Miskolc Mathematical Notes
Vol. 26 (2025), No. 2, pp. 785–795

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4758

ON THE BALANCING-BILINEAR SYSTEM OF DIFFERENCE EQUATIONS: SOLUTIONS VIA SOME NUMBER SEQUENCES AND GLOBAL STABILITY

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Received 17 September, 2023

Abstract. In the current paper, we present a new class of systems called the Balancing-bilinear system of difference equations to investigate some theoretical proprieties. In this class of systems, the coefficients are allowed to depend on the Balancing sequence. One feature of this new class of systems is theoretically solvable in closed form. For this purpose, our interest is focused firstly to show that the solutions of this system are also related to Balancing numbers. As a consequence, we find that these solutions are also related to some number sequences (Pell numbers, Pell-Lucas numbers and Lucas-Balancing numbers). Secondly, we invest in the essential problems related to the Balancing-bilinear system, i.e. the global stability of positive solutions.

2010 *Mathematics Subject Classification:* 39A10; 40A05

Keywords: system of difference equations, balancing numbers, Pell numbers, Lucas-Balancing numbers, Binet formula, stability

1. INTRODUCTION

Despite the development of difference equations and systems, the class of standard bilinear difference equations and systems introduced by (Adamović [2], Boole [3], Brand [4], Jordan [23], Krechmar [32], Mitrinović and Keckić [33] and references therein) remains very popular in many areas of science, mathematical, and economic domains. These equations and systems are employed to model the temporal evolution of variables such as exchange rates, especially when the values of these variables are only observed at discrete time intervals. In econometric applications, difference equations are often modeled with stochastic terms [8–15, 17, 21]. The standard problem is to find formulas for their solutions in closed-form, in this context, can be found several classical solvable nonlinear difference equations and systems [1, 5–7, 16, 18–20, 24–31], such that the general solutions via the Fibonacci sequence [19, 34], and/or Pell sequence [19]. Here we continue the investigation of solvability of a bilinear system of difference equations but the coefficients depend on

The work was supported by *DGRSDT – MESRS* (*Dz*).

the Balancing sequence, as well as the general solutions are given via the Balancing sequence and some other sequences. For this purpose, we consider the following m-dimensional system of difference equations with Balancing-coefficients,

$$v_{n+1}^{(i)} = \frac{B_{k+2} - B_{k+1} v_{n-p}^{(i+1) \bmod m}}{B_{k+3} - B_{k+2} v_{n-p}^{(i+1) \bmod m}},\ n,k,p,m \in \mathbb{N}_0,\ i \in \{1,\ldots,m\}, \tag{1.1}$$

and the initial values $v_{-j}^{(i)}$, $i \in \{1,\ldots,m\}$, $j \in \{0,1,\ldots,p\}$.

2. Main results

To solve system (1.1) we require to utilize the following lemmas.

Lemma 1. *Let $(B_n, n \geq 0)$ the Balancing sequence, defined as follows*

$$B_{n+1} = 6B_n - B_{n-1}, n \geq 1,$$

with initial conditions $B_0 = 0$ and $B_1 = 1$. The following Binet formula of the Balancing numbers gives $B_n = \dfrac{a^n - b^n}{a-b}$ where $a = 3 + 2\sqrt{2}$ and $b = 3 - 2\sqrt{2}$. Moreover, the Lucas-Balancing sequence has the same homogeneous linear difference equation with constant coefficients as the Balancing sequence,

$$C_{n+1} = 6C_n - C_{n-1}, n \geq 1,$$

with distinct initial conditions $C_0 = 3$, $C_1 = 17$ and the closed-form expressions for the Lucas-Balancing numbers are $2C_n = a^n + b^n$. So, we have some important relations for $n, m \in \mathbb{N}$,

- $B_m B_{n+1} - B_{m-1} B_n = B_{m+n}$,
- $B_{n-1} B_{n+1} - B_n^2 = -1$,
- $B_{n-1} - B_{n+1} = 2C_n$,
- $B_{m(n+1)} B_{m+1} - B_{mn} = B_m B_{m(n+1)+1}$,
- $B_{m(n+1)} - B_{m+1} B_{mn} = -B_{mn-1} B_m$,
- $B_{m(n+1)+1} - B_{m+1} B_{mn+1} = -B_{mn} B_m$.

Moreover, the sequence $(B_n / B_{n+1}, n \geq 1)$ converges to $a^{-1} = b$.

