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A NOTE ON (σ,τ)-DERIVATIONS OF RINGS WITH INVOLUTION

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Abstract. Let R be a 2−torsion free simple *−ring and $D : R \to R$ be an additive mapping satisfiying $D(xx^*) = D(x)\sigma(x^*) + \tau(x)D(x^*)$, for all $x \in R$. Then D is a (σ,τ)−derivation of R or R is S_4 ring. Also, if R is a 2−torsion free semiprime ring and $G : R \to R$ is an additive mapping related with some (σ,τ)−derivation D of R such that $G(xx^*) = G(x)\sigma(x^*) + \tau(x)D(x^*)$, for all $x \in R$, then G is generalized (σ,τ)−derivation of R.

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1. INTRODUCTION

Let R be an associative ring with center Z. Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. A ring R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A left (right) centralizer of R is an additive mapping $T : R \to R$ which satisfies $T(xy) = T(x)y$ $(T(xy) = xT(y))$ for all $x, y \in R$. If $a \in R$, then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer. Inspired by the definition derivation and left (right) centralizer, the notion of (σ,τ)−derivation and σ−centralizer were extended as follow:

Let σ and τ be any two functions of R. An additive mapping $d : R \to R$ is called a (σ,τ)−derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. A left (right) σ−centralizer of R is an additive mapping $T : R \to R$ which satisfies $T(xy) = T(x)\sigma(y)$ $(T(xy) = \sigma(x)T(y))$ for all $x, y \in R$. Of course a $(1,1)$−derivation where 1 is the identity map on R is a derivation and a left (right) 1−centralizer is a left (right) centralizer. An additive mapping $x \to x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ holds for all $x, y \in R$. A ring equipped with an involution is called a ring with involution, or a *−ring. Let $S = \{x \in R \mid x^* = x\}$ be the set of symmetric elements of R and $K = \{x \in R \mid x^* = -x\}$ the set of skew elements of R. If A and B are nonempty subsets of R, then AB and $[A,B]$ will be additive subgroups of R generated respectively by ab and $[a,b] = ab - ba$ for all $a \in A, b \in B$.

It is well known that a prime ring R satisfies the standard identity

$$S_{2n}(x_1,x_2,...,x_{2n}) = \sum_{\sigma\in S_{2n}} (-1)^\sigma x_{\sigma(1)}, x_{\sigma(2)},...,x_{\sigma(2n)}$$

if and only if R is an order in a simple algebra of dimension at most n^2 over its center. Since such a condition appears from time to time in what follows, we shall say that "R satisfies S_{2n}" for simplicity. Otherwise, as in Lanski [13], R will be called $S_{2n}-$free.

Recently, in [4], Bresar introduced the following definition: An additive mapping $f : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$f(xy) = f(x)y + xd(y), \text{ for all } x, y \in R.$$

One may observe that the concept of generalized derivation includes the concept of derivations and the left centralizers when $d = 0$. The main examples are the derivations and generalized inner derivations a functions $f_{a,b} : R \to R$, the type $f_{a,b}(x) = ax + xb$ for some fixed $a, b \in R$. Given an arbitrary mapping $f : R \to R$ and additive mapping $d : R \to R$ of a semiprime (or prime) ring R such that $f(xy) = f(x)y + xd(y)$, for all $x, y \in R$, we note that f is uniquely defined by d, which should be a derivation by [4, Remark 2]. The notion of generalized derivation was extended as follows: Let σ,τ two functions of R. An additive mapping $f : R \to R$ is called a generalized $(\sigma,\tau)-$derivation on R if there exists a $(\sigma,\tau)-$derivation $d : R \to R$ such that

$$f(xy) = f(x)\sigma(y) + \tau(x)d(y), \text{ for all } x, y \in R.$$

On the other hand, an additive mapping $d : R \to R$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is false in general. Herstein's result [8] states that each Jordan derivation of a prime 2-torsion free ring is a derivation. M. Bresar extended this result to the case of Jordan derivations of a semiprime 2-torsion free rings in [5]. In [6], under same conditions it was shown that each of Jordan $(\sigma,\tau)-$derivation of a prime 2-torsion free ring is a $(\sigma,\tau)-$derivation. C. Lanski showed the same theorem for semiprime rings in [14]. Following [2], M. Ashraf and N. Rehman proved it for a generalized derivation of a prime 2-torsion free ring.

