Miskolc Mathematical Notes
Vol. 26 (2025), No. 1, pp. 195–207
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4752

PREDICTOR/ESTIMATOR COMPUTATIONS UNDER A CONSTRAINED MULTIVARIATE LINEAR MODEL AND SOME RELATED REDUCED MODELS

MELEK ERIŞ BÜYÜKKAYA AND NESRIN GÜLER

Received 31 May, 2023

Abstract. This study deals with basic prediction/estimation issues involving a constrained multivariate linear model (CMLM) and some related reduced models. By reparameterizing these models, the authors create unconstrained multivariate linear models (UMLMs). After that, the authors use some quadratic matrix optimization methods to derive analytical formulas for computing the best linear unbiased predictors/estimators (BLUPs/BLUEs) of all unknown parameter matrices. This provides a broad perspective on BLUPs.

2010 *Mathematics Subject Classification:* 15A10; 62J05; 62H12

Keywords: BLUE, BLUP, constrained multivariate linear model, reduced models

1. INTRODUCTION

As follows, we introduce the notations that will be used throughout this study. We will write $\Gamma \in \mathbb{R}_{k,t}$ if Γ is a $k \times t$ real matrix, $\Gamma \in \mathbb{R}_k^s$ if $\Gamma \in \mathbb{R}_{k,k}$ and is symmetric, and $\Gamma \in \mathbb{R}_k^{\succcurlyeq}$ if $\Gamma \in \mathbb{R}_k^s$ and is positive semi-definite. We will use Γ', $\mathscr{C}(\Gamma)$, and $r(\Gamma)$ as symbols to present the transpose, the column space, and the rank of $\Gamma \in \mathbb{R}_{k,t}$, respectively. $\Gamma^{\perp} = \mathbf{I}_k - \Gamma\Gamma^{+}$ and $\mathbf{F}_{\Gamma} = \mathbf{I}_t - \Gamma^{+}\Gamma$ stand for orthogonal projectors, where $\mathbf{I}_k \in \mathbb{R}_k^s$ and $\mathbf{I}_t \in \mathbb{R}_t^s$ are the identity matrices and Γ^{+} is the Moore–Penrose generalized inverse of $\Gamma \in \mathbb{R}_{k,t}$. The symbol $i_{+}(\Gamma)$ represents positive inertia of $\Gamma \in \mathbb{R}_k^s$. The vectorization operation (vec operation) of a matrix $\Gamma = [\gamma_1, \ldots, \gamma_t]$ is defined to be $\overrightarrow{\Gamma} = [\gamma_1', \ldots, \gamma_t']'$, where $\gamma_1, \ldots, \gamma_t$ are the columns of $\Gamma \in \mathbb{R}_{k,t}$. A well-known property on the vec operation of a triple matrix product is $\overrightarrow{\Gamma_1 \Lambda \Gamma_2} = (\Gamma_2' \otimes \Gamma_1)\overrightarrow{\Lambda}$ for matrices Λ, Γ_1, and Γ_2.

One of the most frequently used essential statistical tools, both in theory and in practice, is linear models (LMs). The multivariate LM (MLM) is a generalization of the univariate LM, which normally means that a response variable on a given set of regressors is regressed to several response variables on regressors. There may be certain restrictions on unknown parameters that are often added to model assumptions in statistical theory and its applications. Such restrictions arise in some cases, such as linear hypothesis testing on parameters. In such cases, MLM becomes a constrained MLM (CMLM).

In the present study, we consider a CMLM, formulated by

$$\mathcal{M}: \mathbf{Y} = \mathbf{X}\Theta + \Psi = \mathbf{X}_1\Theta_1 + \mathbf{X}_2\Theta_2 + \Psi,\ \mathbf{C}\Theta = \mathbf{C}_1\Theta_1 + \mathbf{C}_2\Theta_2 = \mathbf{D}, \tag{1.1}$$

where $\mathbf{Y} \in \mathbb{R}_{n,m}$ is a matrix of observable dependent variables, $\mathbf{X} = \begin{bmatrix}\mathbf{X}_1, & \mathbf{X}_2\end{bmatrix} \in \mathbb{R}_{n,p}$ with $\mathbf{X}_i \in \mathbb{R}_{n,p_i}$, $\mathbf{C} = \begin{bmatrix}\mathbf{C}_1, & \mathbf{C}_2\end{bmatrix} \in \mathbb{R}_{s,p}$ with $\mathbf{C}_i \in \mathbb{R}_{s,p_i}$, and $\mathbf{D} \in \mathbb{R}_{s,m}$ are known matrices of arbitrary ranks, $\Theta = \begin{bmatrix}\Theta_1', & \Theta_2'\end{bmatrix}' \in \mathbb{R}_{p,m}$ with $\Theta_i \in \mathbb{R}_{p_i,m}$ is a matrix of fixed but unknown parameters, $i = 1,2$, $p_1 + p_2 = p$, $\Psi \in \mathbb{R}_{n,m}$ is a matrix of randomly distributed error terms with mean matrix $\mathrm{E}(\Psi) = \mathbf{0}$ and dispersion matrix $\mathrm{D}(\overrightarrow{\Psi}) = \sigma^2(\Sigma_2 \otimes \Sigma_1)$, where $\Sigma_1 = (\sigma_{1ij}) \in \mathbb{R}_n^{\succcurlyeq}$, $\Sigma_2 = (\sigma_{2ij}) \in \mathbb{R}_m^{\succcurlyeq}$, and σ^2 is an unknown positive number. Further, $\Sigma_2 \otimes \Sigma_1$ means that $\overrightarrow{\Psi}$ has a Kronecker product structured covariance matrix.

A CMLM in partitioned form is usually considered for statistical inferences of partial parameters in regression analysis. Model transformations are a common occurrence, and we utilize them to extrapolate and infer the model's parameter space. One of the simplest transformations is a linear transformation, which is achieved by premultiplying both sides of (1.1) by two matrices $\mathbf{X}_1^{\perp}$ and $\mathbf{X}_2^{\perp}$ to yield the following pair of transformed CMLMs:

$$\begin{gathered}\mathcal{M}_1: \mathbf{X}_2^{\perp}\mathbf{Y} = \mathbf{X}_2^{\perp}\mathbf{X}_1\Theta_1 + \mathbf{X}_2^{\perp}\Psi,\ \mathbf{C}_1\Theta_1 = \mathbf{D},\\ \mathrm{E}(\mathbf{X}_2^{\perp}\Psi) = \mathbf{0},\ \mathrm{D}(\mathbf{X}_2^{\perp}\Psi) = \sigma^2(\Sigma_2 \otimes \mathbf{X}_2^{\perp}\Sigma_1\mathbf{X}_2^{\perp})\end{gathered} \tag{1.2}$$

and

$$\begin{gathered}\mathcal{M}_2: \mathbf{X}_1^{\perp}\mathbf{Y} = \mathbf{X}_1^{\perp}\mathbf{X}_2\Theta_2 + \mathbf{X}_1^{\perp}\Psi, \mathbf{C}_2\Theta_2 = \mathbf{D},\\ \mathrm{E}(\mathbf{X}_1^{\perp}\Psi) = \mathbf{0},\ \mathrm{D}(\mathbf{X}_1^{\perp}\Psi) = \sigma^2(\Sigma_2 \otimes \mathbf{X}_1^{\perp}\Sigma_1\mathbf{X}_1^{\perp}),\end{gathered} \tag{1.3}$$

which are frequently referred to as correctly-reduced versions of $\mathcal{M}$ in (1.1); for the related expositions, see, [14]. Each of the model equations in $\mathcal{M}_1$ and $\mathcal{M}_2$ is not equal to the whole model equation in $\mathcal{M}$ due to the two linear transformations $\mathbf{X}_1^{\perp}\mathbf{Y}$ and $\mathbf{X}_2^{\perp}\mathbf{Y}$ are unique. It is advantageous to formulate $\mathcal{M}_1$ and $\mathcal{M}_2$ since they do not contain the partial parameter matrices Θ_1 and Θ_2 found in $\mathcal{M}$. As a result, using $\mathcal{M}_1$ and $\mathcal{M}_2$, we may estimate Θ_1 and Θ_2 in $\mathcal{M}$ separately.

In many cases, we are forced to use these derived models in the hopes that the inference outcomes obtained from $\mathcal{M}_1$ and $\mathcal{M}_2$ are equal to those that would correspond to $\mathcal{M}$. The linear sufficiency problem, initially stated by [2] and taken into consideration in the statistical literature, is the search for relationships between estimators under original models and their reduced models.