Lemma 2. *Consider the homogeneous linear difference equation with constant coefficients*

$$u_{n+1} - 2C_m u_n + u_{n-1} = 0,\ n \geq 0, \tag{2.1}$$

with initial conditions $u_0, u_{-1} \in \mathbb{R}^*$. Then,

$$\forall n \geq 0,\ u_n = B_{m(n+1)} B_m^{-1} u_0 - B_{mn} B_m^{-1} u_{-1},$$

where $(B_n, n \geq 0)$ is the Balancing sequence and $(C_n, n \geq 0)$ is the Lucas-Balancing sequence.

Proof. Difference equation (2.1) is ordinarily solved by using the following characteristic polynomial

$$\lambda^2 - 2C_m\lambda + 1 = \lambda^2 - (a^m + b^m)\lambda + a^m b^m = (\lambda - a^m)(\lambda - b^m) = 0,$$

roots of this equation are $\lambda_1 = a^m$ and $\lambda_2 = b^m$. These roots are linked the roots of the Balancing number sequence. Then the closed form of general solution of equation (2.1) is

$$\forall n \geq -1,\ u_n = c_1 a^{mn} + c_2 b^{mn},$$

where u_0, u_{-1} are initial values such that

$$\begin{cases} u_0 = c_1 + c_2, \\ u_{-1} = \dfrac{c_1}{a^m} + \dfrac{c_2}{b^m}, \end{cases}$$

and we have

$$c_1 = \frac{a^m u_0 - u_{-1}}{a^m - b^m}, \quad c_2 = \frac{u_{-1} - b^m u_0}{a^m - b^m}.$$

After some calculations, we get

$$\begin{aligned} u_n &= \frac{a^m u_0 - u_{-1}}{a^m - b^m} a^{mn} + \frac{u_{-1} - b^m u_0}{a^m - b^m} b^{mn} \\ &= \frac{\left(a^{m(n+1)} - b^{m(n+1)}\right) u_0 - \left(a^{mn} - b^{mn}\right) u_{-1}}{a^m - b^m} \\ &= \frac{B_{m(n+1)} u_0 - B_{mn} u_{-1}}{B_m}. \end{aligned}$$

The lemma is proved. □

Lemma 3. *Consider the following rational difference equation,*

$$v_{n+1} = \frac{B_m - B_{m-1} v_n}{B_{m+1} - B_m v_n},\ n \geq 0. \tag{2.2}$$

Then,

$$\forall n \geq 1,\ v_n = \frac{-B_{mn} + B_{mn-1} v_0}{B_{mn+1} - B_{mn} v_0}.$$

Proof. Using the change of variables $w_n = B_{m+1} - B_m v_n$, we can write (2.2) as

$$w_{n+1} = \frac{\left(B_{m+1} - B_{m-1}\right) w_n - \left(B_m^2 - B_{m-1} B_{m+1}\right)}{w_n} = \frac{2C_m w_n - 1}{w_n},\ n \geq 0, \tag{2.3}$$

and $w_n = \dfrac{u_n}{u_{n-1}}$, we get $u_{n+1} - 2C_m u_n + u_{n-1} = 0,\ n \geq 0$, by Lemma 2, the closed form of general solution of the equation (2.3) is

$$w_n = \frac{B_{m(n+1)} u_0 - B_{mn} u_{-1}}{B_{mn} u_0 - B_{m(n-1)} u_{-1}} = \frac{B_{m(n+1)} w_0 - B_{mn}}{B_{mn} w_0 - B_{m(n-1)}},\ n \geq 1,$$

then,

$$\begin{aligned} v_n &= B_m^{-1}\left(w_n - B_{m+1}\right) \\ &= B_m^{-1}\left(\frac{B_{m(n+1)}w_0 - B_{mn}}{B_{mn}w_0 - B_{m(n-1)}} - B_{m+1}\right) \\ &= B_m^{-1}\left(\frac{\left(B_{m(n+1)}B_{m+1} - B_{mn}\right) - B_{m(n+1)}B_m v_0}{\left(B_{mn}B_{m+1} - B_{m(n-1)}\right) - B_{mn}B_m v_0} - B_{m+1}\right) \\ &= B_m^{-1}\left(\frac{B_{m(n+1)+1} - B_{m(n+1)}v_0}{B_{mn+1} - B_{mn}v_0} - B_{m+1}\right) \\ &= \frac{-B_{mn} + B_{mn-1}v_0}{B_{mn+1} - B_{mn}v_0},\ n \geq 1. \end{aligned}$$

