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SOME NEW INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR FUNCTIONS WHOSE THIRD DERIVATIVES ARE (α,s,m)-CONVEX

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Abstract. In the paper, the authors establish several new integral inequalities of the Hermite–Hadamard type for functions whose third derivatives are (α,s,m)-convex.

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1. A BRIEF REVIEW

We first recall definitions of several convex functions.

Definition 1. A function $h\colon J\subseteq\mathbb{R}=(-\infty,\infty)\to\mathbb{R}$ is said to be convex if

$$h(\vartheta\varrho+(1-\vartheta)\rho)\le\vartheta h(\varrho)+(1-\vartheta)h(\rho)$$

for all $\varrho,\rho\in J$ and $\vartheta\in[0,1]$.

In the papers [2, 10], the concept of s-convex functions was innovated as follows.

Definition 2 ([2, p. 13] and [10, p. 100])**.** Let $s\in(0,1]$ be a real number. A function $h\colon\mathbb{R}_0=[0,\infty)\to\mathbb{R}_0$ is said to be s-convex (in the second sense) if

$$h(\vartheta\varrho+(1-\vartheta)\rho)\le\vartheta^s h(\varrho)+(1-\vartheta)^s h(\rho)$$

for all $\varrho,\rho\in J$ and $\vartheta\in[0,1]$.

In the conference paper [26], the concept of m-convex functions was innovated as follows.

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Definition 3 ([26]). Let $h\colon [0,b] \to \mathbb{R}$ for $b>0$ and let $m \in (0,1]$. If the inequality

$$h(\vartheta\varrho + m(1-\vartheta)\rho) \le \vartheta h(\varrho) + m(1-\vartheta)h(\rho)$$

is valid for all $\varrho, \rho \in [0,b]$ and $\vartheta \in [0,1]$, then we call that h is an m-convex function on $[0,b]$.

Moreover, we recite the following several definitions of various convex functions.

Definition 4 ([31, Definition 2.1]). A function $h\colon J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex for some fixed $s \in [-1,1]$ if the inequality

$$h(\vartheta\varrho + (1-\vartheta)\rho) \le \vartheta^s h(\varrho) + (1-\vartheta)^s h(\rho)$$

holds for all $\varrho, \rho \in J$ and $\vartheta \in (0,1)$.

Definition 5 ([29, Definition 2.1]). For some $s \in [-1,1]$ and $\alpha \in (0,1]$, a function $h\colon J \subseteq \mathbb{R} \to \mathbb{R}$ is said to be (α,s)-convex if

$$h(\vartheta\varrho + (1-\vartheta)\rho) \le \vartheta^{\alpha s} h(\varrho) + (1-\vartheta^{\alpha})^s h(\rho)$$

for all $\varrho, \rho \in J$ and $\vartheta \in (0,1)$.

Definition 6 ([29, Definition 2.2]). For some $s \in [-1,1]$ and $(\alpha, m) \in (0,1]^2$, a function $h\colon [0,b] \to \mathbb{R}$ is said to be (α,s,m)-convex if

$$h(\vartheta\varrho + m(1-\vartheta)\rho) \le \vartheta^{\alpha s} h(\varrho) + m(1-\vartheta^{\alpha})^s h(\rho)$$

for all $\varrho, \rho \in [0,b]$ and $\vartheta \in (0,1)$.

To establish integral inequalities of the Hermite–Hadamard type is a very active research topic. We would like to recite the following integral inequalities of the Hermite–Hadamard type.

Theorem 1 ([4, Theorem 2.2]). *Let $h\colon J^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on the interior J° of J and let $\varrho, \rho \in J^\circ$ with $\varrho < \rho$. If $|h'|$ is convex on $[\varrho,\rho]$, then*

$$\left|\frac{h(\varrho)+h(\rho)}{2} - \frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x\right| \le \frac{(\rho-\varrho)(|h'(\varrho)|+|h'(\rho)|)}{8}.$$