We can take the following matrix into consideration in order to develop some results on predictors of all unknown matrices under $\mathscr{M}$.

$$\begin{aligned} \mathbf{\Phi} &= \mathbf{H}\Theta + \mathbf{J}\Psi = \mathbf{H}_1\Theta_1 + \mathbf{H}_2\Theta_2 + \mathbf{J}\Psi, \text{ or,} \\ \overrightarrow{\Phi} &= (\mathbf{I}_m \otimes \mathbf{H})\overrightarrow{\Theta} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\Psi} \\ &= (\mathbf{I}_m \otimes \mathbf{H}_1)\overrightarrow{\Theta_1} + (\mathbf{I}_m \otimes \mathbf{H}_2)\overrightarrow{\Theta_2} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\Psi}, \end{aligned} \tag{1.4}$$

in accordance with the partition considered in (1.1) for given matrices $\mathbf{H} = \begin{bmatrix}\mathbf{H}_1, & \mathbf{H}_2\end{bmatrix} \in \mathbb{R}_{k,p}$ with $\mathbf{H}_i \in \mathbb{R}_{k,p_i}$, $\Theta = \begin{bmatrix}\Theta_1, & \Theta_2\end{bmatrix} \in \mathbb{R}_{p,m}$ with $\Theta_i \in \mathbb{R}_{p_i,m}$, $i = 1,2$, $p_1 + p_2 = p$, and $\mathbf{J} \in \mathbb{R}_{k,n}$.

For statistical inference under the CMLM and its two reduced models, in addition to (1.4), we are also able to create the next three matrices linked to the constrained reduced models $\mathscr{M}_i$, $i = 1,2$, respectively.

$$\begin{aligned} \Phi_1 &= \mathbf{H}_1\Theta_1 + \mathbf{J}\Psi = \begin{bmatrix}\mathbf{H}_1, & \mathbf{0}\end{bmatrix}\Theta + \mathbf{J}\Psi, \text{ or,} \\ \overrightarrow{\Phi}_1 &= (\mathbf{I}_m \otimes \mathbf{H}_1)\overrightarrow{\Theta_1} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\Psi}, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} \Phi_2 &= \mathbf{H}_2\Theta_2 + \mathbf{J}\Psi = \begin{bmatrix}\mathbf{0}, & \mathbf{H}_2\end{bmatrix}\Theta + \mathbf{J}\Psi, \text{ or,} \\ \overrightarrow{\Phi}_2 &= (\mathbf{I}_m \otimes \mathbf{H}_2)\overrightarrow{\Theta_2} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\Psi}. \end{aligned} \tag{1.6}$$

Then, we obtain

$$\begin{aligned} \mathrm{E}(\Phi) = \mathbf{H}\Theta,\ \mathrm{D}(\overrightarrow{\Phi}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})', \\ \mathrm{cov}(\overrightarrow{\Phi}, \overrightarrow{\mathbf{Y}}) = \mathrm{cov}(\overrightarrow{\Phi_1}, \overrightarrow{\mathbf{Y}}) = \mathrm{cov}(\overrightarrow{\Phi_2}, \overrightarrow{\mathbf{Y}}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1), \end{aligned} \tag{1.7}$$

$$\begin{aligned} \mathrm{E}(\Phi_1) = \mathbf{H}_1\Theta_1,\ \mathrm{D}(\overrightarrow{\Phi_1}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})', \\ \mathrm{cov}(\overrightarrow{\Phi_1}, \overrightarrow{\mathbf{X}_1^{\perp}\mathbf{Y}}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_1^{\perp}\Sigma_1\mathbf{X}_1^{\perp}), \\ \mathrm{cov}(\overrightarrow{\Phi_1}, \overrightarrow{\mathbf{X}_2^{\perp}\mathbf{Y}}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_2^{\perp}\Sigma_1\mathbf{X}_2^{\perp}), \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \mathrm{E}(\Phi_2) = \mathbf{H}_2\Theta_2,\ \mathrm{D}(\overrightarrow{\Phi_2}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})', \\ \mathrm{cov}(\overrightarrow{\Phi_2}, \overrightarrow{\mathbf{X}_1^{\perp}\mathbf{Y}}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_1^{\perp}\Sigma_1\mathbf{X}_1^{\perp}), \\ \mathrm{cov}(\overrightarrow{\Phi_2}, \overrightarrow{\mathbf{X}_2^{\perp}\mathbf{Y}}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_2^{\perp}\Sigma_1\mathbf{X}_2^{\perp}). \end{aligned} \tag{1.9}$$

Note that the linear restriction equations $\mathbf{C}\Theta = \mathbf{D}$ and $\mathbf{C}_i\Theta_i = \mathbf{D}$ in $\mathscr{M}$ and $\mathscr{M}_i$, $i = 1,2$ are consistent, respectively. The general solutions of these matrix equations can be written as $\Theta = \mathbf{C}^{+}\mathbf{D} + \mathbf{F}_{\mathbf{C}}\Omega$ and $\Theta_i = \mathbf{C}_i^{+}\mathbf{D} + \mathbf{F}_{\mathbf{C}_i}\Omega_i$, respectively, where $\Omega \in \mathbb{R}_{p,m}$ and $\Omega_i \in \mathbb{R}_{p_i,m}$ are reparameterized but arbitrary matrices. Substituting these solutions into the model equations in $\mathcal{M}$ and $\mathcal{M}_i$, $i=1,2$, yields the following reparameterized MLMs:

$$\mathcal{R}: \ \Upsilon = \mathbf{X}\mathbf{F}_{\mathbf{C}}\Omega + \Psi, \tag{1.10}$$

$$\mathcal{R}_1: \ \mathbf{X}_2^{\perp}\Upsilon_1 = \mathbf{X}_2^{\perp}\mathbf{X}_1\mathbf{F}_{\mathbf{C}_1}\Omega_1 + \mathbf{X}_2^{\perp}\Psi, \tag{1.11}$$

$$\mathcal{R}_2: \ \mathbf{X}_1^{\perp}\Upsilon_2 = \mathbf{X}_1^{\perp}\mathbf{X}_2\mathbf{F}_{\mathbf{C}_2}\Omega_2 + \mathbf{X}_1^{\perp}\Psi, \tag{1.12}$$

where $\Upsilon = \mathbf{Y} - \mathbf{X}\mathbf{C}^{+}\mathbf{D}$ and $\Upsilon_i = \mathbf{Y} - \mathbf{X}_i\mathbf{C}_i{}^{+}\mathbf{D}$, $i=1,2$, thus, predictions under $\mathcal{M}$ and $\mathcal{M}_i$ can be derived from $\mathcal{R}$ and $\mathcal{R}_i$, $i=1,2$, respectively. Correspondingly, Φ in (1.4) and Φ_i in (1.5)-(1.6) become the following reparameterized matrices:

$$\begin{aligned}
\Delta &= \mathbf{H}\mathbf{F}_{\mathbf{C}}\Omega + \mathbf{J}\Psi = \mathbf{H}_1\mathbf{F}_{\mathbf{C}_1}\Omega_1 + \mathbf{H}_2\mathbf{F}_{\mathbf{C}_2}\Omega_2 + \mathbf{J}\Psi, \text{ or,}\\
\overrightarrow{\Delta} &= (\mathbf{I}_m \otimes \mathbf{H}\mathbf{F}_{\mathbf{C}})\overrightarrow{\Omega} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\Psi}, \qquad (1.13)\\
&= (\mathbf{I}_m \otimes \mathbf{H}_1\mathbf{F}_{\mathbf{C}_1})\overrightarrow{\Omega_1} + (\mathbf{I}_m \otimes \mathbf{H}_2\mathbf{F}_{\mathbf{C}_2})\overrightarrow{\Omega_2} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\Psi},\\
\Delta_1 &= \mathbf{H}_1\mathbf{F}_{\mathbf{C}_1}\Omega_1 + \mathbf{J}\Psi = \begin{bmatrix}\mathbf{H}_1\mathbf{F}_{\mathbf{C}_1}, & \mathbf{0}\end{bmatrix}\Omega + \mathbf{J}\Psi, \text{ or,}\\
\overrightarrow{\Delta_1} &= (\mathbf{I}_m \otimes \mathbf{H}_1\mathbf{F}_{\mathbf{C}_1})\overrightarrow{\Omega_1} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\Psi}, \qquad (1.14)
\end{aligned}$$

and

$$\left.\begin{aligned}
\Delta_2 &= \mathbf{H}_2\mathbf{F}_{\mathbf{C}_2}\Omega_2 + \mathbf{J}\Psi = \begin{bmatrix}\mathbf{0}, & \mathbf{H}_2\mathbf{F}_{\mathbf{C}_2}\end{bmatrix}\Omega + \mathbf{J}\Psi, \text{ or,}\\
\overrightarrow{\Delta_2} &= (\mathbf{I}_m \otimes \mathbf{H}_2\mathbf{F}_{\mathbf{C}_2})\overrightarrow{\Omega_2} + (\mathbf{I}_m \otimes \mathbf{J})\overrightarrow{\Psi},
\end{aligned}\right. \tag{1.15}$$