The lemma is proved. □

2.1. *On the system (2.4)*

In this subsection, we consider the following system of difference equations of 1st-order

$$v_{n+1}^{(i)} = \frac{B_{k+2} - B_{k+1}v_n^{(i+1)\bmod m}}{B_{k+3} - B_{k+2}v_n^{(i+1)\bmod m}},\ n,k,m \in \mathbb{N}_0,\ i \in \{1,\ldots,m\}. \tag{2.4}$$

Now, using the last difference equation in (2.4), we get

$$v_{n+1}^{(m-1)} = \frac{B_{k+2} - B_{k+1}v_n^{(m)}}{B_{k+3} - B_{k+2}v_n^{(m)}} = \frac{B_{2k+4} - B_{2k+3}v_{n-1}^{(1)}}{B_{2k+5} - B_{2k+4}v_{n-1}^{(1)}},\ n \geq 1,$$

similarly, we get

$$v_{n+1}^{(m-2)} = \frac{B_{k+2} - B_{k+1}v_n^{(m-1)}}{B_{k+3} - B_{k+2}v_n^{(m-1)}} = \frac{B_{3k+6} - B_{3k+5}v_{n-1}^{(1)}}{B_{3k+7} - B_{3k+6}v_{n-1}^{(1)}},\ n \geq 2,$$

and recursively for the above, we can get

$$v_{n+1}^{(1)} = \frac{B_{m(k+2)} - B_{m(k+2)-1}v_{n-(m-1)}^{(1)}}{B_{m(k+2)+1} - B_{m(k+2)}v_{n-(m-1)}^{(1)}},\ n \geq m-1.$$

System (2.4) can be written as the following rational difference equation of mth-order

$$v_{n+1} = \frac{B_{m_k} - B_{m_k-1}v_{n-(m-1)}}{B_{m_k+1} - B_{m_k}v_{n-(m-1)}},\ n \geq m-1. \tag{2.5}$$

where $m_k = m(k+2)$. Let $v_{n,h} = v_{mn+h}$, $h \in \{0,1,\ldots,m-1\}$. For this, we have

$$v_{n+1,h} = \frac{B_{m_k} - B_{m_k-1}v_{n,h}}{B_{m_k+1} - B_{m_k}v_{n,h}},\ n \geq 0,\ h \in \{0,1,\ldots,m-1\}.$$

By Lemma 3, the closed form of the general solution of equation (2.5) is easily obtained in the following corollary.

Corollary 1. *Let* $\{v_n, n \geq 0\}$ *be a solution of equation (2.5). Then*

$$\forall n \geq m-1,\ v_{mn+h} = \frac{-B_{m_k n} + B_{m_k n-1} v_h}{B_{m_k n+1} - B_{m_k n} v_h},\ h \in \{0,1,..,m-1\},$$

where $(B_n, n \geq 0)$ *is the Balancing sequence.*

Through the above discussion, we can introduce the following theorem.

Theorem 1. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (2.4). Then,*

$$v_{mn+h}^{(i)} = \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_0^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_0^{(h+i) \bmod (m)}},\ h \in \{0,1,\ldots,m-1\},\ i \in \{1,\ldots,m\},$$

where $(B_n, n \geq 0)$ *is the Balancing sequence.*

Proof. From Corollary 1, we have

$$\forall n \geq m-1,\ v_{mn+h}^{(1)} = \frac{-B_{m_k n} + B_{m_k n-1} v_h^{(1)}}{B_{m_k n+1} - B_{m_k n} v_h^{(1)}},\ h \in \{0,1,\ldots,m-1\},$$

and by system (2.4), we get

$$v_h^{(1)} = \frac{B_{h_k} - B_{h_k-1} v_0^{(h+1) \bmod (m)}}{B_{h_k+1} - B_{h_k} v_0^{(h+1) \bmod (m)}},\ h \in \{0,1,\ldots,m-1\}.$$

Now, using Lemma 2, we obtain

$$\begin{aligned} v_{mn+h}^{(1)} &= \frac{\left(B_{m_k n-1} B_{h_k} - B_{m_k n} B_{h_k+1}\right) - \left(B_{m_k n-1} B_{h_k-1} - B_{m_k n} B_{h_k}\right) v_0^{(h+1) \bmod (m)}}{-\left(B_{m_k n} B_{h_k} - B_{m_k n+1} B_{h_k+1}\right) + \left(B_{m_k n} B_{h_k-1} - B_{m_k n+1} B_{h_k}\right) v_0^{(h+1) \bmod (m)}} \\ &= \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_0^{(h+1) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_0^{(h+1) \bmod (m)}}, h \in \{0,1,\ldots,m-1\}. \end{aligned}$$

