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$\overrightarrow{p}(\cdot)-$ANISOTROPIC DEGENERATE ELLIPTIC EQUATIONS HAVING NATURAL EXPONENTIAL FUNCTION

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Abstract. In this work, we establish existence results for distributional solutions of anisotropic nonlinear elliptic equations with variable growth, L^1-data, and degenerate coercivity involving natural exponential function, where the nonlinear anisotropic differential operator is well defined between $\overrightarrow{p}(x)-$anisotropic Sobolev space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and its dual, yet it lacks coercivity as $|u|$ becomes large, i.e., when $e^{|u|}$ is large.

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1. INTRODUCTION

In this paper we will prove the existence of distributional solutions to a class of anisotropic nonlinear elliptic equations with variable exponents

$$\begin{aligned} -\sum_{i=1}^{N} D_i\left(\frac{a(x,u)|D_iu|^{p_i(x)-2}D_iu}{1+e^{|u|}}\right)+u\sum_{i=1}^{N}|u|^{p_i(x)-2}=f, \quad &\text{in } \Omega,\\ u=0, \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N\geq 2$) with Lipschitz boundary $\partial\Omega$, f is a function in $L^1(\Omega)$, and $a\colon \Omega\times\mathbb{R}\to\mathbb{R}^+$ is a Carathéodory function, such that

$$0<\alpha\leq a(x,s)\leq\beta,\ \forall x\in\Omega,\ \forall s\in\mathbb{R}, \tag{1.2}$$

where α and β are positive constants.

The nonlinear anisotropic operator A defined between the space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ (will be mentioned later in Section 2) and its dual by

$$A(u)=-\sum_{i=1}^{N} D_i\left(\frac{a(x,u)|D_iu|^{p_i(x)-2}D_iu}{1+e^{|u|}}\right),$$

it fails to be coercive if $|u|$ is large (i.e., $e^{|u|}$ is large). This difficulty arises in proving the existence of solutions to the approximate problems, as well as when taking the limit, and classical existence theorems cannot be applied. The existence and regularity results for distributional solutions of degenerate anisotropic nonlinear elliptic problems with variable exponents given (for example, but not limited to) in the works [10–12, 14, 20].

In our proof we establish uniform anisotropic estimates on the sequence of solutions (u_n) and its derivative, where we have proven the a.e. convergence of the partial derivatives $D_i u_n$, $i = 1, \dots, N$, which can be turned into strong L^1-convergence. Equipped with this convergence we pass to the limit in the strong $L^1(\Omega)$ sense in

$$\frac{a_n(x,u_n)|D_i u_n|^{p_i(x)-2}D_i u_n}{1+e^{|u_n|}} \quad \text{and} \quad u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2},$$

finally conclude that the approximate solutions u_n converge to the solution of (1.1).

The work has been organized in the following form: Section 2 for some mathematical preliminaries, where we remind the p(x)-anisotropic Lebesgue-Sobolev spaces. The main theorem and its proof are in Section 3.

2. Mathematical preliminaries

In this section we recall the Lebesgue-Sobolev spaces with variable exponent in both cases isotropic and anisotropic, with mentioning some of its properties and facts (see [4, 5, 8]).

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function } p(\cdot)\colon \overline{\Omega} \longmapsto \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < \infty\},$$

where

$$p^+ = \max_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u\colon \Omega \mapsto \mathbb{R}; \rho_{p(\cdot)}(u) < \infty\},$$

where

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} dx \quad \text{the convex modular.}$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$u \mapsto \|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \mid \rho_{p(\cdot)}(u/\lambda) \leq 1\right\}$$

becomes a reflexive Banach space. $L^{p'(\cdot)}(\Omega)$ symbolizes to the dual of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

$\forall u \in L^{p(\cdot)}(\Omega)$, $\forall v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}$$

holds true.

The variable exponents Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined as follows:

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |Du| \in L^{p(\cdot)}(\Omega)\right\},$$

it becomes a Banach space when equipped with the norm

$$u \mapsto \|u\|_{W^{1,p(\cdot)}(\Omega)} := \|Du\|_{p(\cdot)}. \tag{2.1}$$

We define also the Banach space $W_0^{1,p(\cdot)}(\Omega)$ by

$$W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)},$$

endowed with the norm (2.1). Moreover, it is reflexive and separable if $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$.

The following lemma will be used later.

Lemma 1 ([4, 5]). *If (u_n), $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold:*

(i) $\|u\|_{p(\cdot)} < 1$ (*respectively* $= 1, > 1$) $\Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ (*respectively* $= 1, > 1$),

(ii) $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \le \|u\|_{p(\cdot)} \le \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right)$,

(iii) $\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \le \rho_{p(\cdot)}(u) \le \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right)$,

(iv) $\|u\|_{p(\cdot)} \le \rho_{p(\cdot)}(u) + 1$,

(v) $\|u_n - u\|_{p(\cdot)} \to 0 \Longleftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0$.

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of problem (1.1).