I. N. Herstein proved that if R is a simple $^*-$ring with characteristic different from two, $dim_Z R > 4$ and an additive mapping $D : R \to R$ such that $D(xx^*) = D(x)x^* + xD(x^*)$, for all $x \in R$, then D must be a derivation in [10, Theorem 4.1.3]. M. N. Daif and M. S. Tammam El-Sayiad extended this result for additive mapping $G : R \to R$ related with some derivation D of R such that $G(xx^*) = G(x)x^* + xD(x^*)$, for all x in a $2-$torsion free semiprime $^*-$ring R in [7]. They showed that G is a Jordan derivation of R. Also, in [15], Vukman and Kosi-Ulbl proved that R is a $2-$torsion free semiprime $^*-$ring and an additive mapping $T : R \to R$ is an additive mapping such that $T(xx^*) = T(x)x^*$ $(T(x^*x) = x^*T(x))$ is fulfilled for all

$x \in R$, then T is a left (right) centralizer. This result was extended for a left (right) $\sigma-$centralizer of R in [1].

The first purpose of this paper is to prove the theorem in [10, Theorem 4.1.3] for $(\sigma,\tau)-$derivation of R. The second aim is to show the theorem in [7, Theorem 2.1] for generalized $(\sigma,\tau)-$derivation of R.

2. Results

Throughout the present paper, σ and τ are automorphisms of R. In order to prove the theorems, we shall require the following lemmas.

Lemma 1 ([15], Lemma 1)**.** *Let R be a semiprime $^*-$ring. Suppose there exists an element $a \in R$ such that $ax^* = ax$ for all $x \in R$. In this case $a \in Z$.*

Lemma 2 ([9], Theorem 1.3)**.** *Let R be a simple ring of characteristic differrent from two and U be a Lie ideal of R. Then either $U \subset Z$ or $[R,R] \subset U$.*

Lemma 3 ([9], Corollary, p.6)**.** *If R is a noncommutative simple ring of characteristic differrent from two. Then the subring generated by $[R,R]$ in R.*

Lemma 4 ([9], Lemma 2.1)**.** *Let R be any ring with involution $R = S + K$, then K^2 is a Lie ideal of R.*

Lemma 5 ([10], Theorem 2.1.2)**.** *Let R be a $2-$torsion free semiprime ring and suppose that A is both a subring of R and a Lie ideal of R. Then $A \subset Z$ or A contains a nonzero ideal of R.*

Lemma 6 ([12], Lemma 2)**.** *Let R be any semiprime ring with involution. If $[K^2, K^2] = (0)$, then R satisfies S_4.*

Lemma 7 ([14], Theorem 2)**.** *Let R be a $2-$torsion free semiprime ring and d a Jordan $(\phi,\theta)-$derivation with ϕ or θ an automorphism of R. Then d is a $(\phi,\theta)-$derivation of R.*

Lemma 8 ([1], Theorem 2.2)**.** *Let R be a $2-$torsion free semiprime ring and α be an automorphism of R. If $T : R \to R$ is an additive mapping such that $T(x^2) = T(x)\sigma(x)$ for all $x \in R$, then T is a left $\alpha-$centralizer.*

Lemma 9 ([3], Lemma 4)**.** *Let R be a $2-$torsion free prime ring, U is a Lie ideal of R and $a,b \in R$. If $aUb = (0)$, then $a = 0$ or $b = 0$ or $U \subset Z$.*

Lemma 10 ([11], 1.1 Lemma)**.** *Let R be a prime ring with characteristic not two and U a nonzero Lie ideal of R. If d is a nonzero $(\sigma,\tau)-$derivation of R such that $d(U) = 0$, then $U \subseteq Z$.*

The following theorem gives a generalization of [10, Theorem 4.1.3] for $(\sigma,\tau)-$derivation of R.