The theorem is proved. □

2.2. *On the system (1.1)*

In this article, we study system (1.1), which is an extension of system (2.4). Therefore, system (1.1) can be written as follows

$$v_{(p+1)(n+1)-s}^{(i)} = \frac{B_{k+2} - B_{k+1} v_{(p+1)n-s}^{(i+1) \bmod m}}{B_{k+3} - B_{k+2} v_{(p+1)n-s}^{(i+1) \bmod m}} \tag{2.6}$$

for $s \in \{0,1,\ldots,p\}$, $i \in \{1,\ldots,m\}$ and $n \in \mathbb{N}$. Now, using the following notation,

$$v_{n,s}^{(i)} = v_{(p+1)n-s}^{(i)}, s \in \{0,1,\ldots,p\},\ i \in \{1,\ldots,m\},$$

we can get $(p+1)$-systems similar to system (2.4),

$$v_{n+1,s}^{(i)} = \frac{B_{k+2} - B_{k+1} v_{n,s}^{(i+1) \bmod m}}{B_{k+3} - B_{k+2} v_{n,s}^{(i+1) \bmod m}},\ i \in \{1,\ldots,m\},\ n \in \mathbb{N}_0 \tag{2.7}$$

for $s \in \{0,1,\ldots,p\}$. Through the above discussion, we can introduce the following theorem.

Theorem 2. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (1.1). Then, for* $s \in \{0,1,\ldots,p\}$

$$v_{(p+1)(mn+h)-s}^{(i)} = \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_{-s}^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_{-s}^{(h+i) \bmod (m)}},$$

$h \in \{0,1,\ldots,m-1\}$, $i \in \{1,\ldots,m\}$, *where* $(B_n, n \geq 0)$ *is the Balancing sequence.*

Proof. Let $\left\{v_{n,s}^{(1)}, v_{n;s}^{(2)}, \ldots, v_{n;s}^{(m)}, n \geq 0, s \in \{0,1,\ldots,p\}\right\}$ be a solution of systems (2.7) with initial values $v_{0,s}^{(i)}$, $i \in \{1,\ldots,m\}$, $s \in \{0,1,\ldots,p\}$. Using Theorem 1, we obtain for $s \in \{0,1,\ldots,p\}$,

$$v_{mn+h,s}^{(i)} = \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_{0,s}^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_{0;s}^{(h+i) \bmod (m)}},\ h \in \{0,1,\ldots,m-1\},\ i \in \{1,\ldots,m\}.$$

Returning to the original notation, we obtain

$$v_{(p+1)(mn+h)-s}^{(i)} = \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_{-s}^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_{-s}^{(h+i) \bmod (m)}},$$

for $h \in \{0,1,\ldots,m-1\}$, $s \in \{0,1,\ldots,p\}$, $i \in \{1,\ldots,m\}$. □

Based on Theorem 2, we can express the solutions of system (1.1) using some sequences.

Corollary 2. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (1.1). Then, for* $s \in \{0,1,\ldots,p\}$,

$$v_{(p+1)(mn+h)-s}^{(i)} = \frac{-P_{2(m_k n+h_k)} + P_{2(m_k n+h_k-1)} v_{-s}^{(h+i) \bmod (m)}}{P_{2(m_k n+h_k+1)} - P_{2(m_k n+h_k)} v_{-s}^{(h+i) \bmod (m)}},$$

where $h \in \{0,1,\ldots,m-1\}$, $i \in \{1,\ldots,m\}$ *and* $(P_n, n \geq 0)$ *is the Pell sequence.*

Proof. We see that it suffices to remark $a = a_1^2\left(1+\sqrt{2}\right)^2$ and $b = b_1^2$, where $a_1 = 1+\sqrt{2}$ and $b_1 = 1-\sqrt{2}$, we get

$$B_n = \frac{a^n - b^n}{a-b} = \frac{a_1^{2n} - b_1^{2n}}{(a_1-b_1)(a_1+b_1)} = \frac{1}{2}P_{2n} \text{ (see [22])}.$$