where $\Delta = \Phi - \mathbf{H}\mathbf{C}^{+}\mathbf{D}$ and $\Delta_i = \Phi_i - \mathbf{H}_i\mathbf{C}_i{}^{+}\mathbf{D}$, $i=1,2$. Then, we obtain

$$\begin{aligned}
&\mathrm{E}(\Delta) = \mathbf{H}\mathbf{F}_{\mathbf{C}}\Omega, \ \mathrm{D}(\overrightarrow{\Delta}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})',\\
&\mathrm{cov}(\overrightarrow{\Delta}, \overrightarrow{\Upsilon}) = \mathrm{cov}(\overrightarrow{\Delta_1}, \overrightarrow{\Upsilon}) = \mathrm{cov}(\overrightarrow{\Delta_2}, \overrightarrow{\Upsilon}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1). \qquad (1.16)
\end{aligned}$$

$$\begin{aligned}
&\mathrm{E}(\Delta_1) = \mathbf{H}_1\mathbf{F}_{\mathbf{C}_1}\Omega_1, \ \mathrm{D}(\overrightarrow{\Delta_1}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})',\\
&\mathrm{cov}(\overrightarrow{\Delta_1}, \overrightarrow{\mathbf{X}_2^{\perp}\Upsilon_1}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_2^{\perp}\Sigma_1\mathbf{X}_2^{\perp}), \qquad (1.17)\\
&\mathrm{cov}(\overrightarrow{\Delta_1}, \overrightarrow{\mathbf{X}_1^{\perp}\Upsilon_2}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_1^{\perp}\Sigma_1\mathbf{X}_1^{\perp}).
\end{aligned}$$

$$\begin{aligned}
&\mathrm{E}(\Delta_2) = \mathbf{H}_2\mathbf{F}_{\mathbf{C}_2}\Omega_2, \ \mathrm{D}(\overrightarrow{\Delta_2}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})',\\
&\mathrm{cov}(\overrightarrow{\Delta_2}, \overrightarrow{\mathbf{X}_2^{\perp}\Upsilon_1}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_2^{\perp}\Sigma_1\mathbf{X}_2^{\perp}), \qquad (1.18)\\
&\mathrm{cov}(\overrightarrow{\Delta_2}, \overrightarrow{\mathbf{X}_1^{\perp}\Upsilon_2}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_1^{\perp}\Sigma_1\mathbf{X}_1^{\perp}).
\end{aligned}$$

In the present study, we consider the CMLM in (1.1) and some related reduced models in (1.2) and (1.3) with general assumptions. After reparameterizing these models given as $\mathcal{R}$ and $\mathcal{R}_i$, $i=1,2$ in (1.10)-(1.12), we derive analytic formulas for computing the best linear unbiased predictor/estimator (BLUP/BLUE).

In the statistical literature, LMs with exact restrictions on unknown parameters have been studied under various model assumptions, see e.g., [3, 5, 8–10, 12, 13, 18, 20, 23, 25]. We may refer to the studies [6, 7, 11, 19, 24], among others, in which both an CMLM and unconstrained MLM (UMLM) on unknown parameters has been considered from different perspectives.

2. Preliminary

For the models that are taken into consideration in the work, consistency is given the first definition; see [17]. In the second definition, predictability is given; see [1]. The last definition is related to the BLUP and the BLUE.

Definition 1. The model $\mathcal{R}$ is consistent $\Longleftrightarrow \Upsilon \in \mathscr{C}\left[\mathbf{XF_C}, \quad \Sigma_2 \otimes \Sigma_1\right]$ holds with probability (wp) 1.

For reduced models $\mathcal{R}_i$, $i = 1, 2$, it is possible to make the following consistency assumptions in accordance with Definition 1.

(1) $\mathcal{R}_1$ is consistent $\Longleftrightarrow \mathbf{X}_2^{\perp}\Upsilon_1 \in \mathscr{C}\left[\mathbf{X}_2^{\perp}\mathbf{X}_1\mathbf{F}_{\mathbf{C}_1}, \quad \Sigma_2 \otimes \mathbf{X}_2^{\perp}\Sigma_1\mathbf{X}_2^{\perp}\right]$ holds wp 1.
(2) $\mathcal{R}_2$ is consistent $\Longleftrightarrow \mathbf{X}_1^{\perp}\Upsilon_2 \in \mathscr{C}\left[\mathbf{X}_1^{\perp}\mathbf{X}_2\mathbf{F}_{\mathbf{C}_2}, \quad \Sigma_2 \otimes \mathbf{X}_1^{\perp}\Sigma_1\mathbf{X}_1^{\perp}\right]$ holds wp 1.

We note that $\mathcal{R}_i$ are consistent under the assumption of consistency of $\mathcal{R}, i = 1, 2$.

Definition 2. The predictability requirement of matrix Δ under $\mathcal{R}$ is described as holding the inclusion $\mathscr{C}((\mathbf{HF_C})') \subseteq \mathscr{C}((\mathbf{XF_C})')$. This requirement also corresponds to the estimability of $\mathbf{HF_C}\Omega$ under $\mathcal{R}$. Further, the matrix Ψ is always predictable under $\mathcal{R}$.

For models $\mathcal{R}$ and $\mathcal{R}_i$, $i = 1, 2$, the following predictability/estimability requirements of Δ_i and its special cases are given in accordance with Definition 2. Let $\widehat{\mathbf{H}}_1 = \left[\mathbf{H}_1\mathbf{F}_{\mathbf{C}_1}, \quad \mathbf{0}\right]$ and $\widehat{\mathbf{H}}_2 = \left[\mathbf{0}, \quad \mathbf{H}_2\mathbf{F}_{\mathbf{C}_2}\right]$. Then

(1) Δ_1 is predictable by Υ in $\mathcal{R} \Longleftrightarrow \mathscr{C}(\widehat{\mathbf{H}}_1') \subseteq \mathscr{C}((\mathbf{XF_C})')$.
(2) Δ_2 is predictable by Υ in $\mathcal{R} \Longleftrightarrow \mathscr{C}(\widehat{\mathbf{H}}_2') \subseteq \mathscr{C}((\mathbf{XF_C})')$.
(3) Δ_1 is predictable by $\mathbf{X}_2^{\perp}\Upsilon_1$ in $\mathcal{R}_1 \Longleftrightarrow \mathscr{C}((\mathbf{H}_1\mathbf{F}_{\mathbf{C}_1})') \subseteq \mathscr{C}((\mathbf{X}_2^{\perp}\mathbf{X}_1\mathbf{F}_{\mathbf{C}_1})')$.
(4) Δ_2 is predictable by $\mathbf{X}_1^{\perp}\Upsilon_2$ in $\mathcal{R}_2 \Longleftrightarrow \mathscr{C}((\mathbf{H}_2\mathbf{F}_{\mathbf{C}_2})') \subseteq \mathscr{C}((\mathbf{X}_1^{\perp}\mathbf{X}_2\mathbf{F}_{\mathbf{C}_2})')$.

We note that if Δ_i is predictable under $\mathcal{R}_i$, $i = 1, 2$, then it is predictable under $\mathcal{R}$.

Definition 3. If a matrix $\mathbf{K} \in \mathbb{R}_{k,n}$ and $\mathbf{K}_1 \in \mathbb{R}_{k,m}$, or $\widehat{\mathbf{K}} \in \mathbb{R}_{km,nm}$ and $\widehat{\mathbf{K}}_1 \in \mathbb{R}_{km,1}$ exists with

$$\begin{aligned} \mathrm{D}(\overrightarrow{\mathbf{KY} + \mathbf{K}_1 - \Phi}) &= \ \min \text{ s.t. } \mathrm{E}(\mathbf{KY} + \mathbf{K}_1 - \Phi) = \mathbf{0}, \text{ or,} \\ \mathrm{D}(\widehat{\mathbf{K}}\overrightarrow{\mathbf{Y}} + \widehat{\mathbf{K}}_1 - \overrightarrow{\Phi}) &= \ \min \text{ s.t. } \mathrm{E}(\widehat{\mathbf{K}}\overrightarrow{\mathbf{Y}} + \widehat{\mathbf{K}}_1 - \overrightarrow{\Phi}) = \mathbf{0} \end{aligned} \tag{2.1}$$

in the Lӧwner partial ordering (LPO), the linear statistic $\mathbf{KY} + \mathbf{K}_1$ is said to be the BLUP of Φ and is represented by $\mathbf{KY} + \mathbf{K}_1 = \mathrm{BLUP}_{\mathcal{M}}(\mathbf{H}\Theta + \mathbf{J}\Psi)$. If $\mathbf{J} = \mathbf{0}$ in Φ,

$\mathbf{KY}+\mathbf{K}_1$ is the well-known the BLUE of $\mathbf{H}\Theta$, represented by $\mathrm{BLUE}_{\mathcal{M}}(\mathbf{H}\Theta)$, under $\mathcal{M}$.