Let $p_i(\cdot)\colon \overline{\Omega} \to [1, +\infty)$ for all $i = 1, \dots, N$ be continuous functions, and we set for all $x \in \overline{\Omega}$

$$\overrightarrow{p}(x) = (p_1(x), \dots, p_N(x)), \quad p_+(x) = \max_{1\le i\le N} p_i(x), \quad p_-(x) = \min_{1\le i\le N} p_i(x),$$

$$p_+^+ = \max_{x\in\overline{\Omega}} p_+(x), \quad p_-^- = \min_{x\in\overline{\Omega}} p_-(x),$$

$$\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i(x)}, \quad \overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N, \\ +\infty, & \text{for } \overline{p}(x) \ge N. \end{cases}$$

The anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined as follows:

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega), D_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \dots, N\right\},$$

which is Banach space with respect to the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}.$$

We define the spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as follows:

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}, \quad \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

Remark 1 ([7]). If Ω is a bounded open set with Lipschitz boundary $\partial\Omega$, then

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{ u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega),\ u_{|\partial\Omega} = 0 \right\},$$

where, $u_{|\partial\Omega}$ denotes the trace on $\partial\Omega$ of u in $W^{1,1}(\Omega)$.

We have the following embedding results.

Lemma 2 ([7, 8]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$. If $r \in \mathcal{C}_+(\overline{\Omega})$ and $\forall x \in \overline{\Omega}$, $r(x) < \max(p_+(x),\ \overline{p}^\star(x))$. Then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \text{ is compact.} \tag{2.2}$$

Lemma 3 ([7, 8]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x \in \overline{\Omega},\ p_+(x) < \overline{p}^\star(x). \tag{2.3}$$

Then the following Poincaré-type inequality holds

$$\|u\|_{p_+(\cdot)} \le C \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{2.4}$$

where C is a positive constant independent of u.

Remark 2. If (2.3) holds, then (2.4) implies that

$$u \mapsto \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}$$

is an equivalent norm on $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

For any $t > 0$, the standard scalar truncation function $T_t : \mathbb{R} \to \mathbb{R}$ (at height $t > 0$) is defined as

$$T_t(r) := \begin{cases} r, & \text{if } |r| \le t, \\ \frac{r}{|r|} t, & \text{if } |r| > t. \end{cases} \tag{2.5}$$

This function will be used to derive a priori estimates for our approximate solutions. We also need its derivative (see [20]):

$$(DT_t)(r) = \begin{cases} 1, & |r| < t, \\ 0, & |r| > t. \end{cases} \tag{2.6}$$

We need further the following function defined for $s \in \mathbb{R}$ by

$$G_t(s) = \begin{cases} 0, & if\ |s| \le t, \\ s-t, & if\ s > t, \quad t > 0, \\ s+t, & if\ s < -t, \end{cases}$$

as a test function in the approximate weak formulation.

3. Statement of results

Definition 1. We say that u is a distributional solution for problem (1.1) if $u \in W_0^{1,1}(\Omega)$, and for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \sum_{i=1}^N \frac{a(x,u)|D_i u|^{p_i(x)-2} D_i u}{1+e^{|u|}} D_i\varphi \, dx + \int_\Omega u \sum_{i=1}^N |u|^{p_i(x)-2}\varphi \, dx = \int_\Omega f(x)\varphi \, dx.$$

Our main result is the following.

Theorem 1. *Let $p_i(\cdot) > 1$, $i = 1,\dots,N$, are continuous functions on Ω such that (2.3) holds and $\overline{p} < N$. Assume that f is in $L^1(\Omega)$, and $a\colon \Omega \times \mathbb{R} \to \mathbb{R}^+$ is a Carathéodory function satisfying (1.2). Then the problem (1.1) has at least one solution $u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ in the sense of distributions.*

3.1. *Approximate solutions*

We are going to prove the existence of solution to problem (1.1). We define

$$f_n(x) = \frac{f(x)}{1+\frac{|f(x)|}{n}}, \quad a_n(x,u) = \frac{a(x,u)}{1+\frac{a(x,u)}{n}}, \quad n \in \mathbf{N}^*. \tag{3.1}$$

We first notice that since $\Theta(x) = \frac{x}{1+\frac{x}{n}}$ is increasing and $a_n(x,u) = \Theta(a(x,u))$, (1.2) implies that

$$\frac{\alpha}{1+\alpha} \le a_n \le \beta. \tag{3.2}$$

Lemma 4. *Let $p_i(\cdot) > 1$, $i = 1,\dots,N$, are continuous functions on Ω such that $\overline{p} < N$, and (2.3) holds. Assume that f is in $L^1(\Omega)$, and $a\colon \Omega \times \mathbb{R} \to \mathbb{R}^+$ is a Carathéodory function satisfying (1.2). Then, there exists at least one weak solution $u_n \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ to the approximated problems*

$$\begin{aligned} -\sum_{i=1}^N D_i\left(\frac{a_n(x,u_n)|D_i u_n|^{p_i(x)-2} D_i u_n}{1+e^{|T_n(u_n)|}}\right) + u_n \sum_{i=1}^N |u_n|^{p_i(x)-2} &= f_n, \quad \text{in } \Omega, \\ u_n &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$

in the sense that

$$\sum_{i=1}^N \int_\Omega \frac{a_n(x,u_n)|D_i u_n|^{p_i(x)-2} D_i u_n}{1+e^{|T_n(u_n)|}} D_i\varphi \, dx + \int_\Omega u_n \sum_{i=1}^N |u_n|^{p_i(x)-2}\varphi \, dx = \int_\Omega f_n\varphi \, dx \tag{3.4}$$

for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$.

