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INCORPORATING VIGILANCE AND HARVESTING INTO A DISCRETE PREDATOR-PREY SYSTEM WITH IMPRECISE PARAMETER ESTIMATES

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Abstract. In this paper, we consider a discrete predator-prey system with imprecise biological parameters, and consider the harvesting of prey and predator. We first study the existence and local stability conditions of the ecological equilibrium points of the system. Secondly, the possible bioeconomic equilibrium points of the system are analyzed. Thirdly, the optimal harvesting strategy and the optimal equilibrium solution are obtained by using the Pontryagin maximum principle. Finally, the feasibility of the theoretical results is verified by numerical simulation, and the influence of various parameters in the model on the population density of prey and predator are analyzed.

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1. INTRODUCTION

As a renewable natural resource, biological resources can evolve and maintain a stable number by means of growth and reproduction under natural and artificial conditions. If it is used and managed reasonably, it will be inexhaustible. Let $x(t)$ and $y(t)$ denote the population densities of prey and predator, respectively, and a general predator-prey system with harvesting can be written as:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1-\frac{x}{K}\right) - f(x)y - q_1E_1x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = ef(x)y - dy - q_2E_2y, \end{cases} \tag{1.1}$$

where r is the prey's intrinsic growth rate and K is the environmental carrying capacity of the prey. d is the predator's natural death rate. e denotes the conversion

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efficiency from prey biomass to predator biomass. q_1, q_2 (>0) are the catchability coefficient of two species. E_1 and E_2 are the harvesting effort for prey and predator populations, respectively. $f(x)$ represents the functional response of the predator. Many scholars have studied the harvesting problem of various population systems on the basis of system (1.1), and have achieved many results in the fields of fish resource management, pest control and so on [4, 6].

In the survival strategy of the prey, the vigilance effect helps the prey to avoid the attack of the predator. The prey can determine when the attack occurs based on high vigilance, so as to take necessary actions, such as escape, hiding, and maintaining a safe distance [7, 14]. We assume that vigilance will have a negative impact on the predation rate, that is, as vigilance increases, the predation rate will decrease, and vice versa. Therefore, the above system (1.1) in the presence of vigilance takes the following form:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1-v-\frac{x}{K}\right) - \frac{\beta xy}{l+\alpha v} - q_1E_1x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{axy}{l+\alpha v} - dy - q_2E_2y, \end{cases} \tag{1.2}$$

where $f(x) = \beta x$, β is the predation rate of the predator, $a = e\beta$ stands for the efficiency rate with which captured prey are converted to new predators, $\frac{1}{l}$ is the lethality of the predator, v is the level of prey vigilance, and α denotes the effectiveness of vigilance.

In the above discussion, all researchers have developed models based on the assumption that biological parameters are precisely known, but in real life the situation is different. Biological parameters may not be fixed, and may change for several reasons. For example, lack of sufficient information data and experimental errors. Therefore, the model with inaccurate biological parameters is more reasonable. In 2013, Pal et al. [12] first proposed the concept of interval number and used it in a Lotka-Volterra prey-predator capture model with uncertain parameters. They investigated the existence of biological and bionomic equilibrium points, the local stability of biological equilibrium points, the optimal harvesting policy and the solution in the interior equilibrium. On the basis of model (1.2), we establish a predator-prey system with vigilance effect and inaccurate biological parameters,

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \bar{r}x\left(1-v-\frac{x}{K}\right) - \frac{\bar{\beta}xy}{l+\alpha v} - q_1E_1x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\bar{a}xy}{l+\alpha v} - \bar{d}y - q_2E_2y, \end{cases} \tag{1.3}$$

where $\bar{r} \in [r_1, r_2]$, $\bar{\beta} \in [\beta_1, \beta_2]$, $\bar{a} \in [a_1, a_2]$, $\bar{d} \in [d_1, d_2]$ and $r_1 > 0$, $r_2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $c_1 > 0$, $c_2 > 0$, $d_1 > 0$, $d_2 > 0$.

Based on Theorem 1 in [12], we easily obtain that the above system (1.3) is equivalent to the system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = r_1^{1-k}r_2^kx - r_2^{1-k}r_1^kx\left(v+\frac{x}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{xy}{l+\alpha v} - q_1E_1x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = a_1^{1-k}a_2^k\frac{xy}{l+\alpha v} - d_2^{1-k}d_1^ky - q_2E_2y, \end{cases} \tag{1.4}$$

where $k \in [0,1]$.

However, in recent years, an increasing number of experts have discovered that the discrete model is more realistic than the continuous model for populations with a short lifetime, non-overlapping generations, or a limited number of populations. At the same time, discrete systems have more dynamic phenomena than continuous systems [1–3, 5, 8–10, 13]. As far as we know, the discrete form of system (1.4) has not been studied. Therefore, we discretize system (1.4) by Euler difference method and obtain the following discrete-time system:

$$\begin{cases} x_{t+1} = (1+r_1^{1-k}r_2^k)x_t - r_2^{1-k}r_1^k x_t\left(v+\frac{x_t}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{x_t y_t}{l+\alpha v} - q_1 E_1 x_t, \\ y_{t+1} = (1-d_2^{1-k}d_1^k)y_t + a_1^{1-k}a_2^k \frac{x_t y_t}{l+\alpha v} - q_2 E_2 y_t. \end{cases} \tag{1.5}$$

The rest of this paper is organized as follows. In Section 2, the existence and stability of the equilibrium points of system (1.5) are analyzed in detail. Also, the existence of bionomic equilibriums of system (1.5) is discussed in Section 3. Furthermore, we study the optimal harvesting policy for system (1.5) in Section 4. In Section 5, we give three numerical examples and two tables to substantiate our analytical results. Finally, some conclusions are made in the last section.

