Miskolc Mathematical Notes
Vol. 26 (2025), No. 1, pp. 229–242

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4747

$M-$ESTIMATION IN PERIODIC THRESHOLD *GARCH* MODELS: CONSISTENCY AND ASYMPTOTIC NORMALITY

AHMED GHEZAL AND IMANE ZEMMOURI

Received 28 May, 2023

Abstract. The present paper derives convergence rates and asymptotic normality of a class of $M-$estimators in the periodic asymmetric *GARCH* model. Simulation studies are conducted for evaluating the performance of the estimator. Finally, an empirical study on the exchange rates of the Algerian Dinar against the U.S-dollar and the single European currency (Euro) illustrates the usefulness of the periodic *TGARCH* model.

2010 *Mathematics Subject Classification:* 62F12; 62M10

Keywords: periodic threshold GARCH model, M-estimator, strong consistency, asymptotic normality, QMLE

1. INTRODUCTION

Periodic Generalized AutoRegressive Conditional Heteroscedastic $(PGARCH)$ models have been vastly used to analyze the volatility in economic and financial time series (c.f., [1], [2]). Later Bibi and Ghezal [3] proposed an asymmetric of the *PGARCH* model called the *PTGARCH* model. A stochastic process $(X_n)_{n\in\mathbb{Z}}$ is said to follow a $PTGARCH(p,q)$ model if

$$X_n = \sigma_n^{\frac{1}{2}}\eta_n, \tag{1.1}$$

where $(\eta_n)_{n\in\mathbb{Z}}$ is a sequence of independent identically distributed $(i.i.d.)$ random variables with zero mean and unit variance and η_k is independent of X_n for $k>n$, and conditionally on the $\sigma-$field $\Im_{n-1} = \sigma(X_{n-i}, i\geq 1)$, σ_n satisfy

$$\sigma_n = a_0(n) + \sum_{i=1}^{q}\left(a_i(n)X_{n-i}^{+2} + b_i(n)X_{n-i}^{-2}\right) + \sum_{j=1}^{p} c_j(n)\,\sigma_{n-j}, \tag{1.2}$$

where $X_n^{\pm} = \max(\pm X_n, 0)$, $X_n^{\pm 2} = (X_n^{\pm})^2$ so, $X_n = X_n^+ - X_n^-$ and $|X_n| = X_n^+ + X_n^-$. Now, setting $n = st+v$, $X_{st+v} = X_t(v)$, $\sigma_{st+v} = \sigma_t(v)$ and $\eta_{st+v} = \eta_t(v)$, Model

The work was supported by DGRSDT-MESRS (DZ)..

(1.1) − (1.2) may be equivalently written as

$$\begin{cases} X_t(v) = \sigma_t^{\frac{1}{2}}(v)\eta_t(v) \\ \sigma_t(v) = a_0(v) + \sum_{i=1}^{q}\left(a_i(v)X_t^+(v-i) + b_i(v)X_t^-(v-i)\right) + \sum_{j=1}^{p} c_j(v)\sigma_t(v-j) \end{cases}. \tag{1.3}$$

In (1.3), $a_0(v)$, $a_i(v)$, $b_i(v)$ and $c_j(v)$ with $i \in \{1,\dots,q\}$ and $j \in \{1,\dots,p\}$ are positive coefficients with $a_0(v) > 0$ for any $v \in \{1,\dots,s\}$, and $X_t(v)$ refers to X_t during the $v-th$ regime $v \in \{1,\dots,s\}$ of cycle t. For the convenience, $X_t(v) = X_{t-1}(v+s)$, $\sigma_t(v) = \sigma_{t-1}(v+s)$ and $\eta_t(v) = \eta_{t-1}(v+s)$ if $v < 0$. The non-periodic notations (X_t), (σ_t), (η_t) etc. will be used interchangeably with the periodic notations $(X_t(v))$, $(\sigma_t(v))$, $(\eta_t(v))$ etc. Some results on $PTGARCH$ model can be found in literature. Guerbyenne and Kessira [10] gave a necessary and sufficient condition which ensure the existence of a strictly periodically stationary (SPS) and periodically ergodic (PE) solution to $PTGARCH$ process, and established its strong consistency and asymptotic normality of the quasi maximum likelihood estimator $(QMLE)$, See also Bibi and Ghezal [3]. As a consequence, $QMLE$ is currently a widely used method for estimating the unknown parameters in symmetric or asymmetric $PGARCH$ models. But, as everyone knows, the $QMLE$ is asymptotic if the innovation has finite four moments, while stringent moment conditions may not hold in many situations, for this reason, we propose an $M-$estimation to reduce the requirement on moments of the innovations. Many symmetric standard or periodic $GARCH$ models have been considered in the literature to estimate parameter using $M-$estimation (see, [13], [18], [20]). Now, in this paper we present a class of $M-$estimators of asymmetric $PGARCH$ models.