Theorem 1. *Let R be a $2-$torsion free simple $^*-$ring. Suppose there exists an additive mapping $D : R \to R$ such that*

$$D(xx^*) = D(x)\sigma(x^*) + \tau(x)D(x^*), \text{ for all } x \in R. \tag{2.1}$$

Then D is $(\sigma,\tau)-$derivation of R or R is S_4 ring.

Proof. Assume that R is S_4 free. A linearization of (2.1) yields that

$$\begin{aligned} D(xx^* + xy^* + yx^* + yy^*) = {} & D(x)\sigma(x^*) + D(x)\sigma(y^*) + D(y)\sigma(x^*) \\ & + D(y)\sigma(y^*) + \tau(x)D(x^*) + \tau(y)D(x^*) \\ & + \tau(x)D(y^*) + \tau(y)D(y^*), \text{ for all } x, y \in R. \end{aligned}$$

Using D is an additive mapping and (2.1), we arrive at

$$D(xy^* + yx^*) = D(x)\sigma(y^*) + D(y)\sigma(x^*) + \tau(y)D(x^*) + \tau(x)D(y^*). \tag{2.2}$$

Taking x^* instead of y in (2.2), we get

$$D(x^2 + (x^*)^2) = D(x)\sigma(x) + D(x^*)\sigma(x^*) + \tau(x^*)D(x^*) + \tau(x)D(x),$$

and so

$$D(x^2) - D(x)\sigma(x) - \tau(x)D(x) + D((x^*)^2) - D(x^*)\sigma(x^*) - \tau(x^*)D(x^*) = 0.$$

This relation reduces to

$$A(x) + A(x^*) = 0, \text{ for all } x \in R$$

where $A(x)$ stands for $A(x) = D(x^2) - D(x)\sigma(x) - \tau(x)D(x)$. Replacing y by $xy^* + yx^*$ in (2.2), we obtain that

$$\begin{aligned} & D(x(xy^* + yx^*)^* + (xy^* + yx^*)x^*) = D(x)\sigma((xy^* + yx^*)^*) \\ & \quad + \tau(x)D((xy^* + yx^*)^*) + D(xy^* + yx^*)\sigma(x^*) + \tau(xy^* + yx^*)D(x^*), \end{aligned}$$

and so

$$\begin{aligned} & D(xyx^* + x^2y^* + xy^*x^* + y(x^*)^2) = D(x)\sigma(yx^*) + D(x)\sigma(xy^*) \\ & \quad + \tau(x)D(yx^* + xy^*) + D(xy^* + yx^*)\sigma(x^*) + \tau(xy^*)D(x^*) + \tau(yx^*)D(x^*). \end{aligned}$$

Using (2.2), we have

$$\begin{aligned} D(x(y + y^*)x^*) + D(x^2y^* + y(x^*)^2) = {} & D(y)\sigma(x^*)\sigma(x^*) + D(x)\sigma(y^*)\sigma(x^*) \\ & + \tau(y)D(x^*)\sigma(x^*) + \tau(x)D(y^*)\sigma(x^*) \\ & + D(x)\sigma(yx^*) + D(x)\sigma(xy^*) \\ & + \tau(xy^*)D(x^*) + \tau(yx^*)D(x^*) \\ & + \tau(x)D(y)\sigma(x^*) + \tau(x)D(x)\sigma(y^*) \\ & + \tau(xy)D(x) + \tau(x^2)D(y^*). \end{aligned}$$

Now the relation (2.2) reduces to $D(x^2y^*+y(x^2)^*) = D(y)\sigma(x^2)^* + D(x^2)\sigma(y^*)$ $+\tau(y)D((x^*)^2)+\tau(x^2)D(y^*)$. Using this in the last equation, we arrive at

$$\begin{aligned} D(x(y+y^*)x^*) = &-A(x)\sigma(y^*)-\tau(y)A(x^*)+D(x)\sigma(y)\sigma(x^*) \\ &+D(x)\sigma(y^*)\sigma(x^*)+\tau(x)D(y^*)\sigma(x^*)+\tau(x)\tau(y^*)D(x^*) \\ &+\tau(x)D(y)\sigma(x^*)+\tau(x)\tau(y)D(x^*). \end{aligned}$$