2. The existence and stability of fixed points

The equilibrium points of system (1.5) satisfy the following equations:

$$\begin{cases} r_1^{1-k}r_2^k x - r_2^{1-k}r_1^k x\left(v+\frac{x}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{xy}{l+\alpha v} - q_1 E_1 x = 0, \\ a_1^{1-k}a_2^k \frac{xy}{l+\alpha v} - d_2^{1-k}d_1^k y - q_2 E_2 y = 0. \end{cases}$$

By calculating we can get the equilibrium points of the system as follows :

(1) Trivial equilibrium: $A_0 = (0,0)$.

(2) Semi-trivial equilibrium $A_1 = (\Theta, 0)$ exists if $r_1^{1-k}r_2^k - r_2^{1-k}r_1^k v - q_1 E_1 > 0$, where $\Theta = \frac{K(r_1^{1-k}r_2^k - r_2^{1-k}r_1^k v - q_1 E_1)}{r_2^{1-k}r_1^k}$.

(3) Interior equilibrium: $A^* = (x^*, y^*)$, where

$$x^* = \frac{(l+\alpha v)(d_2^{1-k}d_1^k + q_2 E_2)}{a_1^{1-k}a_2^k}, y^* = \frac{(l+\alpha v)[r_1^{1-k}r_2^k - r_2^{1-k}r_1^k\left(v+\frac{x^*}{K}\right) - q_1 E_1]}{\beta_2^{1-k}\beta_1^k}$$

exists if

$$r_1^{1-k}r_2^k - r_2^{1-k}r_1^k\left(v+\frac{x^*}{K}\right) - q_1 E_1 > 0.$$

Assuming that λ_1 and λ_2 are the eigenvalues of system (1.5) at the equilibrium point (x,y), there is the following definition:

Definition 1 ([15, Definition 2.1]). The equilibrium point (x,y) is called

(1) sink point if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and it is locally asymptotically stable;

(2) source point if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and it is locally unstable;

(3) saddle point if either ($|\lambda_1|>1$ and $|\lambda_2|<1$) or ($|\lambda_1|<1$ and $|\lambda_2|>1$);
(4) non-hyperbolic point if either $|\lambda_1|=1$ or $|\lambda_2|=1$.

Jacobian matrix can be evaluated at $A_0(0,0)$ as

$$J(E_0)=\begin{pmatrix}1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1 & 0\\ 0 & 1-d_2^{1-k}d_1^k-q_2E_2\end{pmatrix}. \tag{2.1}$$

The eigenvalues of the Jacobian are $\lambda_1=1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1$ and $\lambda_2=1-d_2^{1-k}d_1^k-q_2E_2$ at trivial equilibrium point $A_0(0,0)$. So, we can get the following conclusions.

Lemma 1.

(1) $A_0(0,0)$ *is a sink if*
$$0<r_2^{1-k}r_1^k v+q_1E_1-r_1^{1-k}r_2^k<2 \text{ and } 0<d_2^{1-k}d_1^k+q_2E_2<2.$$
(2) $A_0(0,0)$ *is a saddle if one of the following conditions holds:*
(II-1) $r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1>0$ *and* $0<d_2^{1-k}d_1^k+q_2E_2<2$;
(II-2) $2+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1<0$ *and* $0<d_2^{1-k}d_1^k+q_2E_2<2$;
(II-3) $0<r_2^{1-k}r_1^k v+q_1E_1-r_1^{1-k}r_2^k<2$ *and* $2-d_2^{1-k}d_1^k-q_2E_2<0$.
(3) $A_0(0,0)$ *is a source if one of the following conditions holds:*
(III-1) $r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1>0$ *and* $2-d_2^{1-k}d_1^k-q_2E_2<0$;
(III-2) $2+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1<0$ *and* $2-d_2^{1-k}d_1^k-q_2E_2<0$.

Proof.

(1): It can be obtained from (2.1) that the eigenvalues of the system at the equilibrium point A_0 are $\lambda_1=1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1$ and $\lambda_2=1-d_2^{1-k}d_1^k-q_2E_2$. It can be seen from Definition 1 that when $\lambda_1<1$ and $\lambda_2<1$, the equilibrium point is the sink. Therefore, when $0<r_2^{1-k}r_1^k v+q_1E_1-r_1^{1-k}r_2^k<2$ and $0<d_2^{1-k}d_1^k+q_2E_2<2$, the equilibrium point A_0 is a sink point and is locally asymptotically stable.
(2) and (3): These can be proved similarly.