□

Corollary 3. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (1.1). Then, for* $s \in \{0,1,\ldots,p\}$,

$$v^{(i)}_{(p+1)(mn+h)-s} = \frac{-P_{(m_kn+h_k)}Q_{(m_kn+h_k)} + P_{(m_kn+h_k-1)}Q_{(m_kn+h_k-1)}v_{-s}^{(h+i) \bmod (m)}}{P_{(m_kn+h_k+1)}Q_{(m_kn+h_k+1)} - P_{(m_kn+h_k)}Q_{(m_kn+h_k)}v_{-s}^{(h+i) \bmod (m)}},$$

where $h \in \{0,1,\ldots,m-1\}$, $i \in \{1,\ldots,m\}$, $(P_n, n \geq 0)$ *is the Pell sequence and* $(Q_n, n \geq 0)$ *is the Pell-Lucas sequence.*

Proof. We see that it suffices to remark

$$B_n = \frac{a_1^n - b_1^n}{a_1 - b_1}\frac{a_1^n + b_1^n}{a_1 + b_1} = \frac{1}{2}P_nQ_n \text{ (see [22])}.$$

□

Corollary 4. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (1.1). Then, for* $s \in \{0,1,\ldots,p\}$,

$$v^{(i)}_{(p+1)(mn+h)-s} = \frac{-C_{m_kn+h_k+1} + C_{m_kn+h_k-1} + \left(C_{m_kn+h_k} - C_{m_kn+h_k-2}\right)v_{-s}^{(h+i) \bmod (m)}}{C_{m_kn+h_k+2} - C_{m_kn+h_k} - \left(C_{m_kn+h_k+1} - C_{m_kn+h_k-1}\right)v_{-s}^{(h+i) \bmod (m)}},$$

where $h \in \{0,1,\ldots,m-1\}$, $i \in \{1,\ldots,m\}$ *and* $(C_n, n \geq 0)$ *is the Lucas-Balancing sequence.*

Proof. We see that it suffices to remark $16B_n = C_{n+1} - C_{n-1}$ (see [22]). □

3. Global stability of positive solutions of (1.1)

In the following, we will study the global stability character of the solutions of system (1.1). Obviously, the unique positive equilibrium of system (1.1) is

$$E = \left(\overline{v}^{(1)}, \overline{v}^{(2)}, \ldots, \overline{v}^{(m)}\right) = -b\underline{1}'_{(m)},$$

where $\underline{1}_{(m)}$ denotes the vector of order $m \times 1$ whose entries are ones. Let the functions $f_i \colon (0,+\infty)^{m(p+1)} \to (0,+\infty)$, $i \in \{1,\ldots,m\}$ defined by

$$f_i\left(\left(\underline{x}_{0:p}^{(1)}\right)', \left(\underline{x}_{0:p}^{(2)}\right)', \ldots, \left(\underline{x}_{0:p}^{(m)}\right)'\right) = \frac{B_{k+2} - B_{k+1}x_p^{(i+1) \bmod m}}{B_{k+3} - B_{k+2}x_p^{(i+1) \bmod m}},\ i \in \{1,\ldots,m\},$$

where $\underline{x}_{0:m} = (x_0, x_1, \ldots, x_m)'$. Now, it is usually useful to linearize system (1.1) around the equilibrium point E in order to facilitate its study. For this purpose, introduce the vectors

$$\underline{X}'_n := \left(\left(\underline{X}_n^{(1)}\right)', \left(\underline{X}_n^{(2)}\right)', \ldots, \left(\underline{X}_n^{(m)}\right)' \right)$$

where $\underline{X}_n^{(i)} = \left(x_n^{(i)}, x_{n-1}^{(i)}, \ldots, x_{n-p}^{(i)}\right)$, $i \in \{1, \ldots, m\}$. With these notations, we obtain the following representation