We can write the last equation such as

$$\begin{aligned} D(x(y+y^*)x^*) = &-A(x)\sigma(y^*)-\tau(y)A(x^*)+D(x)\sigma(y+y^*)\sigma(x^*) \\ &+\tau(x)\tau(y+y^*)D(x^*)+\tau(x)D(y+y^*)\sigma(x^*). \end{aligned} \tag{2.3}$$

Replacing $y-y^*$ by y in (2.3), we get

$$-A(x)\sigma(y^*-y)-\tau(y-y^*)A(x^*)=0,$$

and so

$$A(x)\sigma(y^*)-A(x)\sigma(y)+\tau(y)A(x^*)-\tau(y^*)A(x^*)=0.$$

Using $A(x)+A(x^*)=0$, for all $x\in R$ in the last equation, we arrive at

$$A(x)\sigma(y^*)+\tau(y^*)A(x) = A(x)\sigma(y)+\tau(y)A(x), \text{ for all } x,y\in R. \tag{2.4}$$

Now, writting $k\in K$ by y in (2.4), we have

$$-A(x)\sigma(k)-\tau(k)A(x) = A(x)\sigma(k)+\tau(k)A(x),$$

and so

$$2(A(x)\sigma(k)+\tau(k)A(x))=0.$$

Since R is $2-$torsion free ring, we get

$$A(x)\sigma(k)+\tau(k)A(x)=0, \text{ for all } k\in K, x\in R. \tag{2.5}$$

Multipliying (2.5) from the right by $\sigma(t), t\in K$ and using (2.5), we obtain that

$$[A(x),kt]_{\sigma,\tau}=0,$$

and so

$$[A(x),K^2]_{\sigma,\tau}=0. \tag{2.6}$$

We know that K^2 is a Lie ideal of R by Lemma 4. So, in view of Lemma 2, we have either $K^2\subset Z$ or $[R,R]\subset K^2$. If $K^2\subset Z$, then R is S_4 ring by Lemma 6. Since R is S_4 free, we have $[R,R]\subset K^2$. Also, by Lemma 3 and (2.6), we conclude that

$$[A(x),R]_{\sigma,\tau}=(0).$$

Hence we obtain that $A(x)\in C_{\sigma,\tau}$, for all $x\in R$. Returning (2.5) and using $A(x)\in C_{\sigma,\tau}$, we arrive at

$$2A(x)\sigma(k)=0, \text{ for all } k\in K, x\in R,$$

and so

$$\sigma^{-1}(A(x))K^2=(0), \text{ for all } x\in R.$$

Using $[R,R] \subset K^2$ and Lemma 3, we have $\sigma^{-1}(A(x))R = (0)$, and so $A(x) = (0)$, for all $x \in R$ by the semiprimeness of R.

$$D(x^2) = D(x)\sigma(x) + \tau(x)D(x), \text{ for all } x \in R.$$

Thus we obtain that D is a Jordan $(\sigma,\tau)-$derivation of R, and so D is $(\sigma,\tau)-$derivation of R by Lemma 7. □

Theorem 2. *Let R be a 2−torsion free prime $^*-$ring. Suppose there exists an additive mapping $D : R \to R$ such that*

$$D(xx^*) = D(x)\sigma(x^*) + \tau(x)D(x^*), \textit{ for all } x \in R.$$

Then D is a $(\sigma,\tau)-$derivation of R or R is S_4 ring.

Proof. Using the same methods in the proof of Theorem 1, we have

$$A(x)\sigma(k) + \tau(k)A(x) = 0, \text{ for all } k \in K, x \in R. \tag{2.7}$$