Suppose that Δ or $\overrightarrow{\Delta}$ in (1.13) is predictable under $\mathcal{R}$. $\mathbf{K}\Upsilon$ is defined to be the BLUP of Δ if there exists $\mathbf{K}\Upsilon$ such that

$$\mathrm{D}(\overrightarrow{\mathbf{K}\Upsilon-\Delta}) = \min \text{ s.t. } \mathrm{E}(\mathbf{K}\Upsilon-\Delta)=\mathbf{0} \tag{2.2}$$

holds in the LPO, and is represented by

$$\mathbf{K}\Upsilon = \mathrm{BLUP}_{\mathcal{R}}(\Delta) = \mathrm{BLUP}_{\mathcal{R}}(\mathbf{HF_C}\Omega+\mathbf{J}\Psi).$$

If $\mathbf{J}=\mathbf{0}$ in Δ, $\mathbf{K}\Upsilon$ corresponds the BLUE of $\mathbf{HF_C}\Omega$, represented by $\mathrm{BLUE}_{\mathcal{R}}(\mathbf{HF_C}\Omega)$. Further, if $\mathbf{H}=\mathbf{0}$ and $\mathbf{J}=\mathbf{I}_n$ in Δ, $\mathbf{K}\Upsilon$ corresponds the BLUP of Ψ, i.e., $\mathrm{BLUP}_{\mathcal{R}}(\Psi)$, under $\mathcal{R}$.

The following results are given in accordance with Definition 3. Let Δ_1 or $\overrightarrow{\Delta_1}$ be predictable under $\mathcal{R}$ and $\mathcal{R}_i$, $i=1,2$. Then

(1) $\mathrm{BLUP}_{\mathcal{R}}(\Delta_1)=\mathbf{K}_1\Upsilon \Longleftrightarrow \mathrm{D}(\overrightarrow{\mathbf{K}_1\Upsilon-\Delta_1}) = \min \text{ s.t. } \mathrm{E}(\mathbf{K}_1\Upsilon-\Delta_1)=\mathbf{0}$.

(2) $\mathrm{BLUP}_{\mathcal{R}}(\Delta_2)=\mathbf{K}_2\Upsilon \Longleftrightarrow \mathrm{D}(\overrightarrow{\mathbf{K}_2\Upsilon-\Delta_2}) = \min \text{ s.t. } \mathrm{E}(\mathbf{K}_2\Upsilon-\Delta_2)=\mathbf{0}$.

(3) $\mathrm{BLUP}_{\mathcal{R}_1}(\Delta_1)=\mathbf{G}_1\mathbf{X}_2^{\perp}\Upsilon_1$
$\Longleftrightarrow \mathrm{D}(\overrightarrow{\mathbf{G}_1\mathbf{X}_2^{\perp}\Upsilon_1-\Delta_1}) = \min \text{ s.t. } \mathrm{E}(\mathbf{G}_1\mathbf{X}_2^{\perp}\Upsilon_1-\Delta_1)=\mathbf{0}$.

(4) $\mathrm{BLUP}_{\mathcal{R}_2}(\Delta_2)=\mathbf{G}_2\mathbf{X}_1^{\perp}\Upsilon_2$
$\Longleftrightarrow \mathrm{D}(\overrightarrow{\mathbf{G}_2\mathbf{X}_1^{\perp}\Upsilon_2-\Delta_2}) = \min \text{ s.t. } \mathrm{E}(\mathbf{G}_2\mathbf{X}_1^{\perp}\Upsilon_2-\Delta_2)=\mathbf{0}$.

Finally, we give the following two lemmas, which are required in the sections that follow. For the first lemma; see, [15], and for the last one; see, [20].

Lemma 1. *The linear matrix equation* $\mathbf{HX}=\mathbf{D}$ *is consistent* $\Leftrightarrow \mathbf{r}\begin{bmatrix}\mathbf{H}, & \mathbf{D}\end{bmatrix}=\mathbf{r}(\mathbf{H})$, *or equivalently,* $\mathbf{HH}^{+}\mathbf{D}=\mathbf{D}$. *Then, the general solution of this equation can be written as* $\mathbf{X}=\mathbf{H}^{+}\mathbf{D}+(\mathbf{I}-\mathbf{H}^{+}\mathbf{H})\mathbf{V}$, *where* $\mathbf{V}$ *is an arbitrary matrix.*

Lemma 2. *Let* $\mathbf{B}\in\mathbb{R}_{m,p}$, $\mathbf{A}\in\mathbb{R}_{n,p}$ *be given matrices, and let* $\mathbf{Q}\in\mathbb{R}_n^{\succcurlyeq}$. *Suppose that there exists* $\mathbf{X}_0\in\mathbb{R}_{m,n}$ *such that* $\mathbf{X}_0\mathbf{B}=\mathbf{A}$. *Then the maximal positive inertia of* $\mathbf{X}_0\mathbf{Q}\mathbf{X}_0'-\mathbf{XQX}'$ *subject to all solutions of* $\mathbf{XB}=\mathbf{A}$ *is*

$$\max_{\mathbf{XB}=\mathbf{A}} i_{+}(\mathbf{X}_0\mathbf{Q}\mathbf{X}_0'-\mathbf{XQX}') = \mathbf{r}\begin{bmatrix}\mathbf{X}_0\mathbf{Q}\\ \mathbf{B}'\end{bmatrix}-\mathbf{r}(\mathbf{B})=\mathbf{r}(\mathbf{X}_0\mathbf{Q}\mathbf{B}^{\perp}).$$

Hence a solution $\mathbf{X}_0$ *of* $\mathbf{X}_0\mathbf{B}=\mathbf{A}$ *exists such that* $\mathbf{X}_0\mathbf{Q}\mathbf{X}_0'\preccurlyeq\mathbf{XQX}'$ *holds for all solutions of* $\mathbf{XB}=\mathbf{A}\Leftrightarrow$ *both the equations* $\mathbf{X}_0\mathbf{B}=\mathbf{A}$ *and* $\mathbf{X}_0\mathbf{Q}\mathbf{B}^{\perp}=\mathbf{0}$ *are satisfied by* $\mathbf{X}$.

3. BLUPs/BLUEs under a CMLM with some related reduced models

The following theorems, respectively, are collections of the basic findings on BLUP of Δ and Δ_i in (1.10)-(1.12) and its sub-cases. These theorems use a method described in [4]. For different approaches; see, e.g., [16, 22].

Theorem 1. *Suppose that Δ is predictable under $\mathcal{R}$ in (1.11). For Δ under $\mathcal{R}$, let $\mathbf{K}\Upsilon$ and $\mathbf{L}\Upsilon$ be unbiased linear predictors . Then the maximal positive inertia of* $\mathrm{D}(\overrightarrow{\mathbf{K}\Upsilon-\Delta})-\mathrm{D}(\overrightarrow{\mathbf{L}\Upsilon-\Delta})$ *subject to* $\mathbf{LXF_C}=\mathbf{HF_C}$ *is*

$$\max_{\mathrm{E}(\mathbf{L}\Upsilon-\Delta)=\mathbf{0}} i_{+}(\mathrm{D}(\overrightarrow{\mathbf{K}\Upsilon-\Delta})-\mathrm{D}(\overrightarrow{\mathbf{L}\Upsilon-\Delta}))=r\left(\begin{bmatrix}\mathbf{K}, & -\mathbf{I}_k\end{bmatrix}\begin{bmatrix}\mathbf{I}_n\\ \mathbf{J}\end{bmatrix}\mathrm{cov}(\Upsilon)\begin{bmatrix}\mathbf{I}_n\\ \mathbf{J}\end{bmatrix}'\begin{bmatrix}\mathbf{XF_C}\\ \mathbf{HF_C}\end{bmatrix}^{\perp}\right). \tag{3.1}$$

Hence,

$$\begin{aligned}\mathrm{D}(\overrightarrow{\mathbf{K}\Upsilon-\Delta})=\mathit{min\ s.t.}\,\mathrm{E}(\mathbf{K}\Upsilon-\Delta)=\mathbf{0}&\Leftrightarrow\mathbf{K}\Upsilon=\mathrm{BLUP}_{\mathcal{R}}(\Delta)\\ &\Leftrightarrow\mathbf{K}\begin{bmatrix}\mathbf{XF_C}, & \Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}=\begin{bmatrix}\mathbf{HF_C}, & \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}.\end{aligned} \tag{3.2}$$