Theorem 2 ([21, Theorems 1 and 2]). *Let $h\colon J \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on J° and let $\varrho, \rho \in J$ such that $\varrho < \rho$. If $|h'|^q$ is convex on $[\varrho,\rho]$ and $q \ge 1$, then*

$$\left|\frac{h(\varrho)+h(\rho)}{2} - \frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x\right| \le \frac{\rho-\varrho}{4}\left(\frac{|h'(\varrho)|^q+|h'(\rho)|^q}{2}\right)^{1/q}$$

and

$$\left|h\left(\frac{\varrho+\rho}{2}\right) - \frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x\right| \le \frac{\rho-\varrho}{4}\left(\frac{|h'(\varrho)|^q+|h'(\rho)|^q}{2}\right)^{1/q}.$$

Theorem 3 ([5, Remark 2])**.** *Let $h\colon \mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be m-convex and $m\in(0,1]$. If $h\in L_1([\varrho,\rho])$ for $0\le\varrho<\rho<\infty$, then*

$$\frac{1}{\rho-\varrho}\int_{\varrho}^{\rho} h(x)\,\mathrm{d}x \le \min\left\{\frac{h(\varrho)+mh(\rho/m)}{2}, \frac{mh(\varrho/m)+h(\rho)}{2}\right\}.$$

Theorem 4 ([13, Theorem 1])**.** *Let $h\colon J\subseteq\mathbb{R}_0\to\mathbb{R}$ be differentiable on J° and let $\varrho,\rho\in J$ with $\varrho<\rho$. If $|h'|^q$ is s-convex on $[\varrho,\rho]$ for some fixed $s\in(0,1]$ and $q\ge1$, then*

$$\begin{aligned}\left|\frac{h(\varrho)+h(\rho)}{2}-\frac{1}{\rho-\varrho}\int_{\varrho}^{\rho}h(x)\,\mathrm{d}x\right| \\ \le \frac{\rho-\varrho}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\left[|h'(\varrho)|^q+|h'(\rho)|^q\right]^{1/q}.\end{aligned}$$

Theorem 5 ([11, Theorem 4])**.** *Let $h\colon J\subseteq\mathbb{R}_0\to\mathbb{R}$ be differentiable on J°, let $\varrho,\rho\in J$ such that $\varrho<\rho$, and let $h'\in L[\varrho,\rho]$. If $|h'|^q$ is s-convex on $[\varrho,\rho]$ for some fixed $s\in(0,1]$ and $q>1$, then*

$$\begin{aligned}&\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_{\varrho}^{\rho}h(x)\,\mathrm{d}x\right| \\ &\quad\le\frac{\rho-\varrho}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}\right)^{1/p}\left\{\left[|h'(\varrho)|^q+(s+1)\left|h'\left(\frac{\varrho+\rho}{2}\right)\right|^q\right]^{1/q}\right. \\ &\qquad\left.+\left[|h'(\rho)|^q+(s+1)\left|h'\left(\frac{\varrho+\rho}{2}\right)\right|^q\right]^{1/q}\right\}, \qquad (1.1)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 1. The inequality (1.1) is a corrected version of [11, Theorem 4].
By the way, the identity in [11, Lemma 3] should also be corrected as

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \\ &\quad=\frac{b-a}{4}\int_0^1\left[(1-t)f'\left(ta+(1-t)\frac{a+b}{2}\right)-tf'\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}t.\end{aligned}$$

Consequently, the whole body of the paper [11] should be rewritten and modified.

Theorem 6 ([25, Theorem 7])**.** *Let $h\colon J\subseteq\mathbb{R}_0\to\mathbb{R}$ be differentiable on J°, $\varrho,\rho\in J$ with $\varrho<\rho$, and $h'\in L[\varrho,\rho]$. If $|h'|$ is s-convex on $[\varrho,\rho]$ for some $s\in(0,1]$ and $p>1$, then*

$$\left|\frac{1}{6}\left[h(\varrho)+4h\left(\frac{\varrho+\rho}{2}\right)+h(\rho)\right]-\frac{1}{\rho-\varrho}\int_{\varrho}^{\rho}h(x)\,\mathrm{d}x\right|$$

$$\le \frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(\rho-\varrho)\big(|h'(\varrho)|+|h'(\rho)|\big),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

For more conclusions on this topic, please refer to the papers [1, 6, 8, 9, 12, 14, 15, 17–20, 22–24, 27, 28, 30, 32–34] and closely related references therein.