Multipliying (2.7) from the right by $\sigma(t), t \in K$ and using (2.5), we obtain that

$$[A(x), kt]_{\sigma,\tau} = 0,$$

and so

$$[A(x), K^2]_{\sigma,\tau} = (0).$$

This yields that $d_{A(x)}\left(K^2\right) = (0)$, where $d_{A(x)} : R \to R$, $d_{A(x)}(y) = [A(x), y]_{\sigma,\tau}$ is an inner $(\sigma,\tau)-$derivation of R. Since K^2 is a Lie ideal of R, we have $K^2 \subset Z$ or $d_{A(x)} = 0$, for all $x \in R$ by Lemma 10. In the first case, R is S_4 ring by Lemma 6. So, we get $d_{A(x)} = 0$, for all $x \in R$. Hence we obtain that $A(x) \in C_{\sigma,\tau}$, for all $x \in R$. Returning (2.7) and using $A(x) \in C_{\sigma,\tau}$, we get $2A(x)\sigma(k) = 0$, and so $A(x)\sigma(ktw) = 0$, for all $k,t,w \in K, x \in R$. Hence $\sigma^{-1}(A(x))K^2\sigma^{-1}(w) = 0$, for all $w \in K, x \in R$. By Lemma 9, we have $A(x) = 0$, for all $x \in R$ or $K = (0)$. If $K = (0)$, then $K^2 \subset Z$, and so R is S_4 ring by Lemma 6. So, we get $A(x) = 0$, for all $x \in R$. That is

$$D(x^2) = D(x)\sigma(x) + \tau(x)D(x), \text{ for all } x \in R.$$

Thus we obtain that D is a Jordan $(\sigma,\tau)-$derivation of R, and so D is $(\sigma,\tau)-$derivation of R by Lemma 7. □

The following theorem extends [7, Theorem 2.1] to generalized $(\sigma,\tau)-$derivations of R.

Theorem 3. *Let R be a semiprime 2−torsion free $*-$ring. Suppose there exists an additive mapping $G : R \to R$ related with some $(\sigma,\tau)-$derivation D such that*

$$G(xx^*) = G(x)\sigma(x^*) + \tau(x)D(x^*), \textit{ for all } x \in R. \tag{2.8}$$

Then G is generalized $(\sigma,\tau)-$derivation of R.

Proof. A linearization of (2.8) yields that

$$G(xx^*+xy^*+yx^*+yy^*)=G(x+y)\sigma(x^*+y^*)+\tau(x+y)D(x^*+y^*),$$

for all $x,y\in R$. Using G is an additive mapping and (2.8), we arrive at

$$G(xy^*+yx^*)=G(y)\sigma(x^*)+G(x)\sigma(y^*)+\tau(y)D(x^*)+\tau(x)D(y^*). \tag{2.9}$$

Substituting x^* for y in (2.9), we get

$$G(x^2+(x^*)^2)=G(x^*)\sigma(x^*)+G(x)\sigma(x)+\tau(x^*)D(x^*)+\tau(x)D(x),$$

and so

$$G(x^2)-G(x)\sigma(x)-\tau(x)D(x)+G((x^*)^2)-G(x^*)\sigma(x^*)-\tau(x^*)D(x^*)=0.$$

This relation reduces to

$$A(x)+A(x^*)=0,\ \text{for all } x\in R$$

where $A(x)$ stands for $A(x)=G(x^2)-G(x)\sigma(x)-\tau(x)D(x)$. Replacing y by xy^*+yx^* in (2.9), we obtain that

$$\begin{aligned}G(x(xy^*+yx^*)^*+(xy^*+yx^*)x^*)&=G(x)\sigma((xy^*+yx^*)^*)\\&+G(xy^*+yx^*)\sigma(x^*)+\tau(xy^*+yx^*)D(x^*)+\tau(x)D((xy^*+yx^*)^*),\end{aligned}$$

and so

$$\begin{aligned}G(x(yx^*+xy^*)+(xy^*+yx^*)x^*)&=G(xy^*+yx^*)\sigma(x^*)+G(x)\sigma(yx^*+xy^*)\\&+\tau(xy^*+yx^*)D(x^*)+\tau(x)D(yx^*+xy^*).\end{aligned}$$

Using (2.9) and D is $(\sigma,\tau)-$derivation of R, we have

$$\begin{aligned}G(xyx^*+x^2y^*+xy^*x^*+y(x^*)^2)=&(G(y)\sigma(x^*)+G(x)\sigma(y^*)+\tau(y)D(x^*)\\&+\tau(x)D(y^*))\sigma(x^*)+G(x)\sigma(yx^*+xy^*)\\&+\tau(xy^*+yx^*)D(x^*)+\tau(x)(D(y)\sigma(x^*)\\&+\tau(y)D(x^*)+D(x)\sigma(y^*)+\tau(x)D(y^*).\end{aligned}$$