□

Jacobian matrix can be evaluated at $A_1(\Theta,0)$ as

$$J(E_K)=\begin{pmatrix}1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k\left(v+\frac{2\Theta}{K}\right)-q_1E_1 & \frac{\beta_2^{1-k}\beta_1^k\Theta}{l+\alpha v}\\ 0 & 1+\frac{a_1^{1-k}a_2^k}{l+\alpha v}-d_2^{1-k}d_1^k-q_2E_2\end{pmatrix}. \tag{2.2}$$

Lemma 2. *The characteristic roots at the boundary equilibrium point* $A_1(\Theta,0)$ *are* $\lambda_1=1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k\left(v+\frac{2\Theta}{K}\right)-q_1E_1, \lambda_2=1+\frac{a_1^{1-k}a_2^k}{l+\alpha v}-d_2^{1-k}d_1^k-q_2E_2$, *then*

(1) *A_1 is a sink if*

$$0 < r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) + q_1 E_1 - r_1^{1-k} r_2^k < 2 \text{ and } 0 < d_2^{1-k} d_1^k + q_2 E_2 - \frac{a_1^{1-k} a_2^k}{l + \alpha v} < 2.$$

(2) *A_1 is a saddle if one of the following conditions holds:*

(II-1) $0 < r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) + q_1 E_1 - r_1^{1-k} r_2^k < 2$ *and* $\frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 > 0$;

(II-2) $0 < r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) + q_1 E_1 - r_1^{1-k} r_2^k < 2$ *and* $2 + \frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 < 0$;

(II-3) $r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 > 0$ *and* $0 < d_2^{1-k} d_1^k + q_2 E_2 - \frac{a_1^{1-k} a_2^k}{l+\alpha v} < 2$;

(II-4) $2 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 < 0$ *and* $0 < d_2^{1-k} d_1^k + q_2 E_2 - \frac{a_1^{1-k} a_2^k}{l+\alpha v} < 2$.

(3) *A_1 is a source if one of the following conditions holds:*

(III-1) $r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 > 0$ *and* $\frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 > 0$;

(III-2) $r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 > 0$ *and* $2 + \frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 < 0$;

(III-3) $2 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 < 0$ *and* $\frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 > 0$;

(III-4) $2 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 < 0$ *and* $2 + \frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 < 0$.

Proof.

(1): It can be obtained from (2.2) that the eigenvalues of the system at the equilibrium point A_1 are $\lambda_1 = 1 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1$ and $\lambda_2 = 1 + \frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2$. It can be seen from Definition 1 that when $\lambda_1 < 1$ and $\lambda_2 < 1$, the equilibrium point is the sink. Therefore, when $0 < r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) + q_1 E_1 - r_1^{1-k} r_2^k < 2$ and $0 < d_2^{1-k} d_1^k + q_2 E_2 - \frac{a_1^{1-k} a_2^k}{l+\alpha v} < 2$, the equilibrium point A_1 is a sink point and is locally asymptotically stable.

(2) and (3): These can be proved similarly.

□

$J(x,y)$ evaluated at the positive equilibrium point $A^* = (x^*, y^*)$ is

$$J(E^*) = \begin{pmatrix} 1 - \frac{x^*}{K} r_2^{1-k} r_1^k & -\frac{\beta_2^{1-k} \beta_1^k x^*}{l+\alpha v} \\ \frac{a_1^{1-k} a_2^k y^*}{l+\alpha v} & 1 \end{pmatrix}. \tag{2.3}$$

Let

$$T = 2 - \frac{x^*}{K} r_2^{1-k} r_1^k,$$

$$D = 1 - \frac{x^*}{K} r_2^{1-k} r_1^k + \frac{a_1^{1-k} a_2^k \beta_2^{1-k} \beta_1^k x^* y^*}{(l+\alpha v)^2}.$$

Then, the characteristic equation corresponding to matrix (2.3) is:

$$\lambda^2 - T\lambda + D = 0.$$

Lemma 3 ([11, Lemma 2.2]). *Assume that $F(\lambda) = \lambda^2 - T\lambda + D$, and $F(1) > 0$ with λ_1, λ_2 are roots of $F(\lambda) = 0$. Then the following results hold true:*

(1) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $D < 1$;
(2) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$, *or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) < 0$;
(3) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $D > 1$;
(4) $\lambda_1 = 1$ *and* $\lambda_2 \neq 1$ *if and only if* $F(-1) = 0$ *and* $D \neq 0, 2$;
(5) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = 1$ *and* $|\lambda_2| = 1$ *if and only if* $T^2 - 4D < 0$ *and* $D = 1$

In summary, we have the following theorem.

Theorem 1. *System (1.5) at the equilibrium point $A^* = (x^*, y^*)$ is local asymptotically stable when the condition*

$$\frac{a_1^{1-k} a_2^k \beta_2^{1-k} \beta_1^k x^* y^*}{(l+\alpha v)^2} < \frac{x^*}{K} r_2^{1-k} r_1^k < \frac{a_1^{1-k} a_2^k \beta_2^{1-k} \beta_1^k x^* y^*}{2(l+\alpha v)^2} + 2 \tag{2.4}$$

holds.