The main purpose of this paper is to establish some new integral inequalities of the Hermite–Hadamard type for (α,s,m)-convex functions.

2. A LEMMA

For attaining the aim of this paper, we need the following lemma.

Lemma 1 ([3, Lemma 2.1])**.** *Let* $h\colon J\subseteq\mathbb{R}\to\mathbb{R}$ *be a three-time differentiable mapping on* J° *and let* $\varrho,\rho\in J^\circ$ *such that* $\varrho<\rho$. *If* $h'''\in L_1([\varrho,\rho])$, *then*

$$h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}\left[h'(\rho)-h'(\varrho)\right]$$
$$=\frac{(\rho-\varrho)^3}{96}\left[\int_0^1\vartheta(1-\vartheta)(2-\vartheta)h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\mathrm{d}\vartheta\right.$$
$$\left.-\int_0^1\vartheta(1-\vartheta)(2-\vartheta)h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\mathrm{d}\vartheta\right].$$

3. INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE

Now we start out to establish some integral inequalities of the Hermite–Hadamard type for (α,s,m)-convex functions.

Theorem 7. *For* $(\alpha,m)\in(0,1]^2$, $s\in(-1,1]$, *and* $\rho>\varrho>0$, *let* $f\colon\big(0,\frac{\rho}{m}\big]\to\mathbb{R}$ *be a three-time differentiable function and* $f'''\in L_1\big(\big[\varrho,\frac{\rho}{m}\big]\big)$. *If* $|f'''|^q$ *is an* (α,s,m)-*convex function on* $\big(0,\frac{\rho}{m}\big]$ *for* $q\ge1$, *then*

$$\left|f\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[f'(\rho)-f'(\varrho)]\right|$$
$$\le\frac{(\rho-\varrho)^3}{96}\left(\frac{1}{4}\right)^{1-1/q}\left\{\left[\frac{2^{1-\alpha s}}{(\alpha s+2)(\alpha s+4)}|f'''(\varrho)|^q+mQ(\alpha,s)\left|f'''\left(\frac{\rho}{m}\right)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[W(\alpha,s)|f'''(\varrho)|^q+mR(\alpha,s)\left|f'''\left(\frac{\rho}{m}\right)\right|^q\right]^{1/q}\right\},$$

where $\Gamma(x)$, $B(x,y)$, *and* ${}_2F_1(c,d;e;z)$ *are the Euler gamma function, the Euler beta function, and the Gauss hypergeometric functions, which can be defined respectively* *by*

$$\Gamma(x)=\int_0^1 z^{x-1}\mathrm{e}^{-z}\,\mathrm{d}z,\quad B(x,y)=\int_0^1 z^{x-1}(1-z)^{y-1}\,\mathrm{d}z,$$

and

$$_2F_1(c,d;e;z)=\frac{\Gamma(e)}{\Gamma(d)\Gamma(e-d)}\int_0^1 t^{d-1}(1-t)^{e-d-1}(1-zt)^{-c}\,\mathrm{d}t$$

for $x,y>0$, $e>d>0$, $|z|<1$, *and* $c\in\mathbb{R}$, *and*

$$Q(\alpha,s)=\frac{2^{1-\alpha s}(\alpha s+3)(\alpha s+9)+4[(\alpha s)^2+7\alpha s-12]}{(\alpha s+1)(\alpha s+2)(\alpha s+3)(\alpha s+4)},$$

$$W(\alpha,s)=\frac{\alpha}{8}{}_2F_1\left(\frac{2}{\alpha},-s;\frac{\alpha+2}{\alpha};\frac{1}{2^\alpha}\right)-\frac{\alpha}{8}{}_2F_1\left(\frac{4}{\alpha},-s;\frac{\alpha+4}{\alpha};\frac{1}{2^\alpha}\right),$$

$$R(\alpha,s)=\frac{4}{\alpha}\Bigg[B\left(\frac{1}{\alpha},s+1\right)-B\left(\frac{2}{\alpha},s+1\right)+\frac{\alpha}{8}{}_2F_1\left(\frac{2}{\alpha},-s;\frac{\alpha+2}{\alpha};\frac{1}{2^\alpha}\right)-\frac{\alpha}{8}{}_2F_1\left(\frac{4}{\alpha},-s;\frac{\alpha+4}{\alpha};\frac{1}{2^\alpha}\right)\Bigg].$$

