equation and to carry out a sensitivity analysis of mathematical models. Future work will consider the case when the initial moment t_0 is non-fixed.

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