Proof. According to Definition 1 and Lemma 3 (1), when $D < 1$ and $F(-1) > 0$ hold, the equilibrium point A^* is the sink point and is locally asymptotically stable. It can be seen from formula (2.3) that

$$T = 2 - \frac{x^*}{K} r_2^{1-k} r_1^k, \quad D = 1 - \frac{x^*}{K} r_2^{1-k} r_1^k + \frac{a_1^{1-k} a_2^k \beta_2^{1-k} \beta_1^k x^* y^*}{(l+\alpha v)^2},$$

so the equilibrium point A^* is locally asymptotically stable when condition (2.4) is established. □

3. Bionomic equilibrium

If the total revenue (TR) from selling harvested organisms is equal to the total cost (TC) of harvesting effort, an economic equilibrium can be obtained. To discuss the bionomic equilibrium of the imprecise prey-predator system, we consider the parameters such as

c_1=fishing cost per unit effort for prey species;
c_2=fishing cost per unit effort for predator species;
p_1=price per unit biomass of the prey;
p_2=price per unit biomass of the predator.
Net income N is expressed as:

$$N = (p_1 q_1 x - c_1) E_1 + (p_2 q_2 y - c_2) E_2 = N_1 + N_2,$$

where N_1 and N_2 represent the net revenues for the prey and predator species, respectively. The bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$ is given by the following simultaneous equations

$$\begin{cases} (1+r_1^{1-k}r_2^k)x - r_2^{1-k}r_1^k x\left(v+\frac{x}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{xy}{l+\alpha v} - q_1E_1x = 0, \\ (1-d_2^{1-k}d_1^k)y + a_1^{1-k}a_2^k \frac{xy}{l+\alpha v} - q_2E_2y = 0, \\ (p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 = 0. \end{cases} \tag{3.1}$$

We discuss the bionomic equilibrium of system (3.1) in the following four cases.

Case I. If $c_1 > p_1q_1x$, that is, the cost of harvesting the prey x is greater than the income from selling them, then the harvesting of x needs to be stopped, and the harvesting of the predator y can continue. So $E_1 = 0$ and $c_2 < p_2q_2y$. We calculate that $y_\infty = \frac{c_2}{p_2q_2}$ and $(x_\infty, E_{2\infty})$ will be any point on the line

$$\frac{a_1^{1-k}a_2^k x}{l+\alpha v} = d_2^{1-k}d_1^k + q_2E_2$$

in the first quadrant of the xE_2-plane.

Case II. If $c_2 > p_2q_2y$, that is, the cost of harvesting the predator y is greater than the income from selling them, then the harvesting of y needs to be stopped, and the harvesting of the prey x can continue. So $E_2 = 0$ and $c_1 < p_1q_1x$. We calculate that $x_\infty = \frac{c_1}{p_1q_1}$ and $(y_\infty, E_{1\infty})$ will be any point on the line

$$\frac{\beta_2^{1-k}\beta_1^k y}{l+\alpha v} + q_1E_1 = r_1^{1-k}r_2^k - r_2^{1-k}r_1^k\left(v+\frac{x_\infty}{K}\right)$$

in the first quadrant of the yE_1-plane if $r_1^{1-k}r_2^k > r_2^{1-k}r_1^k\left(v+\frac{x_\infty}{K}\right)$.

Case III. If $c_1 > p_1q_1x$ and $c_2 > p_2q_2y$, that is, the cost of harvesting the prey x and the predator y is greater than the income from selling them, then the harvesting of x and y needs to be stopped. So $E_1 = 0$ and $E_2 = 0$.

Case IV. If $c_1 < p_1q_1x$ and $c_2 < p_2q_2y$, that is, both the prey x and the predator y can be harvested. This yields that $x_\infty = \frac{c_1}{p_1q_1}$ and $y_\infty = \frac{c_2}{p_2q_2}$. We have

$$E_{1\infty} = \frac{1}{q_1}\left[1 + r_1^{1-k}r_2^k - r_2^{1-k}r_1^k(v+\frac{x}{K}) - \beta_2^{1-k}\beta_1^k\frac{y_\infty}{l+\alpha v}\right]$$

and

$$E_{2\infty} = \frac{1}{q_2}\left[1 - d_2^{1-k}d_1^k + a_1^{1-k}a_2^k\frac{x_\infty}{l+\alpha v}\right].$$

We easily know that $E_{1\infty} > 0$ and $E_{2\infty} > 0$ provided that

$$1 + r_1^{1-k}r_2^k > r_2^{1-k}r_1^k\left(v+\frac{x}{K}\right) + \beta_2^{1-k}\beta_1^k\frac{y_\infty}{l+\alpha v} \tag{3.2}$$

and

$$1 + a_1^{1-k}a_2^k\frac{x_\infty}{l+\alpha v} > d_2^{1-k}d_1^k. \tag{3.3}$$

In summary, we have the following theorem.

Theorem 2.

(1) *The trivial bionomic equilibrium* $(x_\infty, y_\infty, 0, E_{2\infty})$ *exists, where* $y_\infty = \frac{c_2}{p_2 q_2}$ *and* $(x_\infty, E_{2\infty})$ *will be any point on the line*

$$\frac{a_1^{1-k} a_2^k x}{l+\alpha v} = d_2^{1-k} d_1^k + q_2 E_2$$

in the first quadrant of the xE_2*-plane.*

(2) *The trivial bionomic equilibrium* $(x_\infty, y_\infty, E_{1\infty}, 0)$ *exists, where* $x_\infty = \frac{c_1}{p_1 q_1}$ *and* $(y_\infty, E_{1\infty})$ *will be any point on the line*

$$\frac{\beta_2^{1-k}\beta_1^k y}{l+\alpha v} + q_1 E_1 = r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{x_\infty}{K}\right)$$

in the first quadrant of the yE_1*-plane if* $r_1^{1-k} r_2^k > r_2^{1-k} r_1^k \left(v + \frac{x_\infty}{K}\right)$.