(3.2) *is consistent and the* BLUP *of Δ under $\mathcal{R}$ can be written as follows by considering the general solution of this equation:*

$$\mathbf{K}\Upsilon=\mathrm{BLUP}_{\mathcal{R}}(\Delta)\Leftrightarrow\mathbf{K}\begin{bmatrix}\mathbf{XF_C}, & \Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}=\begin{bmatrix}\mathbf{HF_C}, & \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}$$

and the general solution of this consistent equation can be written as

$$\begin{aligned}\mathrm{BLUP}_{\mathcal{R}}(\Delta)=\mathbf{K}\Upsilon&=\left(\begin{bmatrix}\mathbf{HF_C}, & \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}\mathbf{W}_r^{+}+\mathbf{U}_r\mathbf{W}_r^{\perp}\right)\Upsilon,\\ \overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta)}&=(\mathbf{I}_m\otimes\mathbf{K})\overrightarrow{\Upsilon}\\ &=\left(\mathbf{I}_m\otimes\left(\begin{bmatrix}\mathbf{HF_C}, & \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}\mathbf{W}_r^{+}+\mathbf{U}_r\mathbf{W}_r^{\perp}\right)\right)\overrightarrow{\Upsilon},\end{aligned} \tag{3.3}$$

where $\mathbf{U}_r\in\mathbb{R}_{k,n}$ *is an arbitrary matrix and* $\mathbf{W}_r=\begin{bmatrix}\mathbf{XF_C}, & \Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}$. *In particular,*

$$\begin{aligned}\mathrm{BLUE}_{\mathcal{R}}(\mathbf{HF_C}\Omega)=\mathbf{K}\Upsilon&=\left(\begin{bmatrix}\mathbf{HF_C}, & \mathbf{0}\end{bmatrix}\mathbf{W}_r^{+}+\mathbf{U}_r\mathbf{W}_r^{\perp}\right)\Upsilon,\\ \overrightarrow{\mathrm{BLUE}_{\mathcal{R}}(\mathbf{HF_C}\Omega)}&=(\mathbf{I}_m\otimes\mathbf{K})\overrightarrow{\Upsilon}\\ &=\left(\mathbf{I}_m\otimes\left(\begin{bmatrix}\mathbf{HF_C}, & \mathbf{0}\end{bmatrix}\mathbf{W}_r^{+}+\mathbf{U}_r\mathbf{W}_r^{\perp}\right)\right)\overrightarrow{\Upsilon},\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathrm{BLUP}_{\mathcal{R}}(\Psi)=\mathbf{K}\Upsilon&=\left(\begin{bmatrix}\mathbf{0}, & \Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}\mathbf{W}_r^{+}+\mathbf{U}_r\mathbf{W}_r^{\perp}\right)\Upsilon,\\ \overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Psi)}&=(\mathbf{I}_m\otimes\mathbf{K})\overrightarrow{\Upsilon}\\ &=\left(\mathbf{I}_m\otimes\left(\begin{bmatrix}\mathbf{0}, & \Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}\mathbf{W}_r^{+}+\mathbf{U}_r\mathbf{W}_r^{\perp}\right)\right)\overrightarrow{\Upsilon}.\end{aligned} \tag{3.5}$$

Further, the matrix $\mathbf{W}_r$ *satisfies*

$$r(\mathbf{W}_r)=r\begin{bmatrix}\mathbf{XF_C}, & \Sigma_1\end{bmatrix}\ \mathit{and}\ \mathscr{C}(\mathbf{W}_r)=\mathscr{C}\begin{bmatrix}\mathbf{XF_C}, & \Sigma_1\end{bmatrix}. \tag{3.6}$$

Also, the following equalities hold.

$$\begin{aligned}\mathrm{D}\left[\overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta)}\right]&=\sigma^2\Sigma_2\otimes\begin{bmatrix}\mathbf{HF_C}, & \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}\mathbf{W}_r^{+}\Sigma_1\\ &\quad\times\left(\begin{bmatrix}\mathbf{HF_C}, & \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\end{bmatrix}\mathbf{W}_r^{+}\right)',\end{aligned}$$

$$\begin{aligned} \operatorname{cov}\{\overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta)}, \overrightarrow{\Delta}\} &= \sigma^2\Sigma_2 \otimes \left[\mathbf{HF_C}, \quad \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\right]\mathbf{W}_r^{+}\Sigma_1\mathbf{J}', \\ \mathrm{D}[\overrightarrow{\Delta} - \overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta)}] &= \sigma^2\Sigma_2 \otimes \left(\left[\mathbf{HF_C}, \quad \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\right]\mathbf{W}_r^{+} - \mathbf{J}\right) \\ &\qquad \times \Sigma_1\left(\left[\mathbf{HF_C}, \quad \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\right]\mathbf{W}_r^{+} - \mathbf{J}\right)'. \end{aligned} \tag{3.7}$$

Proof of Theorem 1. Suppose that $\mathbf{K}\Upsilon$ and $\mathbf{L}\Upsilon$ are two unbiased linear predictors for Δ in $\mathcal{R}$. Then, the expected value and covariance matrix of $\mathbf{K}\Upsilon - \Delta$ are written as

$$\mathrm{E}(\mathbf{K}\Upsilon - \Delta) = \mathbf{0} \Leftrightarrow \mathbf{KXF_C} = \mathbf{HF_C} \Leftrightarrow \left[\mathbf{K}, \quad -\mathbf{I}_k\right]\begin{bmatrix}\mathbf{XF_C}\\ \mathbf{HF_C}\end{bmatrix} = \mathbf{0} \tag{3.8}$$

and

$$\begin{aligned} \mathrm{D}\left(\overrightarrow{\mathbf{K}\Upsilon - \Delta}\right) &= \left(\mathbf{I}_m \otimes (\mathbf{K}-\mathbf{J})\right)\operatorname{cov}(\Psi)\left(\mathbf{I}_m \otimes (\mathbf{K}-\mathbf{J})\right)' \\ &= \sigma^2\left(\mathbf{I}_m \otimes (\mathbf{K}-\mathbf{J})\right)(\Sigma_2 \otimes \Sigma_1)\left(\mathbf{I}_m \otimes (\mathbf{K}-\mathbf{J})\right)' \\ &= \sigma^2\Sigma_2 \otimes (\mathbf{K}-\mathbf{J})\Sigma_1(\mathbf{K}-\mathbf{J})' \\ &= \sigma^2\Sigma_2 \otimes \left[\mathbf{K}, \quad -\mathbf{I}_k\right]\begin{bmatrix}\mathbf{I}_n\\ \mathbf{J}\end{bmatrix}\Sigma_1\begin{bmatrix}\mathbf{I}_n\\ \mathbf{J}\end{bmatrix}'\left[\mathbf{K}, \quad -\mathbf{I}_k\right]' := \Sigma_2 \otimes f(\mathbf{K}), \end{aligned} \tag{3.9}$$

where

$$f(\mathbf{K}) = \left[\mathbf{K}, \quad -\mathbf{I}_k\right]\begin{bmatrix}\mathbf{I}_n\\ \mathbf{J}\end{bmatrix}\Sigma_1\begin{bmatrix}\mathbf{I}_n\\ \mathbf{J}\end{bmatrix}'\left[\mathbf{K}, \quad -\mathbf{I}_k\right]'.$$

By using $\mathbf{L}$ in place of $\mathbf{K}$, the equivalent formulas as in (3.8) and (3.9) may also be given for the other unbiased linear predictor $\mathbf{L}\Upsilon$ for Δ under $\mathcal{R}$. In order to obtain solution $\mathbf{K}$ of the consistent linear matrix equation $\mathbf{KXF_C} = \mathbf{HF_C}$, the matrix minimization problem described in Definition 3 for finding the BLUP of Δ under $\mathcal{R}$ can be expressed such that

$$\Sigma_2 \otimes f(\mathbf{K}) \preccurlyeq \Sigma_2 \otimes f(\mathbf{L}) \text{ s.t. } \mathbf{LXF_C} = \mathbf{HF_C} \tag{3.10}$$

or equivalently,

$$f(\mathbf{K}) \preccurlyeq f(\mathbf{L}) \text{ s.t. } \mathbf{LXF_C} = \mathbf{HF_C}$$

because Σ_2 is a non-null matrix. According to Lemma 2, (3.10) is a typical constrained quadratic matrix-valued function optimization problem in the LPO. Lemma 2 gives us the basic formula for the BLUP of Δ in (3.2), and Lemma 1 gives us the expression for the BLUP of Δ under $\mathcal{R}$ in (3.3). Setting $\mathbf{J} = \mathbf{0}$, and $\mathbf{H} = \mathbf{0}$ and $\mathbf{J} = \mathbf{I}_n$ at (3.3) yields (3.4) and (3.5), respectively. The expressions in (3.6) are well-known results; see also [21, Lemma 2.1(a)]. From (1.17) and (3.3),