Proof. Using Lemma 1 and Hölder's integral inequality [7], we obtain

$$\begin{aligned}
&\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[h'(\rho)-h'(\varrho)]\right|\\
&\le\frac{(\rho-\varrho)^3}{96}\left\{\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\right.\\
&\qquad\left.+\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\right\}\\
&\le\frac{(\rho-\varrho)^3}{96}\left[\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\,\mathrm{d}\vartheta\right]^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\right]^{1/q}\right\}.
\end{aligned}\tag{3.1}$$

By the (α,s,m)-convexity of $|h'''|^q$, we have

$$\begin{aligned}
&\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\\
&\le\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left(\frac{1-\vartheta}{2}\right)^{\alpha s}|h'''(\varrho)|^q\,\mathrm{d}\vartheta\\
&\quad+m\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left[1-\left(\frac{1-\vartheta}{2}\right)^\alpha\right]^s\left|h'''\left(\frac{\rho}{m}\right)\right|^q\mathrm{d}\vartheta
\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}\int_0^1 &\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\\ &\le \int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left(\frac{1+\vartheta}{2}\right)^{\alpha s}|h'''(\varrho)|^q\,\mathrm{d}\vartheta \\ &\quad + m\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left[1-\left(\frac{1+\vartheta}{2}\right)^{\alpha}\right]^s\left|h'''\left(\frac{\rho}{m}\right)\right|^q\,\mathrm{d}\vartheta.\end{aligned} \tag{3.3}$$

Straightforward computation gives

$$\begin{aligned}\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left(\frac{1-\vartheta}{2}\right)^{\alpha s}\mathrm{d}\vartheta &= \frac{2^{1-\alpha s}}{(\alpha s+2)(\alpha s+4)},\\ \int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left(\frac{1+\vartheta}{2}\right)^{\alpha s}\mathrm{d}\vartheta &= Q(\alpha,s),\\ \int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left[1-\left(\frac{1-\vartheta}{2}\right)^{\alpha}\right]^s\mathrm{d}\vartheta &= W(\alpha,s),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left[1-\left(\frac{1+\vartheta}{2}\right)^{\alpha}\right]^s\mathrm{d}\vartheta &\le \int_0^1 (1-\vartheta)\left[1-\left(\frac{1+\vartheta}{2}\right)^{\alpha}\right]^s\mathrm{d}\vartheta\\ &= R(\alpha,s),\end{aligned} \tag{3.5}$$

where

$$\vartheta(1-\vartheta)(2-\vartheta)\le(1-\vartheta)\left[\frac{\vartheta+(2-\vartheta)}{2}\right]^2=1-\vartheta$$

for $\vartheta\in[0,1]$. Substituting those in (3.2), (3.3), (3.4), (3.5) into (3.1) gives

$$\begin{aligned}&\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[h'(\rho)-h'(\varrho)]\right|\\ &\quad\le\frac{(\rho-\varrho)^3}{96}\left(\frac14\right)^{1-1/q}\left\{\left[\frac{2^{1-\alpha s}}{(\alpha s+2)(\alpha s+4)}|h'''(\varrho)|^q+m|h'''(\rho)|^qQ(\alpha,s)\right]^{1/q}\right.\\ &\qquad\left.+\left[|h'''(\varrho)|^qW(\alpha,s)+m|h'''(\rho)|^qR(\alpha,s)\right]^{1/q}\right\}.\end{aligned}$$

The proof of Theorem 7 is thus proved. ◻

Theorem 8. *For $(\alpha,m)\in(0,1]^2$, $s\in(-1,1]$, and $\rho>\varrho>0$, let $h\colon\left(0,\frac{\rho}{m}\right]\to\mathbb{R}$ be a three-time differentiable function and $h'''\in L_1\left(\left[\varrho,\frac{\rho}{m}\right]\right)$. If $|h'''|^q$ is an (α,s,m)-convex function on $\left(0,\frac{\rho}{m}\right]$ for $q>1$, then*

$$\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[h'(\rho)-h'(\varrho)]\right|$$