(3) *The nontrivial bionomic equilibrium* $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$ *exists if* (3.2) *and* (3.3) *hold, where* $x_\infty = \frac{c_1}{p_1 q_1}$, $y_\infty = \frac{c_2}{p_2 q_2}$,

$$E_{1\infty} = \frac{1}{q_1}\left[1 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{x}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{y_\infty}{l+\alpha v}\right]$$

and

$$E_{2\infty} = \frac{1}{q_2}\left[1 - d_2^{1-k} d_1^k + a_1^{1-k} a_2^k \frac{x_\infty}{l+\alpha v}\right].$$

4. Optimal harvesting policy

In this section, we will study the optimal harvesting strategy of system (1.5) in three cases.

Case I. $E_1 = 0$ and $E_2 \neq 0$. That is, only the predator y is harvested. Then system (1.5) becomes:

$$\begin{cases} x_{t+1} = (1 + r_1^{1-k} r_2^k) x_t - r_2^{1-k} r_1^k x_t \left(v + \frac{x_t}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{x_t y_t}{l+\alpha v}, \\ y_{t+1} = (1 - d_2^{1-k} d_1^k) y_t + a_1^{1-k} a_2^k \frac{x_t y_t}{l+\alpha v} - q_2 E_2 y_t. \end{cases}$$

We define the present value of the net revenue function as

$$J = \max \sum_{t=1}^{k} e^{-\delta t} (p_2 q_2 y - c_2) E_2(t)$$

We first construct the following Hamiltonian function:

$$H_t = \lambda_{1(t+1)} \left[(1 + r_1^{1-k} r_2^k) x_t - r_2^{1-k} r_1^k x_t \left(v + \frac{x_t}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{x_t y_t}{l+\alpha v} - x_{t+1}\right]$$
$$+ \lambda_{2(t+1)} \left[(1 - d_2^{1-k} d_1^k) y_t + a_1^{1-k} a_2^k \frac{x_t y_t}{l+\alpha v} - q_2 E_2 y_t - y_{t+1}\right]$$

$$+e^{-\delta t}(p_2q_2y-c_2)E_2(t),$$

where λ_1 and λ_2 are adjoint variables. By Pontryagin's maximum principle, we obtain

$$\lambda_{1(t+1)} = -\frac{\partial H_t}{\partial x_t} = -\lambda_{1(t+1)}\left[r_1^{1-k}r_2^k - r_2^{1-k}r_1^k\left(v+\frac{2x_t}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{y_t}{l+\alpha v}\right] - \lambda_{2(t+1)}\cdot\frac{a_1^{1-k}a_2^k y}{l+\alpha v}, \tag{4.1}$$

$$\lambda_{2(t+1)} = -\frac{\partial H_t}{\partial y_t} = -e^{\delta t}p_2q_2E_2 + \lambda_{1(t+1)}\cdot\beta_2^{1-k}\beta_1^k\frac{x_t}{l+\alpha v} - \lambda_{2(t+1)}\left[a_1^{1-k}a_2^k\frac{x_t}{l+\alpha v} - d_2^{1-k}d_1^k - q_2E_2\right], \tag{4.2}$$

$$\frac{\partial H_t}{\partial E_2} = e^{-\delta t}(p_2q_2y-c_2) - q_2y\lambda_{2(t+1)} = 0. \tag{4.3}$$

From (4.3) we have $\lambda_{2(t+1)} = \frac{1}{q_2y}e^{-\delta t}(p_2q_2y-c_2)$. Using $\lambda_{2(t+1)}$ in (4.1) we get $\lambda_{1(t+1)}$. Now substituting $\lambda_{1(t+1)}$ and $\lambda_{2(t+1)}$ in (4.2) we get the value of the control variable as E_2. Thus the optimal harvesting is obtained as E_2^* at the optimal population level (x_*, y_*).

Case II. $E_1 \neq 0$ and $E_2 = 0$. That is, only the prey x is harvested. Then system (1.5) becomes:

$$\begin{cases} x_{t+1} = (1+r_1^{1-k}r_2^k)x_t - r_2^{1-k}r_1^k x_t\left(v+\frac{x_t}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{x_ty_t}{l+\alpha v} - q_1E_1x_t, \\ y_{t+1} = (1-d_2^{1-k}d_1^k)y_t + a_1^{1-k}a_2^k\frac{x_ty_t}{l+\alpha v}. \end{cases}$$

We define the present value of the net revenue function as

$$J = \max\sum_{t=1}^{k} e^{-\delta t}(p_1q_1x-c_1)E_1(t).$$

We first construct the following Hamiltonian function:

$$H_t = \lambda_{1(t+1)}\left[(1+r_1^{1-k}r_2^k)x_t - r_2^{1-k}r_1^k x_t\left(v+\frac{x_t}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{x_ty_t}{l+\alpha v} - q_1E_1x_t - x_{t+1}\right] + \lambda_{2(t+1)}\left[(1-d_2^{1-k}d_1^k)y_t + a_1^{1-k}a_2^k\frac{x_ty_t}{l+\alpha v} - y_{t+1}\right] + e^{-\delta t}(p_1q_1x-c_1)E_1(t),$$

where λ_1 and λ_2 are adjoint variables. By Pontryagin's maximum principle, we obtain