$$\begin{aligned} \mathrm{D}[\overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta)}] &= (\mathbf{I}_m \otimes \mathbf{K})\operatorname{cov}(\Psi)(\mathbf{I}_m \otimes \mathbf{K})' = \sigma^2(\mathbf{I}_m \otimes \mathbf{K})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{K})' \\ &= \sigma^2\Sigma_2 \otimes \mathbf{K}\Sigma_1\mathbf{K}' \\ &= \sigma^2\Sigma_2 \otimes \left(\left[\mathbf{HF_C}, \quad \mathbf{J}\Sigma_1(\mathbf{XF_C})^{\perp}\right]\mathbf{W}_r^{+} + \mathbf{U}_r\mathbf{W}_r^{\perp}\right) \end{aligned}$$

$$
\begin{aligned}
&\times \Sigma_1 \left(\left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp} \right] \mathbf{W}_r^{+} + \mathbf{U}_r \mathbf{W}_r^{\perp} \right)' \\
&= \sigma^2 \Sigma_2 \otimes \left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp} \right] \mathbf{W}_r^{+} \\
&\quad \times \Sigma_1 \left(\left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp} \right] \mathbf{W}_r^{+} \right)',
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{cov}\{\overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta)}, \overrightarrow{\Delta}\} &= (\mathbf{I}_m \otimes \mathbf{K}) \operatorname{cov}(\Psi) (\mathbf{I}_m \otimes \mathbf{J})' \\
&= \sigma^2 (\mathbf{I}_m \otimes \mathbf{K}) (\Sigma_2 \otimes \Sigma_1) (\mathbf{I}_m \otimes \mathbf{J})' = \sigma^2 \Sigma_2 \otimes \mathbf{K}\Sigma_1 \mathbf{J}' \\
&= \sigma^2 \Sigma_2 \otimes \left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp} \right] \mathbf{W}_r^{+} \Sigma_1 \mathbf{J}',
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}[\overrightarrow{\Delta} - \overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta)}] &= \sigma^2 \Sigma_2 \otimes \left(\left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp} \right] \mathbf{W}_r^{+} + \mathbf{U}_r \mathbf{W}_r^{\perp} - \mathbf{J} \right) \\
&\quad \times \Sigma_1 \left(\left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp} \right] \mathbf{W}_r^{+} + \mathbf{U}_r \mathbf{W}_r^{\perp} - \mathbf{J} \right)' \\
&= \sigma^2 \Sigma_2 \otimes \left(\left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp} \right] \mathbf{W}_r^{+} - \mathbf{J} \right) \\
&\quad \times \Sigma_1 \left(\left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp} \right] \mathbf{W}_r^{+} - \mathbf{J} \right)'.
\end{aligned}
$$

Thus establishing equalities in (3.7). □

Theorem 2. *Suppose that Δ_i is predictable under $\mathcal{R}_i$, $i = 1,2$ in (1.11)-(1.12) (also predictable under $\mathcal{R}$ in (1.10)). Thus, the results that follow are provided.*

(1) $\overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta_1)} = \left(\mathbf{I}_m \otimes \left(\left[\widehat{\mathbf{H}}_1, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} + \mathbf{P}_1 \mathbf{W}_r^{\perp}\right)\right) \overrightarrow{\Upsilon}$, *where* $\widehat{\mathbf{H}}_1 = \left[\mathbf{H}_1 \mathbf{F}_{\mathbf{C}_1}, \quad \mathbf{0}\right]$, $\mathbf{W}_r = \left[\mathbf{X}\mathbf{F}_{\mathbf{C}}, \quad \Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right]$ *and* $\mathbf{P}_1 \in \mathbb{R}_{s,n}$ *is an arbitrary matrix. Then,*

$$
\begin{aligned}
\mathrm{D}\left[\overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta_1)}\right] &= \sigma^2 \Sigma_2 \otimes \left[\widehat{\mathbf{H}}_1, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} \Sigma_1 \\
&\quad \times \left(\left[\widehat{\mathbf{H}}_1, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} \right)', \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}[\overrightarrow{\Delta_1} - \overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta_1)}] &= \sigma^2 \Sigma_2 \otimes \left(\left[\widehat{\mathbf{H}}_1, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} - \mathbf{J} \right) \\
&\quad \times \Sigma_1 \left(\left[\widehat{\mathbf{H}}_1, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} - \mathbf{J} \right)'. \qquad (3.12)
\end{aligned}
$$

(2) $\overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta_2)} = \left(\mathbf{I}_m \otimes \left(\left[\widehat{\mathbf{H}}_2, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} + \mathbf{P}_2 \mathbf{W}_r^{\perp}\right)\right) \overrightarrow{\Upsilon}$, *where* $\widehat{\mathbf{H}}_2 = \left[\mathbf{0}, \quad \mathbf{H}_2 \mathbf{F}_{\mathbf{C}_2}\right]$, $\mathbf{W}_r = \left[\mathbf{X}\mathbf{F}_{\mathbf{C}}, \quad \Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right]$ *and* $\mathbf{P}_2 \in \mathbb{R}_{s,n}$ *is an arbitrary matrix. Then,*

$$
\begin{aligned}
\mathrm{D}\left[\overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta_2)}\right] &= \sigma^2 \Sigma_2 \otimes \left[\widehat{\mathbf{H}}_2, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} \Sigma_1 \\
&\quad \times \left(\left[\widehat{\mathbf{H}}_2, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} \right)', \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}[\overrightarrow{\Delta_2} - \overrightarrow{\mathrm{BLUP}_{\mathcal{R}}(\Delta_2)}] &= \sigma^2 \Sigma_2 \otimes \left(\left[\widehat{\mathbf{H}}_2, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} - \mathbf{J} \right) \\
&\quad \times \Sigma_1 \left(\left[\widehat{\mathbf{H}}_2, \quad \mathbf{J}\Sigma_1 (\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right] \mathbf{W}_r^{+} - \mathbf{J} \right)'. \qquad (3.14)
\end{aligned}
$$

(3) $\overrightarrow{\mathrm{BLUP}_{\mathcal{R}_1}(\Delta_1)} = \left(\mathbf{I}_m \otimes \left(\mathbf{M}_1 \mathbf{W}_{r_1}^{+} + \mathbf{U}_1 \mathbf{W}_{r_1}^{\perp}\right)\right) \overrightarrow{\Upsilon_1}$,
where $\mathbf{U}_1 \in \mathbb{R}_{s,n}$ *is arbitrary matrix,* $\mathbf{M}_1 = \left[\mathbf{H}_1 \mathbf{F}_{\mathbf{C}_1}, \quad \mathbf{J}\Sigma_1 \mathbf{X}_2^{\perp} (\mathbf{X}_2^{\perp} \mathbf{X}_1 \mathbf{F}_{\mathbf{C}_1})^{\perp}\right]$

and $\mathbf{W}_{r_1} = \left[\mathbf{X}_2^{\perp}\mathbf{X}_1\mathbf{F}_{\mathbf{C}_1}, \quad \mathbf{X}_2^{\perp}\Sigma_1\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1\mathbf{F}_{\mathbf{C}_1})^{\perp}\right]$. *Then,*

$$\mathrm{D}\left[\overrightarrow{\mathrm{BLUP}_{\mathcal{R}_1}(\Delta_1)}\right] = \sigma^2\Sigma_2 \otimes \mathbf{M}_1\mathbf{W}_{r_1}^{+}\mathbf{X}_2^{\perp}\Sigma_1\mathbf{X}_2^{\perp}\left(\mathbf{M}_1\mathbf{W}_{r_1}^{+}\right)', \tag{3.15}$$

$$\mathrm{D}[\overrightarrow{\Delta_1} - \overrightarrow{\mathrm{BLUP}_{\mathcal{R}_1}(\Delta_1)}] = \sigma^2\Sigma_2 \otimes \left(\mathbf{M}_1\mathbf{W}_{r_1}^{+}\mathbf{X}_2^{\perp} - \mathbf{J}\right)\Sigma_1\left(\mathbf{M}_1\mathbf{W}_{r_1}^{+}\mathbf{X}_2^{\perp} - \mathbf{J}\right)'. \tag{3.16}$$