Proof. Consider the problem

$$-\sum_{i=1}^{N} D_i\left(\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}\right)+\sum_{i=1}^{N} T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)=f_n, \quad \text{in } \Omega,$$
$$u_{n_k}=0, \quad \text{on } \partial\Omega. \tag{3.5}$$

Let's prove that the operator defined between $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and its dual space by

$$\mathbf{A}\colon u_{n_k} \mapsto \left(v \mapsto \sum_{i=1}^{N}\int_{\Omega}\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}D_iv\,\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega}T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)v\,\mathrm{d}x\right),$$

is bounded, pseudomonotone and coercive.

First of all we put $\mathbf{A}=\mathbf{B}+\mathbf{C}$, with $\mathbf{B},\mathbf{C}\colon \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \left(\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)^{*}$ defined by

$$\langle \mathbf{B}(u_{n_k}),v\rangle = \sum_{i=1}^{N}\int_{\Omega}\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}D_iv\,\mathrm{d}x,$$
$$\langle \mathbf{C}(u_{n_k}),v\rangle = \sum_{i=1}^{N}\int_{\Omega}T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)v\,\mathrm{d}x.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and its dual.

From (3.2), the fact that $1+e^{|T_n(u_{n_k})|}\geq 2$, Lemma 1, the fact that $\frac{1}{p_-^-}\leq 1$, Remark (2) (due (2.3)) and Hölder inequality, we obtain

$$\begin{aligned}
|\langle \mathbf{B}(u_{n_k}),v\rangle| &\leq \frac{\beta}{2}n\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)-1}|D_iv|\,\mathrm{d}x\\
&\leq \beta n\sum_{i=1}^{N}\left\||D_iu_{n_k}|^{p_i(x)-1}\right\|_{p_i'(\cdot)}\|D_iv\|_{p_i(\cdot)}\\
&\leq \beta n\sum_{i=1}^{N}\left(1+\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\right)^{\frac{1}{p_-^-}}\sum_{i=1}^{N}\|D_iv\|_{p_i(\cdot)}\\
&\leq \beta n\sum_{i=1}^{N}\left(1+\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\right)\sum_{i=1}^{N}\|D_iv\|_{p_i(\cdot)}\\
&\leq \beta n\sum_{i=1}^{N}\left(2+\|D_iu_{n_k}\|_{p_i(\cdot)}^{p_+^+}\right)\sum_{i=1}^{N}\|D_iv\|_{p_i(\cdot)}\\
&\leq \beta n\left(2N+\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right)\|v\|_{\overrightarrow{p}(\cdot)},
\end{aligned}$$

which implies the boundedness of $\mathbf{B}$.

On the other hand, thanks to the Hölder inequality and Lemma 1, we have

$$\begin{aligned}\forall u_{n_k}, v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) : |\langle \mathbf{C}(u_{n_k}), v\rangle| &\leq \sum_{i=1}^{N} \int_\Omega |T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)||v|\,\mathrm{d}x \\ &\leq 2\sum_{i=1}^{N} \|T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\|_{p'_+(\cdot)}\|v\|_{p_+(\cdot)} \\ &\leq C\big(1+\int_\Omega k^{p'_+(x)}\,\mathrm{d}x\big)^{1/(p'_+)^-}\|v\|_{\overrightarrow{p}(\cdot)} \\ &\leq C\big(1+k^{(p'_+)^+}|\Omega|\big)^{1/(p'_+)^-}\|v\|_{\overrightarrow{p}(\cdot)}.\end{aligned}$$

Hence $\mathbf{C}$ is bounded.

Through (3.2), the fact that $1+e^{|T_n(u_{n_k})|} \leq e^n$, the fact that $T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)u_{n_k} \geq 0$, and Lemma 1, we have

$$\begin{aligned}\frac{\langle \mathbf{A}(u_{n_k}), u_{n_k}\rangle}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} &\geq \frac{\alpha}{n(1+\alpha)}\frac{\sum_{i=1}^{N}\int_\Omega |D_i u_{n_k}|^{p_i(x)}\,\mathrm{d}x}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &\geq \frac{\alpha}{n(1+\alpha)}\frac{\sum_{i=1}^{N}\|D_i u_{n_k}\|^{p_-^-}_{p_i(\cdot)}-N}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &\geq \frac{\alpha}{n(1+\alpha)}\frac{\left(\frac{1}{N}\sum_{i=1}^{N}\|D_i u_{n_k}\|_{p_i(\cdot)}\right)^{p_-^-}-N}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &= \frac{\alpha}{n(1+\alpha)N^{p_-^-}}\|u_{n_k}\|^{p_-^- -1}_{\overrightarrow{p}(\cdot)} - \frac{N}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}}.\end{aligned}$$

This implies that $\mathbf{A}$ is coercive.