$$\underline{X}_{n+1} = \Delta_p \underline{X}_n, \tag{3.1}$$

where $\Delta_p =$

$$\begin{pmatrix}
\underline{O}'_{(p-1)} & 0 & \underline{O}'_{(p-1)} & \frac{1}{(B_{k+3}+bB_{k+2})^2} & \cdots & O_{(p-1)} & 0 & O_{(p-1)} & 0 \\
I_{(p-1)} & \underline{O}_{(p-1)} & O_{(p-1)} & \underline{O}_{(p-1)} & \cdots & O_{(p-1)} & \underline{O}_{(p-1)} & O_{(p-1)} & \underline{O}_{(p-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
O_{(p-1)} & 0 & \underline{O}'_{(p-1)} & 0 & \cdots & \underline{O}'_{(p-1)} & 0 & \underline{O}'_{(p-1)} & \frac{1}{(B_{k+3}+bB_{k+2})^2} \\
O_{(p-1)} & \underline{O}_{(p-1)} & I_{p-1} & \underline{O}_{(p-1)} & \cdots & I_{(p-1)} & \underline{O}_{(p-1)} & O_{(p-1)} & \underline{O}_{(p-1)} \\
\underline{O}'_{(p-1)} & \frac{1}{(B_{k+3}+bB_{k+2})^2} & O_{(p-1)} & 0 & \cdots & O_{(p-1)} & 0 & \underline{O}'_{(p-1)} & 0 \\
O_{(p-1)} & \underline{O}_{(p-1)} & O_{(p-1)} & \underline{O}_{(p-1)} & \cdots & O_{(p-1)} & \underline{O}_{(p-1)} & I_{(p-1)} & \underline{O}_{(p-1)}
\end{pmatrix},$$

where $O_{(k,l)}$ denotes the matrix of order $k \times l$ whose entries are zeros, for simplicity, and we set $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$ and $I_{(m)}$ is the $m \times m$ identity matrix. We summarize the above discussion in the following theorem.

Theorem 3. *The positive equilibrium point E is locally asymptotically stable.*

Proof. After some priliminary calculations, the characteristic polynomial of Δ_p is

$$P_{\Delta_p}(\lambda) = \det\left(\Delta_p - \lambda I_{(m(p+1))}\right) = (-1)^{m(p+1)} \Omega_1(\lambda) + (-1)^p \Omega_{2,k}(\lambda),$$

where $\Omega_1(\lambda) = \lambda^{m(p+1)}$ and $\Omega_{2,k}(\lambda) = \dfrac{1}{(B_{k+3} + B_{k+2}b)^{2m}}$, then $|\Omega_{2,k}(\lambda)| < |\Omega_1(\lambda)|$, $\forall \lambda\colon\ |\lambda| = 1$. So, according to Rouche's Theorem, all zeros of $\Omega_1(\lambda) - \Omega_{2,k}(\lambda) = 0$, lie in the unit disc $|\lambda| < 1$. Thus, the positive equilibrium point E is locally asymptotically stable. □

Corollary 5. *For every well defined solution of system (1.1), we have*

$$\lim v_n^{(i)} = -b,\ i \in \{1, \ldots, m\}.$$

Proof. From Theorem 2, we have

$$\begin{aligned}
\lim v^{(i)}_{(p+1)(mn+h)-h} &= \lim \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_{-s}^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_{-s}^{(h+i) \bmod (m)}},\ h \in \{0, 1, \ldots, m-1\} \\
&= \lim \frac{-1 + \frac{B_{m_k n+h_k-1}}{B_{m_k n+h_k}} v_{-s}^{(h+i) \bmod (m)}}{\frac{B_{m_k n+h_k+1}}{B_{m_k n+h_k}} - v_{-s}^{(h+i) \bmod (m)}}
\end{aligned}$$

$$= \frac{-1+bv_{-j}^{(h+i)\bmod(m)}}{a-v_{-j}^{(h+i)\bmod(m)}}$$

for $i \in \{1,\dots,m\}$, hence

$$\begin{aligned}
\lim v_{(p+1)(mn+h)-h}^{(i)} &= \frac{-1+\left(3-2\sqrt{2}\right)v_{-j}^{(h+i)\bmod(m)}}{\left(3+2\sqrt{2}\right)-v_{-j}^{(h+i)\bmod(m)}} \\
&= \frac{\left(\left(-1+3v_{-j}^{(h+i)\bmod(m)}\right)-2\sqrt{2}v_{-j}^{(h+i)\bmod(m)}\right)\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)-2\sqrt{2}\right)}{\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)+2\sqrt{2}\right)\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)-2\sqrt{2}\right)} \\
&= \frac{-3\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)^2-8\right)-2\sqrt{2}\left(8-\left(3-v_{-j}^{(h+i)\bmod(m)}\right)^2\right)}{\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)^2-8\right)} \\
&= -3+2\sqrt{2} \\
&= -b,\ i \in \{1,\dots,m\}.
\end{aligned}$$

□

The following Corollary is an immediate consequence of Theorem 3 and Corollary 5.

Corollary 6. *The unique positive equilibrium point E is globally asymptotically stable.*

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