(4) $\overrightarrow{\mathrm{BLUP}_{\mathcal{R}_2}(\Delta_2)} = \left(\mathbf{I}_m \otimes \left(\mathbf{M}_2\mathbf{W}_{r_2}^{+} + \mathbf{U}_2\mathbf{W}_{r_2}^{\perp}\right)\right)\overrightarrow{\Upsilon_2}$,
where $\mathbf{U}_2 \in \mathbb{R}_{s,n}$ *is arbitrary matrix,* $\mathbf{M}_2 = \left[\mathbf{H}_2\mathbf{F}_{\mathbf{C}_2}, \quad \mathbf{J}\Sigma_1\mathbf{X}_1^{\perp}(\mathbf{X}_1^{\perp}\mathbf{X}_2\mathbf{F}_{\mathbf{C}_2})^{\perp}\right]$ *and* $\mathbf{W}_{r_2} = \left[\mathbf{X}_1^{\perp}\mathbf{X}_2\mathbf{F}_{\mathbf{C}_2}, \quad \mathbf{X}_1^{\perp}\Sigma_1\mathbf{X}_1^{\perp}(\mathbf{X}_1^{\perp}\mathbf{X}_2\mathbf{F}_{\mathbf{C}_2})^{\perp}\right]$. *Then,*

$$\mathrm{D}\left[\overrightarrow{\mathrm{BLUP}_{\mathcal{R}_2}(\Delta_2)}\right] = \sigma^2\Sigma_2 \otimes \mathbf{M}_2\mathbf{W}_{r_2}^{+}\mathbf{X}_1^{\perp}\Sigma_1\mathbf{X}_1^{\perp}\left(\mathbf{M}_2\mathbf{W}_{r_2}^{+}\right)', \tag{3.17}$$

$$\mathrm{D}[\overrightarrow{\Delta_2} - \overrightarrow{\mathrm{BLUP}_{\mathcal{R}_2}(\Delta_2)}] = \sigma^2\Sigma_2 \otimes \left(\mathbf{M}_2\mathbf{W}_{r_2}^{+}\mathbf{X}_1^{\perp} - \mathbf{J}\right)\Sigma_1\left(\mathbf{M}_2\mathbf{W}_{r_2}^{+}\mathbf{X}_1^{\perp} - \mathbf{J}\right)'. \tag{3.18}$$

Proof of Theorem 2. For BLUP of Δ_i under $\mathcal{R}_i$, the fundamental equations can be proved similarly to Theorem 1. □

The basic conclusions are reached by changing (3.3) in Theorem 1 into (3.9), accordingly.

Corollary 1. *$\mathcal{M}$ and $\mathcal{R}$ be as given (1.1) and (1.11), respectively. Let Φ and Δ be as given in (1.4) and (1.13), respectively, and suppose that Δ is predictable under $\mathcal{R}$. Then*

$$\begin{aligned}\mathrm{BLUP}_{\mathcal{M}}(\Phi) &= \mathbf{H}\mathbf{C}^{+}\mathbf{D} + \mathrm{BLUP}_{\mathcal{R}}(\Delta)\\ &= \mathbf{H}\mathbf{C}^{+}\mathbf{D} + \left(\left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right]\mathbf{W}_r^{+} + \mathbf{U}_r\mathbf{W}_r^{\perp}\right)\\ &\quad \times\left(\mathbf{Y} - \mathbf{X}\mathbf{C}^{+}\mathbf{D}\right)\end{aligned}$$

where $\mathbf{U}_r \in \mathbb{R}_{k,n}$ *is arbitrary matrix, and* $\mathbf{W}_r = \left[\mathbf{X}\mathbf{F}_{\mathbf{C}}, \quad \Sigma_1(\mathbf{X}\mathbf{F}_{\mathbf{C}})^{\perp}\right]$. *In particular,*

$$\begin{aligned}\mathrm{BLUP}_{\mathcal{M}}(\mathbf{H}\Theta) &= \mathbf{H}\mathbf{C}^{+}\mathbf{D} + \mathrm{BLUP}_{\mathcal{R}}(\mathbf{H}\mathbf{F}_{\mathbf{C}}\Omega)\\ &= \mathbf{H}\mathbf{C}^{+}\mathbf{D} + \left(\left[\mathbf{H}\mathbf{F}_{\mathbf{C}}, \quad \mathbf{0}\right]\mathbf{W}_r^{+} + \mathbf{U}_r\mathbf{W}_r^{\perp}\right)\left(\mathbf{Y} - \mathbf{X}\mathbf{C}^{+}\mathbf{D}\right).\end{aligned}$$

Corollary 2. *$\mathcal{M}_i$ and $\mathcal{R}_i$ be as given (1.2)-(1.3) and (1.11)-(1.12), respectively. Let Φ_i and Δ_i be as given in (1.5)-(1.6) and (1.14)-(1.15), respectively, and suppose that Δ_i, $i = 1,2$ is predictable under $\mathcal{R}_i$. Then*

$$\begin{aligned}\mathrm{BLUP}_{\mathcal{M}_1}(\Phi_1) &= \mathbf{H}_1\mathbf{C}_1{}^{+}\mathbf{D} + \mathrm{BLUP}_{\mathcal{R}_1}(\Delta_1)\\ &= \mathbf{H}_1\mathbf{C}_1{}^{+}\mathbf{D} + \left(\mathbf{M}_1\mathbf{W}_{r_1}^{+} + \mathbf{U}_1\mathbf{W}_{r_1}^{\perp}\right)\left(\mathbf{Y} - \mathbf{X}_1\mathbf{C}_1{}^{+}\mathbf{D}\right)\end{aligned}$$

where $\mathbf{U}_1 \in \mathbb{R}_{k,n}$ *is arbitrary matrix,* $\mathbf{M}_1 = \left[\mathbf{H}_1\mathbf{F}_{\mathbf{C}_1}, \quad \mathbf{J}\Sigma_1\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1\mathbf{F}_{\mathbf{C}_1})^{\perp}\right]$ *and* $\mathbf{W}_{r_1} = \left[\mathbf{X}_2^{\perp}\mathbf{X}_1\mathbf{F}_{\mathbf{C}_1}, \quad \mathbf{X}_2^{\perp}\Sigma_1\mathbf{X}_2^{\perp}(\mathbf{X}_2^{\perp}\mathbf{X}_1\mathbf{F}_{\mathbf{C}_1})^{\perp}\right]$. *In particular,*

$$\mathrm{BLUP}_{\mathcal{M}_1}(\mathbf{H}_1\Theta_1) = \mathbf{H}_1\mathbf{C}_1^{+}\mathbf{D} + \mathrm{BLUP}_{\mathcal{R}_1}(\mathbf{H}_1\mathbf{F}_{\mathbf{C}_1}\Omega_1)$$

$$= \mathbf{H}_1\mathbf{C}_1^{+}\mathbf{D} + \left(\begin{bmatrix} \mathbf{H}_1\mathbf{F}_{\mathbf{C}_1}, & \mathbf{0} \end{bmatrix} \mathbf{W}_{r1}^{+} + \mathrm{U}_1\mathbf{W}_{r1}^{\perp} \right) \left(\mathbf{Y} - \mathbf{X}_1\mathbf{C}_1^{+}\mathbf{D} \right)$$

and

$$\begin{aligned} \mathrm{BLUP}_{\mathcal{M}_2}(\Phi_2) &= \mathbf{H}_2{\mathbf{C}_2}^{+}\mathbf{D} + \mathrm{BLUP}_{\mathcal{R}_2}(\Delta_2) \\ &= \mathbf{H}_2{\mathbf{C}_2}^{+}\mathbf{D} + \left(\mathbf{M}_2\mathbf{W}_{r_2}^{+} + \mathbf{U}_2\mathbf{W}_{r_2}^{\perp} \right) \left(\mathbf{Y} - \mathbf{X}_2{\mathbf{C}_2}^{+}\mathbf{D} \right) \end{aligned}$$

where $\mathbf{U}_2 \in \mathbb{R}_{k,n}$ *is arbitrary matrix,* $\mathbf{M}_2 = \begin{bmatrix} \mathbf{H}_2\mathbf{F}_{\mathbf{C}_2}, & \mathbf{J}\Sigma_1\mathbf{X}_1^{\perp}(\mathbf{X}_1^{\perp}\mathbf{X}_2\mathbf{F}_{\mathbf{C}_2})^{\perp} \end{bmatrix}$ *and*

$$\mathbf{W}_{r_2} = \begin{bmatrix} \mathbf{X}_1^{\perp}\mathbf{X}_2\mathbf{F}_{\mathbf{C}_2}, & \mathbf{X}_1^{\perp}\Sigma_1\mathbf{X}_1^{\perp}(\mathbf{X}_1^{\perp}\mathbf{X}_2\mathbf{F}_{\mathbf{C}_2})^{\perp} \end{bmatrix}.$$

In particular,

$$\begin{aligned} \mathrm{BLUP}_{\mathcal{M}_2}(\mathbf{H}_2\Theta_2) &= \mathbf{H}_2\mathbf{C}_2^{+}\mathbf{D} + \mathrm{BLUP}_{\mathcal{R}_2}(\mathbf{H}_2\mathbf{F}_{\mathbf{C}_2}\Omega_2) \\ &= \mathbf{H}_2\mathbf{C}_2^{+}\mathbf{D} + \left(\begin{bmatrix} \mathbf{H}_2\mathbf{F}_{\mathbf{C}_2}, & \mathbf{0} \end{bmatrix} \mathbf{W}_{r2}^{+} + \mathbf{U}_2\mathbf{W}_{r2}^{\perp} \right) \left(\mathbf{Y} - \mathbf{X}_2\mathbf{C}_2^{+}\mathbf{D} \right). \end{aligned}$$