Let us prove the pseudo-monotonicity of $\mathbf{A}$. Let $(u_{n_k})_k$ be a sequence in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_{n_k} \rightharpoonup u_n \text{ in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)|, \\ \limsup\limits_{k\to\infty}\langle \mathbf{A}(u_{n_k}), u_{n_k}-u_n\rangle \leq 0. \end{cases}$$

We have to prove that

$$\liminf_{k\to\infty}\langle \mathbf{A}(u_{n_k}), u_{n_k}-v\rangle \geq \langle \mathbf{A}(u_n), u_n-v\rangle, \quad \forall\, v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)|.$$

Here we need the following known inequalities, that hold for all two real vectors ξ, ξ' $((\xi,\xi') \neq (0,0))$ and a real $p>1$:

$$(|\xi|^{p-2}\xi - |\xi'|^{p-2}\xi')(\xi-\xi') \geq \begin{cases} 2^{2-p}|\xi-\xi'|^p, & \text{if } p \geq 2, \\ (p-1)\frac{|\xi-\xi'|^2}{(|\xi|+|\xi'|)^{2-p}}, & \text{if } 1<p<2. \end{cases} \tag{3.6}$$

Like in [14, 19] with using (3.6) we can easily prove the required result. Then, the main Theorem on pseudo-monotone operators (Theorem 27.A in [22] and see

also [3, 6, 13]) applies and ensures the existence of at least one weak solution $u_{n_k} \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ to problem (3.5) in the sense that

$$\sum_{i=1}^{N}\int_{\Omega}\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}D_i\varphi\,\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega}T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\varphi\,\mathrm{d}x = \int_{\Omega}f_n\varphi\,\mathrm{d}x,\quad \forall\varphi\in\mathring{W}^{1,\vec{p}(\cdot)}(\Omega). \tag{3.7}$$

Now, we choose $\varphi = u_{n_k}$ as a test function in (3.7) and using (3.2), the fact that $1+e^{|T_n(u_{n_k})|}\le 1+e^n$, and after dropping the nonegative term (due $T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)u_{n_k}\ge 0$), we obtain

$$\frac{\alpha}{(1+\alpha)(1+e^n)}\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\le n\int_{\Omega}|u_{n_k}|\,\mathrm{d}x. \tag{3.8}$$

Using the fact that $|D_iu_{n_k}|^{p_-}\le 1+|D_iu_{n_k}|^{p_i(x)}$, $i=1,\dots,N$, Poincaré inequality and Young's inequality for all $\varepsilon>0$, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x &\le \frac{(1+\alpha)n(1+e^n)}{\alpha}\int_{\Omega}|u_{n_k}|\,\mathrm{d}x\\ &\le \frac{(1+\alpha)n(1+e^n)}{\alpha}\left(\varepsilon\int_{\Omega}|u_{n_k}|^{p_-}\,\mathrm{d}x+C(\varepsilon)\right)\\ &\le \frac{(1+\alpha)n(1+e^n)}{\alpha}\left(\varepsilon c\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_-}\,\mathrm{d}x+C(\varepsilon)\right)\\ &\le \frac{(1+\alpha)n(1+e^n)}{\alpha}\left(\varepsilon c\left(N|\Omega|+\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\right)+C(\varepsilon)\right).\end{aligned} \tag{3.9}$$

Choosing $\varepsilon=\frac{\alpha}{2c(1+\alpha)n(1+e^n)}$, we obtain

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\le c(n). \tag{3.10}$$

On the other hand, by (iii) of Lemma 1, we get

$$1+\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\ge\left\|D_iu_{n_k}\right\|_{p_i(x)}^{p_-}.$$

Then, we conclude

$$N+\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\ge\sum_{i=1}^{N}\left\|D_iu_{n_k}\right\|_{p_i(x)}^{p_-}.$$

This implies

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\geq\left(\frac{1}{N}\sum_{i=1}^{N}\left\|D_iu_{n_k}\right\|_{p_i(x)}\right)^{p^-_-}-N.$$

Since (2.3) holds, Remark 2 gives us $\sum\limits_{i=1}^{N}\left\|D_iu_{n_k}\right\|_{p_i(x)}=\left\|u_{n_k}\right\|_{\overrightarrow{p}(\cdot)}$, thus the last inequality is equivalent to

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\geq\left(\frac{\left\|u_{n_k}\right\|_{\overrightarrow{p}(\cdot)}}{N}\right)^{p^-_-}-N. \tag{3.11}$$