$$\lambda_{1(t+1)} = -\frac{\partial H_t}{\partial x_t} = -\lambda_{1(t+1)}\left[r_1^{1-k}r_2^k - r_2^{1-k}r_1^k\left(v+\frac{2x_t}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{y_t}{l+\alpha v} - q_1E_1\right]$$

$$-\lambda_{2(t+1)}\cdot\frac{c_1^{1-k}c_2^k y}{l+\alpha v}-e^{\delta t}p_1q_1E_1, \tag{4.4}$$

$$\lambda_{2(t+1)}=-\frac{\partial H_t}{\partial y_t}=\lambda_{1(t+1)}\cdot\beta_2^{1-k}\beta_1^k\frac{x_t}{l+\alpha v}-\lambda_{2(t+1)}\left[a_1^{1-k}a_2^k\frac{x_t}{l+\alpha v}-d_2^{1-k}d_1^k\right], \tag{4.5}$$

$$\frac{\partial H_t}{\partial E_1}=e^{-\delta t}(p_1q_1x-c_1)-q_1x\lambda_{1(t+1)}=0. \tag{4.6}$$

From (4.6) we have $\lambda_{1(t+1)}=\frac{1}{q_1x}e^{-\delta t}(p_1q_1x-c_1)$. Using $\lambda_{1(t+1)}$ in (4.5) we get $\lambda_{2(t+1)}$. Now substituting $\lambda_{1(t+1)}$ and $\lambda_{2(t+1)}$ in (4.4) we get the value of the control variable as E_1. Thus the optimal harvesting is obtained as E_1^* at the optimal population level (x_*,y_*).

Case III. $E_1\neq 0$ and $E_2\neq 0$.

$$\begin{cases}x_{t+1}=(1+r_1^{1-k}r_2^k)x_t-r_2^{1-k}r_1^kx_t\left(v+\frac{x_t}{K}\right)-\beta_2^{1-k}\beta_1^k\frac{x_ty_t}{l+\alpha v}-q_1E_1x_t,\\ y_{t+1}=(1-d_2^{1-k}d_1^k)y_t+a_1^{1-k}a_2^k\frac{x_ty_t}{l+\alpha v}-q_2E_2y_t.\end{cases}$$

We define the present value of the net revenue function as

$$J=\max\sum_{t=1}^{k}e^{-\delta t}\left[(p_1q_1x-c_1)E_1(t)+(p_2q_2x-c_2)E_2(t)\right].$$

We first construct the following Hamiltonian function:

$$\begin{aligned}H_t=&\lambda_{1(t+1)}\left[(1+r_1^{1-k}r_2^k)x_t-r_2^{1-k}r_1^kx_t\left(v+\frac{x_t}{K}\right)-\beta_2^{1-k}\beta_1^k\frac{x_ty_t}{l+\alpha v}-q_1E_1x_t-x_{t+1}\right]\\&+\lambda_{2(t+1)}\left[(1-d_2^{1-k}d_1^k)y_t+a_1^{1-k}a_2^k\frac{x_ty_t}{l+\alpha v}-q_2E_2y_t-y_{t+1}\right]\\&+e^{-\delta t}\left[(p_1q_1x-c_1)E_1(t)+(p_2q_2x-c_2)E_2(t)\right],\end{aligned}$$

where λ_1 and λ_2 are adjoint variables. By Pontryagin's maximum principle, we obtain

$$\begin{aligned}\lambda_{1(t+1)}=-\frac{\partial H_t}{\partial x_t}=&-\lambda_{1(t+1)}\left[r_1^{1-k}r_2^k-r_2^{1-k}r_1^k\left(v+\frac{2x_t}{K}\right)-\beta_2^{1-k}\beta_1^k\frac{y_t}{l+\alpha v}-q_1E_1\right]\\&-e^{\delta t}p_1q_1E_1-\lambda_{2(t+1)}\cdot\frac{c_1^{1-k}c_2^k y}{l+\alpha v},\end{aligned} \tag{4.7}$$

$$\begin{aligned}\lambda_{2(t+1)}=-\frac{\partial H_t}{\partial y_t}=&\lambda_{1(t+1)}\cdot\beta_2^{1-k}\beta_1^k\frac{x_t}{l+\alpha v}-\lambda_{2(t+1)}\left[a_1^{1-k}a_2^k\frac{x_t}{l+\alpha v}-d_2^{1-k}d_1^k-q_2E_2\right]\\&-e^{\delta t}p_2q_2E_2+,\end{aligned} \tag{4.8}$$

$$\frac{\partial H_t}{\partial E_1}=e^{-\delta t}(p_1q_1x-c_1)-q_1x\lambda_{1(t+1)}=0, \tag{4.9}$$

$$\frac{\partial H_t}{\partial E_2} = e^{-\delta t}(p_2 q_2 x - c_2) - q_2 y \lambda_{2(t+1)} = 0. \tag{4.10}$$

From (4.9) and (4.10) we have

$$\lambda_{1(t+1)} = \frac{1}{q_1 x} e^{-\delta t}(p_1 q_1 x - c_1), \quad \lambda_{2(t+1)} = \frac{1}{q_2 y} e^{-\delta t}(p_2 q_2 x - c_2).$$

Now substituting $\lambda_{1(t+1)}$ and $\lambda_{2(t+1)}$ in (4.7) and (4.8) we get the value of the control variable as E_1 and E_2. Thus the optimal harvesting is obtained as E_1^* and E_2^* at the optimal population level (x_*, y_*).