The rest of the paper is organized as follows. In section 2, we define the class of $M-$estimators. Section 3 is devoted to the asymptotic properties of $M-$estimation and gives an estimator of asymptotic variance. Simulation results and an illustrative application on real data are reported in Sections 4 and 5. Section 6 concludes the paper.

2. $M-$ESTIMATION

In this section, we investigate the problem of $M-$estimation of some function of the model parameter vector $\underline{\theta}' := (\underline{a}', \underline{b}', \underline{c}') := (\underline{\theta}'(1), \dots, \underline{\theta}'(s)) \in \Theta \subset]0,+\infty]^s \times [0,+\infty[^{s(2q+p)}$ where $\underline{a}' := (\underline{a}_0', \underline{a}_1', \dots, \underline{a}_q')$, $\underline{b}' := (\underline{b}_1', \dots, \underline{b}_q')$, $\underline{c}' := (\underline{c}_1', \dots, \underline{c}_p')$ and $\underline{\theta}'(v) := (a_0(v), a_1(v), \dots, a_q(v), b_1(v), \dots, b_q(v), c_1(v), \dots, c_p(v))$, $v = 1,\dots,s$ with $\underline{a}_i' := (a_i(1), \dots, a_i(s))$, $\underline{b}_k' := (b_k(1), \dots, b_k(s))$ and $\underline{c}_j' := (c_j(1), \dots, c_j(s))$ for all $0 \le i \le q$, $1 \le k \le q$ and $1 \le j \le p$ based on the realization $\{X_1, \dots, X_n; n = sN\}$ from the unique, causal and SPS solution of (1.3), and let $\sigma_t^2(\underline{\theta})$ be the conditional variance of X_t given $\mathcal{F}_{t-1}$ where $\mathcal{F}_t := \sigma(e_u; u \le t)$. The true parameter value is unknown and is denoted by $\underline{\theta}_0' := (\underline{a}_0', \underline{b}_0', \underline{c}_0') \in \Theta \subset]0,+\infty[^s \times [0,+\infty[^{s(2q+p)}$. Now, by a similar argument to Lemma 2.3 and Theorem 2.1 of Berkes et al. [12], then $PTGARCH(p,q)$

can be transformed to an infinite order $PTARCH(\infty)$, i.e., almost surely

$$\sigma_t(v) = \kappa_0(v) + \sum_{i\geq 1}\left(\kappa_i(v)X_t^{+2}(v-i) + \tau_i(v)X_t^{-2}(v-i)\right),$$

where the coefficients $\kappa_i(v)$, $\tau_k(v)$, $i \geq 0$, $k \geq 1$ are given by

$$\kappa_i(v) = \frac{a_0(v)}{1-\sum\limits_{j=1}^{p} c_j(v)}\mathbb{I}_{\{i=0\}} + \frac{d^i}{dx^i}\left(\frac{\sum\limits_{j=1}^{q} a_j(v)x^j}{1-\sum\limits_{j=1}^{p} c_j(v)x^j}\right)_{x=0}\mathbb{I}_{\{i>0\}},\ i \geq 0,$$

$$\tau_k(v) = \frac{d^k}{dx^k}\left(\frac{\sum\limits_{j=1}^{q} b_j(v)x^j}{1-\sum\limits_{j=1}^{p} c_j(v)x^j}\right)_{x=0}\mathbb{I}_{\{k>0\}},\ k \geq 1,$$

where $\mathbb{I}_{\{.\}}$ denotes the indicator function. If f denotes the error density of η_t, then the conditional density of X_t given the information available up to time $t-1$ will be $\sigma_{t,\underline{\theta}_0} f\left(\sigma_{t,\underline{\theta}_0}^{-1}X_t\right)$, $1 \leq t \leq sN$. Hence, we can get a minimizer of the negative $\log-$likelihood function

$$(sN)^{-1}\sum_{t=0}^{N-1}\sum_{v=1}^{s}\left(\frac{1}{2}\log\sigma_{t,\underline{\theta}}^2(v) - f\left(\sigma_{t,\underline{\theta}}^{-1}(v)X_{st+v}\right)\right), \underline{\theta} \in \Theta,$$