By combining (3.10) and (3.11), we conclude

$$\left(\frac{\left\|u_{n_k}\right\|_{\overrightarrow{p}(\cdot)}}{N}\right)^{p^-_-}-N\leq c(n),$$

implying that

$$\left\|u_{n_k}\right\|_{\overrightarrow{p}(\cdot)}\leq C(n). \tag{3.12}$$

Relation (3.12) guarantees the existence of a subsequence (still denoted by u_{n_k}) and $u_n\in\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$u_{n_k}\rightharpoonup u_n\ \text{ weakly in }\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)\text{ and a.e. in }\Omega. \tag{3.13}$$

Taking $T_t(u_{n_k})$ as a test function in (3.7), we obtain

$$\sum_{i=1}^{N}\int_{\Omega}\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}D_iT_t(u_{n_k})\,\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega}T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)T_t(u_{n_k})\,\mathrm{d}x$$
$$=\int_{\Omega}f_nT_t(u_{n_k})\,\mathrm{d}x.$$

From (2.6), (3.2) and the fact that $1+e^{|T_n(u_{n_k})|}\leq 1+e^n$, we get

$$\frac{\alpha}{(1+\alpha)(1+e^n)}\sum_{i=1}^{N}\int_{\{|u_{n_k}|<t\}}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\leq\int_{\Omega}|f_n||T_t(u_{n_k})|\,\mathrm{d}x$$
$$+\sum_{i=1}^{N}\int_{\Omega}\left|T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\right||T_t(u_{n_k})|\,\mathrm{d}x.$$

Since $\int_\Omega |D_i(T_t(u_{n_k}))|^{p_i(x)}\,dx = \int_{\{|u_{n_k}|<t\}} |D_i u_{n_k}|^{p_i(x)}\,dx$, $|T_t(u_{n_k})| \le t$, $|f_n| \le |f|$ (due (3.1)), then by Young's inequality we obtain for any $\varepsilon > 0$

$$c(n)\sum_{i=1}^{N}\int_\Omega |D_i(T_t(u_{n_k}))|^{p_i(x)}\,dx \le t\|f\|_{L^1(\Omega)} + C(\varepsilon)\sum_{i=1}^{N}\int_\Omega \left|T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\right|^{p_i'(x)}\,dx + \varepsilon\sum_{i=1}^{N}\int_\Omega |T_t(u_{n_k})|^{p_i(x)}\,dx. \tag{3.14}$$

Since $|T_t(u_{n_k})|^{p_i(x)} \le 1 + t^{p_+^+}$, and $T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right) \in L^{p_i'(x)}(\Omega)$ (due $u_{n_k}|u_{n_k}|^{p_i(x)-2} \in L^{p_i'(x)}(\Omega)$), we deduce for all $t > 0$ that

$$T_t(u_{n_k}) \rightharpoonup T_t(u_n) \ \text{ weakly in } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \tag{3.15}$$

Now, let's prove that for all $i = 1,\dots,N$

$$D_i u_{n_k} \to D_i u_n, \text{ a.e. in } \Omega. \tag{3.16}$$

For that let us define for all $i = 1,\dots,N$ and $t > 0$ fixed

$$I_{i,n,k}^t(x) = \left(|D_i(T_t(u_{n_k}))|^{p_i(x)-2}D_i(T_t(u_{n_k})) - D_i(T_t(u_n))|^{p_i(x)-2}D_i(T_t(u_n))\right)\left(D_i(T_t(u_{n_k})) - D_i(T_t(u_n))\right).$$

For $0 < \theta < 1$, $0 < h < t$, and all $i = 1,\dots,N$, we can get

$$\begin{aligned}\int_\Omega \left(I_{i,n,k}^t(x)\right)^\theta dx &= \int_{\{|T_t(u_{n_k})-T_t(u_n)|>h\}} \left(I_{i,n,k}^t(x)\right)^\theta dx + \int_{\{|T_t(u_{n_k})-T_t(u_n)|\le h\}} \left(I_{i,n,k}^t(x)\right)^\theta dx \\ &\le \left(\int_\Omega I_{i,n,k}^t(x)\,dx\right)^\theta \left|\{|T_t(u_{n_k}) - T_t(u_n)| > h\}\right|^{1-\theta} \\ &\quad + \left(\int_{\{|T_t(u_{n_k})-T_t(u_n)|\le h\}} I_{i,n,k}^t(x)\,dx\right)^\theta |\Omega|^{1-\theta}.\end{aligned} \tag{3.17}$$

Then, we can write

$$\int_\Omega \left(I_{i,n,k}^t(x)\right)^\theta dx \le J_1 + J_2, \tag{3.18}$$

where

$$J_1 = \left(\int_\Omega I_{i,n,k}^t(x)\,dx\right)^\theta \left|\{|T_t(u_{n_k}) - T_t(u_n)| > h\}\right|^{1-\theta},$$

$$J_2 = \left(\int_{\{|T_t(u_{n_k})-T_t(u_n)|\le h\}} I_{i,n,k}^t(x)\,dx\right)^\theta |\Omega|^{1-\theta}.$$