5. Numerical Simulations

In this section, we will analyze the effects of imprecise parameter and harvest on ecological dynamics through numerical simulation.

Example 1. Consider the following system with imprecise parameters

$$\begin{cases} x_{t+1} = (1 + 2^{1-k}2.2^k)x_t - 2.2^{1-k}2^k x_t \left(v + \frac{x_t}{4}\right) - 1.6^{1-k}1.4^k \frac{x_t y_t}{2+0.6v} - 0.2E_1 x_t, \\ y_{t+1} = (1 - 0.2^{1-k}0.1^k)y_t + 1.2^{1-k}1.4^k \frac{x_t y_t}{2+0.6v} - 0.1E_2 y_t, \end{cases} \tag{5.1}$$

where $E_1 = E_2 = 0$ and v is bifurcation parameter.

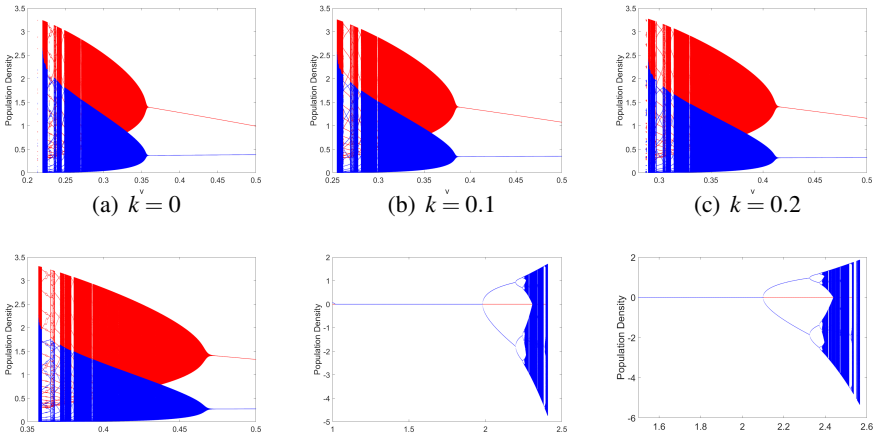
Fig 5 is the bifurcation diagram of system (5.1). We obtain the dynamic behavior of the system when k takes different values. Firstly, we find that when $k = 0$, the Neimark-Sacker bifurcation will occur in system (5.1), and as k increases, the critical value of bifurcation will gradually increase. Combined with Fig 5, it can be seen that when $v = 0.45$, the oscillation time of the solution will be extended with the increase of k, and the periodic solution will be gradually generated. When $k = 0.6$ and $k = 1$, system (5.1) will produce flip bifurcation. However, it can be seen from the image that the system will produce negative values. This is because the discrete system (1.5) is obtained by the continuous system (1.4) through the Euler difference method. Because the random values of the parameters in system (1.5) do not guarantee that the population size is always positive, so the Euler difference method has some limitations.

Example 2. Consider the following system

$$\begin{cases} x_{t+1} = x_t + r x_t \left(1 - v - \frac{x_t}{K}\right) - \frac{\beta x_t y_t}{l + \alpha v} - q_1 E_1 x_t, \\ y_{t+1} = y_t + \frac{a x_t y_t}{l + \alpha v} - d y_t - q_2 E_2 y_t, \end{cases}$$

where $E_1 = E_2 = 0$ and v is bifurcation parameter.

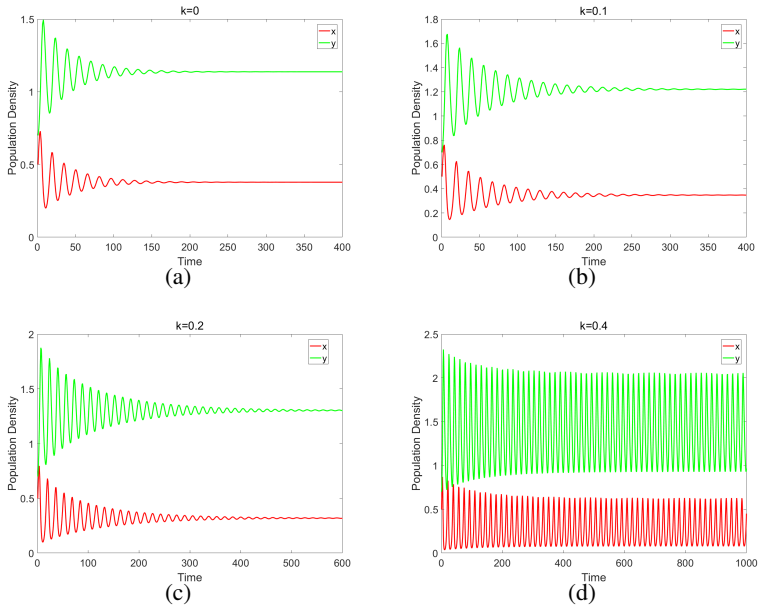
It is easy to find from Fig 5 that when the harvesting effort E_2 of the predator remains unchanged, the harvesting effort E_1 of the prey gradually increases, the number of prey populations gradually decreases and tends to extinction, and the number



(a) $k = 0$ (b) $k = 0.1$ (c) $k = 0.2$

(d) $k = 0.4$ (e) $k = 0.6$ (f) $k = 1$

FIGURE 1. The bifurcation diagram of system (5.1) when k takes different values. Red represents prey x and blue represents predator y.