4. Summary Comments

This study presents a general approach to CMLMs and some related reduced models. We use the approach of reparameterization of CMLMs subject to exact linear restrictions. Another popular approach to handling a CMLM is to construct a new combined model by merging two given parts of the CMLM, i.e., the model part and the restriction part, into a combined form. According to this approach, the explicitly constrained model $\mathcal{M}$ and some related reduced models $\mathcal{M}_i$, $i = 1,2$ are converted into the following implicitly CMLM and some related reduced models:

$$\widehat{\mathcal{M}} : \begin{bmatrix} \mathbf{Y} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{C} \end{bmatrix} \Theta + \begin{bmatrix} \Psi \\ \mathbf{0} \end{bmatrix} \quad \text{with} \quad \mathrm{D} \begin{bmatrix} \mathbf{Y} \\ \mathbf{D} \end{bmatrix} = \sigma^2 \begin{bmatrix} \Sigma_2 \otimes \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{4.1}$$

$$\widehat{\mathcal{M}_i} : \begin{bmatrix} \mathbf{X}_j^{\perp}\mathbf{Y} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_j^{\perp}\mathbf{X}_i \\ \mathbf{C}_i \end{bmatrix} \Theta_i + \begin{bmatrix} \mathbf{X}_j^{\perp}\Psi \\ \mathbf{0} \end{bmatrix} \quad \text{with} \quad \mathrm{D} \begin{bmatrix} \mathbf{X}_j^{\perp}\mathbf{Y} \\ \mathbf{D} \end{bmatrix} = \sigma^2 \begin{bmatrix} \Sigma_2 \otimes \mathbf{X}_j^{\perp}\Sigma_1\mathbf{X}_j^{\perp} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{4.2}$$

$i, j = 1,2$ and $i \neq j$. Then, by taking into account the model $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{M}_i}$, $i = 1,2$ in (4.1) and (4.2), respectively, equivalent results for the BLUP of Φ may be established.

References

[1] I. S. Alalouf and G. P. H. Styan, "Characterizations of estimability in the general linear model." *Ann. Stat.*, vol. 7, no. 1, pp. 194–200, 1979, doi: 10.1214/aos/1176344564.

[2] J. K. Baksalary and R. Kala, "Linear transformations preserving best linear unbiased estimators in a general Gauss–Markoff model." *Ann. Stat.*, vol. 9, no. 4, pp. 913–916, 1981, doi: 10.1214/aos/1176345533.

[3] M. E. Büyükkaya and N. Güler, "On predictors and estimators under a constrained partitioned linear model and its reduced models." *Fundam. J. Math. Appl.*, vol. 5, no. 3, pp. 192–200, 2022, doi: 10.33401/fujma.1134600.

[4] N. Güler and M. E. Büyükkaya, "Inertia and rank approach in transformed linear mixed models for comparison of BLUPs." *Commun. Stat. - Theory Methods*, vol. 52, no. 9, pp. 3108–3123, 2023, doi: 10.1080/03610926.2021.1967397.

[5] N. Güler and M. E. Büyükkaya, "Further remarks on constrained over-parameterized linear models." *Stat. Pap.*, vol. 65, pp. 975–988, 2024, doi: 10.1007/s00362-023-01426-z.

[6] B. Jiang and Y. Tian, "On additive decompositions of estimators under a multivariate general linear model and its two submodels." *J. Multivar. Anal.*, vol. 162, pp. 193–214, 2017a, doi: 10.1016/j.jmva.2017.09.007.

[7] B. Jiang and Y. Tian, "On equivalence of predictors/estimators under a multivariate general linear model with augmentation." *J. Korean Stat. Soc.*, vol. 46, pp. 551–561, 2017b, doi: 10.1016/j.jkss.2017.04.001.

[8] B. Jiang and Y. Tian, "On best linear unbiased estimation and prediction under a constrained linear random-effects model." *J. Ind. Manag. Optim.*, vol. 19, no. 2, pp. 852–867, 2023, doi: 10.3934/jimo.2021209.

[9] B. Jiang, Y. Tian, and X. Zhang, "On decompositions of estimators under a general linear model with partial parameter restrictions." *Open Math.*, vol. 15, no. 1, pp. 1300–1322, 2017, doi: 10.1515/math-2017-0109.

[10] W. Li, Y. Tian, and R. Yuan, "Statistical analysis of a linear regression model with restrictions and superfluous variables." *J. Ind. Manag. Optim.*, vol. 19, no. 5, pp. 3107–3127, 2023, doi: 10.3934/jimo.2022079.

[11] Y. Liu and C. Xia, "Fundamental equations of BLUE and BLUP in the multivariate linear model with applications." *Commun. Stat. - Theory Methods*, vol. 42, no. 3, pp. 398–412, 2013, doi: 10.1080/03610926.2011.585007.

[12] C. Lu, Y. Sun, and Y. Tian, "A comparison between two competing fixed parameter constrained general linear models with new regressors." *Statistics*, vol. 52, no. 4, pp. 769–781, 2018, doi: 10.1080/02331888.2018.1469021.

[13] R. Ma and Y. Tian, "A matrix approach to a general partitioned linear model with partial parameter restrictions." *Linear Multilinear Algebra*, vol. 70, no. 13, pp. 2513–2532, 2022, doi: 10.1080/03081087.2020.1804521.

[14] M. Nurhonen and S. Puntanen, "A property of partitioned generalized regression." *Commun. Stat. - Theory Methods*, vol. 21, no. 6, pp. 1579–1583, 1992, doi: 10.1080/03610929208830865.

[15] R. Penrose, "A generalized inverse for matrices." *Math. Proc. Camb. Philos. Soc.*, vol. 51, no. 3, pp. 406–413, 1955, doi: 10.1017/S0305004100030401.

[16] S. Puntanen, G. P. H. Styan, and J. Isotalo, *Matrix Tricks for Linear Statistical Models: Our Personal Top Twenty.* Heidelberg: Springer, 2011. doi: 10.1007/978-3-642-10473-2.

[17] C. R. Rao, "Representations of best linear unbiased estimators in the Gauss-Markoff model with a singular dispersion matrix." *J. Multivar. Anal.*, vol. 3, no. 3, pp. 276–292, 1973, doi: 10.1016/0047-259X(73)90042-0.

[18] Y. Sun, B. Jiang, and H. Jiang, "Computations of predictors/estimators under a linear random-effects model with parameter restrictions." *Commun. Stat. - Theory Methods*, vol. 48, no. 14, pp. 3482–3497, 2019, doi: 10.1080/03610926.2018.1476714.

[19] Y. Sun, H. Jiang, and Y. Tian, "A prediction analysis in a constrained multivariate general linear model with future observations." *Commun. Stat. - Theory Methods*, vol. 50, no. 2, pp. 345–357, 2021, doi: 10.1080/03610926.2019.1634819.

[20] Y. Tian, "Solving optimization problems on ranks and inertias of some constrained nonlinear matrix functions via an algebraic linearization method." *Nonlinear Anal. Theory Methods Appl.*, vol. 75, no. 2, pp. 717–734, 2012, doi: 10.1016/j.na.2011.09.003.

[21] Y. Tian, "On properties of BLUEs under general linear regression models." *J. Stat. Plan. Inference*, vol. 143, no. 4, pp. 771–782, 2013, doi: 10.1016/j.jspi.2012.10.005.

[22] Y. Tian, “A new derivation of BLUPs under random-effects model.” *Metrika*, vol. 78, no. 8, pp. 905–918, 2015, doi: 10.1007/s00184-015-0533-0.
[23] Y. Tian and B. Jiang, “Equalities for estimators of partial parameters under linear model with restrictions.” *J. Multivar. Anal.*, vol. 143, pp. 299–313, 2016, doi: 10.1016/j.jmva.2015.09.007.
[24] Y. Tian and C. Wang, “On simultaneous prediction in a multivariate general linear model with future observations.” *Stat. Probab. Lett.*, vol. 128, pp. 52–59, 2017, doi: 10.1016/j.spl.2017.04.007.
[25] X. Zhang and Y. Tian, “On decompositions of BLUEs under a partitioned linear model with restrictions.” *Stat. Pap.*, vol. 57, no. 2, pp. 345–364, 2016, doi: 10.1007/s00362-014-0654-y.

Authors' addresses

Melek Eriş Büyükkaya
(**Corresponding author**) Karadeniz Technical University, Department of Statistics and Computer Sciences, 61080, Trabzon, Turkey
E-mail address: `melekeris@ktu.edu.tr`

Nesrin Güler
Sakarya University, Department of Econometrics, 54187 Sakarya, Turkey
E-mail address: `nesring@sakarya.edu.tr`