For every fixed h, thanks to (3.15), and the convergence in measure of $T_t(u_{n_k})$, we can get

$$\lim_{k\to+\infty} J_1 = 0. \tag{3.19}$$

Now, choosing $T_h(u_{n_k} - T_t(u_n))$ (with $0 < h < t$) as a test function in (3.7), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \frac{a_n(x,u_{n_k})}{1+e^{|T_n(u_{n_k})|}} |D_i(T_t(u_{n_k}))|^{p_i(x)-2} D_i(T_t(u_{n_k})) D_i T_h\big(T_t(u_{n_k}) - T_t(u_n)\big)\,\mathrm{d}x \\
&- \sum_{i=1}^{N} \int_{\{|u_{n_k} - T_t(u_n)|<h\}} \frac{a_n(x,u_{n_k})}{1+e^{|T_n(u_{n_k})|}} |D_i(G_t(u_{n_k}))|^{p_i(x)-2} D_i(G_t(u_{n_k})) D_i\big(T_t(u_n)\big)\,\mathrm{d}x \\
&+ \sum_{i=1}^{N} \int_\Omega T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right) T_h(u_{n_k} - T_t(u_n))\,\mathrm{d}x \le ch. \qquad (3.20)
\end{aligned}$$

By (3.2), we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |D_i(T_t(u_{n_k}))|^{p_i(x)-2} D_i(T_t(u_{n_k})) D_i T_h\big(T_t(u_{n_k}) - T_t(u_n)\big)\,\mathrm{d}x \\
&\le c_1(n)h + c_2(n) \sum_{i=1}^{N} \int_{\{|u_{n_k} - T_t(u_n)|<h\}} |D_i(G_t(u_{n_k}))|^{p_i(x)-2} D_i(G_t(u_{n_k})) D_i\big(T_t(u_n)\big)\,\mathrm{d}x \\
&- c_3(n) \sum_{i=1}^{N} \int_\Omega T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right) T_h(u_{n_k} - T_t(u_n))\,\mathrm{d}x. \qquad (3.21)
\end{aligned}$$

Then, by using (3.21) and the fact that $I^t_{i,n}(x) \ge 0$ (since (3.6)), we deduce

$$\begin{aligned}
0 &\le \sum_{i=1}^{N} \int_{\{|T_t(u_{n_k}) - T_t(u_n)|\le h\}} I^t_{i,n,k}(x)\,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_\Omega \big(|D_i(T_t(u_{n_k}))|^{p_i(x)-2} D_i(T_t(u_{n_k})) \\
&\qquad - |D_i(T_t(u_n))|^{p_i(x)-2} D_i(T_t(u_n))\big) D_i T_h\big(T_t(u_{n_k}) - T_t(u_n)\big)\,\mathrm{d}x \\
&\le c_1(n)h - c_3(n) \sum_{i=1}^{N} \int_\Omega T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right) T_h(u_{n_k} - T_t(u_n))\,\mathrm{d}x \\
&\quad + c_2(n) \sum_{i=1}^{N} \int_{\{|u_{n_k}|>t\}\cap\{|u_{n_k} - T_t(u_n)|<h\}} |D_i(u_{n_k})|^{p_i(x)-2} D_i(u_{n_k}) D_i\big(T_t(u_n)\big)\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{N} \int_\Omega |D_i(T_t(u_n))|^{p_i(x)-2} D_i(T_t(u_n)) D_i T_h\big(T_t(u_{n_k}) - T_t(u_n)\big)\,\mathrm{d}x. \qquad (3.22)
\end{aligned}$$

Since $\{|u_{n_k} - T_t(u_n)| < h\} \subset \{|u_{n_k}| \le h + t\} \subset \{|u_{n_k}| \le 2t\}$, the boundedness of both $\big\{|D_i(T_{2t}(u_{n_k}))|^{p_i(x)-2} D_i(T_{2t}(u_{n_k}))\big\}_k$, $\big\{T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\big\}_k$ in $L^{p_i'(x)}(\Omega)$, and the use of (3.15), we can pass to the limit in (3.22) when $k \to +\infty$. So, we get

$$\limsup_{k\to+\infty} \sum_{i=1}^{N} \int_{\{|T_t(u_{n_k}) - T_t(u_n)|\le h\}} I^t_{i,n,k}(x)\,\mathrm{d}x \le c_1(n)h - c_3(n) \sum_{i=1}^{N} \int_\Omega \nu T_h(G_t(u_n))\,\mathrm{d}x$$

$$+c_2(n)\sum_{i=1}^{N}\int_{\{t<|u_n|<t+h\}}\tau_t D_i\big(T_t(u_n)\big), \tag{3.23}$$

where $\tau_t, \nu \in L^{p_i'(x)}(\Omega)$ are the weak limits of $\big\{|D_i(T_{2t}(u_{n_k}))|^{p_i(x)-2}D_i(T_{2t}(u_{n_k}))\big\}_k$ and $\big\{T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\big\}_k$ respectively.