Again using

$$G(x^2y^*+y(x^2)^*)=G(y)\sigma(x^2)^*+G(x^2)\sigma(y^*)+\tau(y)D((x^*)^2)+\tau(x^2)D(y^*)$$

in the last equation, we arrive at

$$\begin{aligned}G(x(y+y^*)x^*)=&-G(x^2)\sigma(y^*)+G(x)\sigma(y^*)\sigma(x^*)+\tau(x)D(y^*)\sigma(x^*)\\&+G(x)\sigma(y)\sigma(x^*)+G(x)\sigma(x)\sigma(y^*)+\tau(x)\tau(y^*)D(x^*)\\&+\tau(x)D(y)\sigma(x^*)+\tau(x)\tau(y)D(x^*)+\tau(x)D(x)\sigma(y^*),\end{aligned}$$

and so

$$G(x(y+y^*)x^*)=-A(x)\sigma(y^*)+G(x)\sigma(y+y^*)\sigma(x^*)+\tau(x)D((y+y^*)x^*). \tag{2.10}$$

Replacing $y-y^*$ by y in (2.10), we get

$$A(x)\sigma(y)=A(x)\sigma(y^*),$$

and so

$$\sigma^{-1}(A(x))y=\sigma^{-1}(A(x))y^*, \text{ for all } x,y\in R.$$

By Lemma 1 and σ is an automorphism of R, we have $A(x)\in Z$, for all $x\in R$.

Now, writing y^* by y in (2.9), we have

$$G(xy+y^*x^*)=G(y^*)\sigma(x^*)+G(x)\sigma(y)+\tau(y^*)D(x^*)+\tau(x)D(y). \quad (2.11)$$

Taking xy instead of y in (2.11), we conclude that

$$G(x^2y+y^*(x^*)^2)=G(y^*x^*)\sigma(x^*)+G(x)\sigma(x)\sigma(y)+\tau(y^*)\tau(x^*)D(x^*)$$
$$+\tau(x)D(x)\sigma(y)+\tau(x)\tau(x)D(y). \quad (2.12)$$

Replacing x^2 by x in (2.11), we get

$$G(x^2y+y^*(x^*)^2)=G(y^*)\sigma((x^*)^2)+G(x^2)\sigma(y)+\tau(y^*)D(x^*)\sigma(x^*)$$
$$+\tau(y^*)\tau(x^*)D(x^*)+\tau(x)\tau(x)D(y). \quad (2.13)$$

Comparing (2.12) and (2.13), we obtain that

$$A(x)\sigma(y)+(G(y^*)\sigma(x^*)+\tau(y^*)D(x^*)-G(y^*x^*)\sigma(x^*))=0.$$

Replacing y by x in this equation, we get

$$A(x)\sigma(x)+(G(x^*)\sigma(x^*)+\tau(x^*)D(x^*)-G(x^*x^*)\sigma(x^*))=0,$$

and so

$$A(x)\sigma(x)-A(x^*)\sigma(x^*)=0.$$

Using $A(x)+A(x^*)=0$, we arrive at

$$A(x)\sigma(x+x^*)=0 \quad (2.14)$$

Returning $A(x)\sigma(y)=A(x)\sigma(y^*)$ and writing x by y in this equation, we have

$$A(x)\sigma(x-x^*)=0. \quad (2.15)$$

Combining (2.14) and (2.15), we arrive at $A(x)\sigma(x)=0$, and so $\sigma(x)A(x)=0$. A linearization of this equation yields that

$$A(x+y)\sigma(x+y)=0,$$

and so

$$(G(x^2)-G(x)\sigma(x)-\tau(x)D(x)+G(xy+yx)-G(x)\sigma(y)-G(y)\sigma(x)$$
$$+G(y^2)-G(y)\sigma(y)-\tau(y)D(y)-\tau(x)D(y)-\tau(y)D(x))\sigma(x+y)=0.$$