(a) (b) (c) (d)

FIGURE 2. When $v = 0.45$, the time sequence diagram corresponding to Fig 5 (a, b, c, d).

of predator populations also gradually decreases and tends to extinction. This phenomenon is caused by the strict dependence of predators on the number of prey populations. When E_1 is small and remains unchanged, as E_2 increases, the number of predator populations gradually decreases. For prey, the reduction of natural enemies will gradually increase the number of their own populations. When E_1 is large, the number of prey population is small, and the predator population is extinct, so the harvest of predator E_2 will not affect the number of prey population.

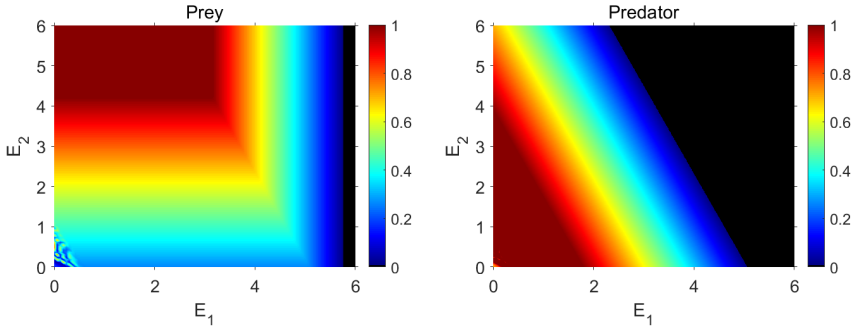


FIGURE 3. Figure shows the variation of densities of all populations in $E_1 - E_2$ space.

In Fig 5, we will analyze the effects of prey growth rate r and vigilance effect v on the population density of prey and predator. By observing the image, it can be found that when r remains unchanged and v gradually increases, the number of prey gradually decreases and the region becomes extinct, and the number of predator populations gradually decreases and the region becomes extinct. This is due to the increase in the vigilance effect of the prey on the predator, which reduces the number of prey foraging, and the resources obtained cannot meet the reproduction of their own populations, thus the number of predator populations gradually reduces and eventually becomes extinction. The reproduction of predators is strictly dependent on the prey population. Therefore, the decrease of prey population will lead to the decrease of predator population and extinction. When v remains unchanged and the growth rate of the prey gradually increases, the number of predators in the population gradually increases. For the prey, the birth rate increases and the number of natural enemies increases, so that the number of its own population will not change much.

It can be found from Fig 5 that when the predation rate β remains unchanged and the natural mortality d of the predator gradually increases, the population density of the predator gradually decreases and tends to be extinct, so that the population density of the prey gradually increases. When the natural mortality d of the predator remains unchanged and the predation rate β increases, it is found that the number of predator populations decreases gradually, while the number of prey populations

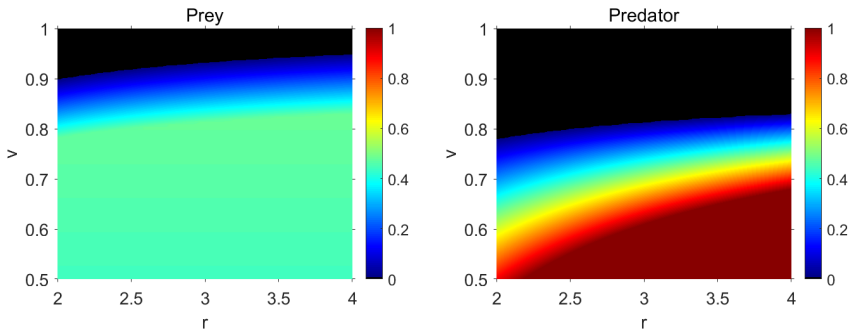


FIGURE 4. Figure shows the variation of densities of all populations in $r - v$ space.

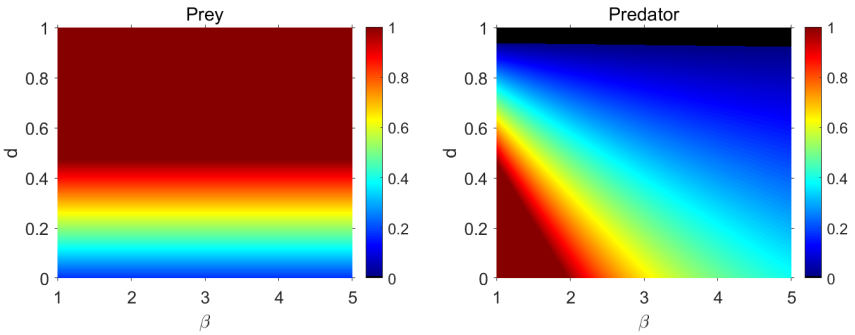


FIGURE 5. Figure shows the variation of densities of all populations in $\beta - d$ space.

remains almost unchanged. In ecology, this is because when the predation rate increases, the number of prey populations gradually decreases, so that the number of predator populations gradually decreases, and the reduction of natural enemies will increase the number of prey populations, so that the number of prey populations will not change much.

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