After letting $h \to 0$ in (3.23), we obtain

$$\lim_{k\to+\infty} J_2 = 0. \tag{3.24}$$

We combine (3.18), (3.19), (3.24) and recalling (3.6), we get

$$\lim_{k\to+\infty}\int_\Omega \left(I^t_{i,n,k}(x)\right)^\theta \,\mathrm{d}x = 0. \tag{3.25}$$

From (3.25) we deduce, like in [1], that for every $t>0$ and every $i=1,\dots,N$

$$D_i(T_t(u_{n_k})) \to D_i(T_t(u_n)), \text{ almost everywhere in } \overline{\Omega}. \tag{3.26}$$

And through the results obtained in [21], we can get (3.16).

By (3.16) we have

$$|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k} \rightharpoonup |D_iu_n|^{p_i(x)-2}D_iu_n \ \text{ weakly in } L^{p_i'(\cdot)}(\Omega).$$

From (3.13) and Lebesgue's dominated convergence theorem, we obtain

$$\frac{a_n(x,u_{n_k})}{1+e^{|T_n(u_{n_k})|}}D_i\varphi \to \frac{a_n(x,u_n)}{1+e^{|T_n(u_n)|}}D_i\varphi \ \text{ strongly in } L^{p_i(\cdot)}(\Omega),\ i=1,\dots,N.$$

We will now prove the equi-integrability of $\left\{\sum_{i=1}^{N} T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\right\}_k$ in $L^1(\Omega)$. First, let's prove that

$$\sum_{i=1}^{N}\int_{|u_{n_k}|>t}|T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)|\,\mathrm{d}x \le \int_{|u_{n_k}|>t}|f_n|\,\mathrm{d}x. \tag{3.27}$$

For that, we can choose $\phi_l(u_{n_k})$ as a test function in (3.7), where $(\phi_l)_l$ is a sequence of real smooth increasing functions with $\phi_l' \in L^\infty(\Omega)$, $\phi_l(0)=0$ and it converges to the function ϕ_t, $t>0$ defined by

$$\phi_t(s) = \begin{cases} 1, & if\ s \ge t,\\ 0, & if\ |s|<t, \quad t>0,\\ -1, & if\ s\le -t. \end{cases}$$

Through this choice and following the technique used in [9], we can easily get (3.27). Let $E\subset\Omega$ be any measurable set, using (3.27), we have

$$\sum_{i=1}^{N}\int_E |T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})|\,\mathrm{d}x = \sum_{i=1}^{N}\int_{E\cap\{|u_{n_k}|\le t\}}|T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})|\,\mathrm{d}x$$

$$+\sum_{i=1}^{N}\int_{E\cap\{|u_{n_k}|>t\}}|T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})|\,\mathrm{d}x$$
$$\leq N(1+t^{p_+^+})|E|+\int_{\{|u_{n_k}|>t\}}|f_n|\,\mathrm{d}x.$$

Then we deduce the desired result.

From (3.13), we obtain

$$\sum_{i=1}^{N}T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})\to\sum_{i=1}^{N}|u_n|^{p_i(x)-2}u_n \text{ a.e. in } \Omega. \tag{3.28}$$

Since (3.28), Vitali's theorem implies that

$$\sum_{i=1}^{N}T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})\to\sum_{i=1}^{N}|u_n|^{p_i(x)-2}u_n \quad \text{in } L^1(\Omega).$$

Therefore, we can get (3.4) by passing to the limit in (3.7). □

3.1.1. *A priori estimates*

Lemma 5. *Let f, a and p_i, $i=1,\dots,N$ be restricted as in Theorem 1. Then*

$$u_n \text{ is bounded in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{3.29}$$

where u_n the weak solution to the problem (3.3).

Proof. By choosing $\varphi=u_n$ as a test function in (3.4) and using the same way as in proving (3.12), we can get (3.29). □

Lemma 6. *There exists a subsequence (still denoted (u_n)) such that, for all $i=1,\dots,N$*

$$D_iu_n\to D_iu \text{ a.e. in } \Omega, \tag{3.30}$$

where u is the weak limit of the sequence (u_n) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proof. From (3.29) we conclude that, there exists a function $u\in\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by (u_n)) such that

$$u_n\rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{3.31}$$

In the same way of proof as in [2] or in [14–19] we can easily get that

$$\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\Omega}\left(|D_iu_n|^{p_i(x)-2}D_iu_n-|D_iu|^{p_i(x)-2}D_iu\right)(D_iu_n-D_iu)\,\mathrm{d}x=0. \tag{3.32}$$

Through (3.6) we conclude that for all $i=1,\dots,N$

$$\left(|D_iu_n|^{p_i(x)-2}D_iu_n-|D_iu|^{p_i(x)-2}D_iu\right)(D_iu_n-D_iu)>0. \tag{3.33}$$

Then, (3.33) and (3.32) gives us for all $i = 1, \dots, N$

$$\left(|D_i u_n|^{p_i(x)-2} D_i u_n - |D_i u|^{p_i(x)-2} D_i u\right)(D_i u_n - D_i u) \to 0, \quad \text{strongly in } L^1(\Omega). \tag{3.34}$$