Defining $B(x,y)=G(xy+yx)-G(x)\sigma(y)-G(y)\sigma(x)-\tau(x)D(y)-\tau(y)D(x)$, we arrive at

$$(A(x)+B(x,y)+A(y))\sigma(x+y)=0.$$

Expanding this equation and using $A(x)\sigma(x)=0$, we have

$$B(x,y)\sigma(x)+A(y)\sigma(x)+A(x)\sigma(y)+B(x,y)\sigma(y)=0. \tag{2.16}$$

Taking $-x$ instead of x in (2.16) and using $A(-x)=A(x)$, $B(-x,y)=-B(x,y)$, we conclude that

$$B(x,y)\sigma(x)-A(y)\sigma(x)+A(x)\sigma(y)-B(x,y)\sigma(y)=0. \tag{2.17}$$

Adding (2.16) and (2.17), we obtain that

$$2B(x,y)\sigma(x)+2A(x)\sigma(y)=0.$$

Since R is 2−torsion free ring, we get $B(x,y)\sigma(x)+A(x)\sigma(y)=0$, for all $x,y\in R$. Multipliying this equation with $A(x)$ from the right and using $\sigma(x)A(x)=0$, we find that

$$A(x)\sigma(y)A(x)=0,\quad \text{for all } x,y\in R.$$

Since R is semiprime ring, we have $A(x)=0$, and so

$$G(x^2)=G(x)\sigma(x)+\tau(x)D(x),\quad \text{for all } x\in R.$$

Hence G is a generalized $(\sigma,\tau)-$Jordan derivation of R.

Now, let assume $T=G-D$. We get

$$\begin{aligned} T(x^2)=(G-D)(x^2) &= G(x)\sigma(x)+\tau(x)D(x)-D(x)\sigma(x)-\tau(x)D(x)\\ &= G(x)\sigma(x)-D(x)\sigma(x)\\ &= (G-D)(x)\sigma(x)=T(x)\sigma(x). \end{aligned}$$

Hence we find that T is a Jordan $\sigma-$centralizer. In view of Lemma 8, T is left $\sigma-$centralizer. On the other hand, since $G=D+T$, we have

$$\begin{aligned} G(xy)=(D+T)(xy) &= D(x)\sigma(y)+\tau(x)D(y)+T(x)\sigma(y)\\ &= (D+T)(x)\sigma(y)+\tau(x)D(y), \end{aligned}$$

and so

$$G(xy)=G(x)\sigma(y)+\tau(x)D(y),\quad \text{for all } x,y\in R.$$

Hence we obtain that G is generalized $(\sigma,\tau)-$derivation of R. □

Corollary 1. *Let R be a 2−torsion free simple $^*-$ring. Suppose there exists an additive mapping $G:R\to R$ related with some additive mapping D such that*

$$D(xx^*)=D(x)\sigma(x^*)+\tau(x)D(x^*),\ \textit{for all } x\in R$$

and

$$G(xx^*)=G(x)\sigma(x^*)+\tau(x)D(x^*),\ \textit{for all } x\in R.$$

Then G is generalized $(\sigma,\tau)-$derivation of R or R is S_4 ring.

Proof. By Theorem 1, we find that D is a $(\sigma,\tau)-$derivation of R or R is S_4 ring. Hence G is generalized $(\sigma,\tau)-$derivation of R by Theorem 3. □

Corollary 2. *Let R be a $2-$torsion free prime $^*-$ring. Suppose there exists an additive mapping $G : R \to R$ related with some additive mapping D such that*

$$D(xx^*) = D(x)\sigma(x^*) + \tau(x)D(x^*), \text{ for all } x \in R$$

and

$$G(xx^*) = G(x)\sigma(x^*) + \tau(x)D(x^*), \text{ for all } x \in R.$$

Then G is generalized $(\sigma,\tau)-$derivation of R or R is S_4 ring.

In particular, if we take $D = 0$ in Theorem 3, we have the following corollary which is in [15].

Corollary 3. *Let R be a $2-$torsion free semiprime $^*-$ring. Suppose there exists an additive mapping $T : R \to R$ an additive mapping such that*

$$T(xx^*) = T(x)\sigma(x^*), \text{ for all } x \in R.$$

Then T is left $\sigma-$centralizer.

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