$$
\begin{aligned}
&\le \frac{(\rho-\varrho)^3}{96}\left[\frac{3}{4}B\left(\frac{2q-1}{2(q-1)},\frac{3q-2}{q-1}\right)\right]^{1-1/q}\\
&\quad\times\left\{\left[\frac{2^{-\alpha s}}{\alpha s+1}|h'''(\varrho)|^q+mB\left(\frac{1}{2^\alpha};\frac{1}{\alpha},s+1\right)\left|h'''\left(\frac{\rho}{m}\right)\right|^q\right]^{1/q}\right.\\
&\qquad+\left[\frac{2-2^{-\alpha s}}{\alpha s+1}|h'''(\varrho)|^q+m\left(B\left(\frac{1}{\alpha},s+1\right)\right.\right.\\
&\qquad\left.\left.\left.-B\left(\frac{1}{2^\alpha};\frac{1}{\alpha},s+1\right)\right)\left|h'''\left(\frac{\rho}{m}\right)\right|^q\right]^{1/q}\right\},
\end{aligned}
\tag{3.6}
$$

where

$$
B(z;\varrho,\rho)=\int_0^z u^{\varrho-1}(1-u)^{\rho-1}\,\mathrm{d}u,\quad 0\le z\le 1
$$

is the incomplete beta function with $B(\varrho;\varrho,\rho)=B(\varrho,\rho)$.

Proof. By Lemma 1 and Hölder's integral inequality [16], we have

$$
\begin{aligned}
&\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[h'(\rho)-h'(\varrho)]\right|\\
&\le\frac{(\rho-\varrho)^3}{96}\left\{\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\right.\\
&\qquad\left.+\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\right\}\\
&\le\frac{(\rho-\varrho)^3}{96}\left(\int_0^1[\vartheta(1-\vartheta)(2-\vartheta)]^{q/(q-1)}\,\mathrm{d}\vartheta\right)^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\right]^{1/q}\right.\\
&\qquad\left.+\left[\int_0^1\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\right]^{1/q}\right\}.
\end{aligned}
\tag{3.7}
$$

Utilizing the (α,s,m)-convexity of $|h'''|^q$, we acquire

$$
\begin{aligned}
&\int_0^1\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\\
&\le\int_0^1\left(\frac{1-\vartheta}{2}\right)^{\alpha s}|h'''(\varrho)|^q\,\mathrm{d}\vartheta+m\int_0^1\left[1-\left(\frac{1-\vartheta}{2}\right)^\alpha\right]^s\left|h'''\left(\frac{\rho}{m}\right)\right|^q\mathrm{d}\vartheta\\
&=\frac{2^{-\alpha s}}{\alpha s+1}|h'''(\varrho)|^q+mB\left(\frac{1}{2^\alpha};\frac{1}{\alpha},s+1\right)\left|h'''\left(\frac{\rho}{m}\right)\right|^q
\end{aligned}
\tag{3.8}
$$

and

$$\begin{aligned}&\int_0^1\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\\&\le\int_0^1\left(\frac{1+\vartheta}{2}\right)^{\alpha s}|h'''(\varrho)|^q\,\mathrm{d}\vartheta+m\int_0^1\left[1-\left(\frac{1+\vartheta}{2}\right)^{\alpha}\right]^s\left|h'''\left(\frac{\rho}{m}\right)\right|^q\mathrm{d}\vartheta\\&=\frac{2-2^{-\alpha s}}{\alpha s+1}|h'''(\varrho)|^q+m\left(B\left(\frac{1}{\alpha},s+1\right)-B\left(\frac{1}{2^\alpha};\frac{1}{\alpha},s+1\right)\right)\left|h'''\left(\frac{\rho}{m}\right)\right|^q.\end{aligned}\tag{3.9}$$

It is not difficult to see that

$$\int_0^1[\vartheta(1-\vartheta)(2-\vartheta)]^{q/(q-1)}\,\mathrm{d}\vartheta=\frac{3}{4}B\left(\frac{2q-1}{2(q-1)},\frac{3q-2}{q-1}\right).\tag{3.10}$$

Substituting (3.8), (3.9), and (3.10) into the inequality (3.7) leads to the inequality (3.6). The proof of Theorem 8 is thus proved. □

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