Extracting a subsequence (still denoted by (u_n)), we have for all $i = 1, \dots, N$

$$\left(|D_i u_n|^{p_i(x)-2} D_i u_n - |D_i u|^{p_i(x)-2} D_i u\right)(D_i u_n - D_i u) \to 0 \quad \text{a.e. in } \Omega. \tag{3.35}$$

Then there exists a subset Γ of Ω of Lebesgue zero measure, such that, for $x \in \Omega \backslash \Gamma$

$$|D_i u(x)| < \infty \text{ and}$$
$$\left(|D_i u_n(x)|^{p_i(x)-2} D_i u_n(x) - |D_i u(x)|^{p_i(x)-2} D_i u(x)\right)(D_i u_n(x) - D_i u(x)) \to 0.$$

From (3.35), we have

$$\left(|D_i u_n|^{p_i(x)-2} D_i u_n - |D_i u|^{p_i(x)-2} D_i u\right)(D_i u_n - D_i u) \leq h(x)$$

for some function h.

Let us prove that there exists a function g such that

$$\left|D_i u_n(x)\right| \leq g(x). \tag{3.36}$$

By inequality (3.6), we obtain

$$h(x) \geq \begin{cases} c\left((|D_i u_n| - |D_i u|)^{p_-^-} - 1\right), & \text{if } p_i(x) \geq 2, \\ c'\left(\frac{|D_i u_n| - |D_i u|}{1 + |D_i u_n| + |D_i u|}\right)^2, & \text{if } 1 < p_i(x) < 2 \end{cases}$$

and this implies (3.36).
We are going to prove that

$$D_i u_n(x) \to D_i u(x) \text{ in } \Omega \backslash \Gamma. \tag{3.37}$$

Assume by contradiction that there exists $x_0 \in \Omega \backslash \Gamma$ such that $D_i u_n(x_0)$ does not converge to $D_i u(x_0)$. The Bolzano-Weierstrass theorem implies that $D_i u_n(x_0) \to b$, $b \in \mathbb{R}$, up to a subsequence. Passing to the limit in

$$\left(\left(|D_i u_n(x_0)|^{p_i(x_0)-2} D_i u_n(x_0) - |D_i u x_0)|^{p_i(x_0)-2} D_i u(x_0)\right)(D_i u_n(x_0) - D_i u(x_0)),\right.$$

we get

$$\left(|b|^{p_i(x_0)-2} b - |D_i u(x_0)|^{p_i(x_0)-2} D_i u(x_0)\right)(b - D_i u(x_0)) = 0,$$

which yields $b = D_i u(x_0)$ using inequality (3.6). Through this we get (3.30). □

3.2. *Proof of Theorem 1*

From (3.30) and the boundedness of $\{|D_iu_n|^{p_i(x)-2}D_iu_n\}$ in $L^{p_i'(\cdot)}(\Omega)$, we can get, for all $i=1,\dots,N$

$$|D_iu_n|^{p_i(x)-2}D_iu_n \rightharpoonup |D_iu|^{p_i(x)-2}D_iu \text{ weakly in } L^{p_i'(\cdot)}(\Omega). \tag{3.38}$$

From (3.31), (3.2), and the fact that $0<\frac{1}{1+e^{|u_n|}}\leq\frac{1}{2}$ we can apply the Lebesgue's dominated convergence theorem to obtain

$$\frac{a_n(x,u_n)}{1+e^{|u_n|}}D_i\varphi\to\frac{a(x,u)}{1+e^{|u|}}D_i\varphi \quad \text{strongly in } L^{p_i(\cdot)}(\Omega),\ 1\leq i\leq N.$$

Now, from (3.31), we conclude that

$$u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}\to u\sum_{i=1}^{N}|u|^{p_i(x)-2} \quad \text{a.e. in } \Omega. \tag{3.39}$$

With similar arguments as in the proof of (3.27), we obtain

$$\int_{|u_n|>t}|u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}|\,\mathrm{d}x\leq\int_{|u_n|>t}|f_n|\,\mathrm{d}x. \tag{3.40}$$

For any measurable set $E\subset\Omega$, by (3.40) we have

$$\begin{aligned}\int_E|u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}|\,\mathrm{d}x&=\int_{E\cap\{|u_n|\leq t\}}|u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}|\,\mathrm{d}x\\&\quad+\int_{E\cap\{|u_n|>t\}}|u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}|\,\mathrm{d}x\\&\leq N(1+t^{p_+^+})|E|+\int_{\{|u_n|>t\}}|f_n|\,\mathrm{d}x.\end{aligned}$$

Then, we conclude the equi-integrability of $\left(u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}\right)$ in $L^1(\Omega)$, and since (3.39), Vitali's theorem implies that

$$u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}\to u\sum_{i=1}^{N}|u|^{p_i(x)-2} \quad \text{strongly in } L^1(\Omega). \tag{3.41}$$

So, we can easily pass to the limit in the weak formulation (3.4). This proves Theorem 1.

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