or, as a solution to the equation

$$\sum_{t=0}^{N-1}\sum_{v=1}^{s}\frac{1}{2}\left(1+K^*\left(\sigma_{t,\underline{\theta}}^{-1}(v)X_{st+v}\right)\right)\sigma_{t,\underline{\theta}}^{-2}(v)\nabla_{\underline{\theta}}\sigma_{t,\underline{\theta}}^2(v) = 0,$$

where $K^*(x) = xf^{-1}(x)\nabla_x f(x)$, $\nabla_x f(x)$ denotes its derivative or gradient of $f(x)$. More generally, since the density f is unknown, for a score function K which satisfies some constraints, we can then define an estimator $\widehat{\underline{\theta}}_{sN}$ as a solution of the following equation

$$\sum_{t=0}^{N-1}\sum_{v=1}^{s}\frac{1}{2}\left(1+K\left(\sigma_{t,\underline{\theta}}^{-1}(v)X_{st+v}\right)\right)\sigma_{t,\underline{\theta}}^{-2}(v)\nabla_{\underline{\theta}}\sigma_{t,\underline{\theta}}^2(v) = 0.$$

Remark 1. We now discuss the score function K, in [18], let $K(x) = x\varphi(x)$, $x \in \mathbb{R}$,where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable in all but finite number of points and an odd, i.e., satisfies the skew-symmetric function $\varphi(-x) = -\varphi(x)$, $\forall x \in \mathbb{R}^*$. Let $\Delta \subset \mathbb{R}$ be the set of points on which φ is differentiable and $\overline{\Delta}$ denote its complement. Examples of such score function are reported in Table 1.

Names	Expressions of φ	Expressions of K	Set $\overline{\Delta}$
1. Least absolute deviation (LAD) score function	$sign(x)$	$\|x\|$	$\{0\}$
2. Huber's $k-$score function, $k>0$	$x\mathbb{I}_{\{\|x\|\leq k\}}+ksign(x)\mathbb{I}_{\{\|x\|>k\}}$	$x^2\mathbb{I}_{\{\|x\|\leq k\}}+k\|x\|\mathbb{I}_{\{\|x\|>k\}}$	$\{-k,k\}$
3. $QMLE$ score function	x	x^2	ϕ
4. Score function for the maximum likelihood estimation (MLE)	$-f^{-1}(x)\nabla_x f(x)$, f is the true density of η_t	$x\left(-f^{-1}(x)\nabla_x f(x)\right)$	According to f
5. Score function for the exponential pseudo MLE	$a\|x\|^{b-1}sign(x)$ $a>0, 1<b\leq 2$	$a\|x\|^b$	$\{0\}$
6. $\lambda-$score function, $\lambda>1$	$\lambda sign(x)(1+\|x\|)^{-1}$	$\lambda\|x\|(1+\|x\|)^{-1}$	$\{0\}$
7. Score function for the Cauchy	$2x\left(1+x^2\right)^{-1}$	$2x^2\left(1+x^2\right)^{-1}$	$\{0\}$

Table 1. Examples of score functions.

Note however that $\widehat{\underline{\theta}}_{sN}$'s are noncomputable because $\sigma^2_{t,\underline{\theta}}$'s are unobservable. We define observable approximations $\left\{\sigma^2_{t,\underline{\theta}}, t\geq 1\right\}$ to the variance functions $\left\{\widetilde{\sigma}^2_{t,\underline{\theta}}, t\geq 1\right\}$ as

$$\widetilde{\sigma}_{t,\underline{\theta}}(v)=\kappa_{0,\underline{\theta}}(v)+\mathbb{I}_{\{st+v\geq 2\}}\sum_{i=1}^{st+v-1}\left(\kappa_{i,\underline{\theta}}(v)X_t^{+2}(v-i)+\tau_{i,\underline{\theta}}(v)X_t^{-2}(v-i)\right),\ \underline{\theta}\in\Theta.$$

Then an $M-$estimator $\widetilde{\underline{\theta}}_{sN}$ based on the score function K is a measurable solution to the following equation,

$$\sum_{t=0}^{N-1}\sum_{v=1}^{s}\frac{1}{2}\left(1+K\left(\widetilde{\sigma}^{-1}_{t,\underline{\theta}}(v)X_{st+v}\right)\right)\widetilde{\sigma}^{-2}_{t,\underline{\theta}}(v)\nabla_{\underline{\theta}}\widetilde{\sigma}^2_{t,\underline{\theta}}(v)=0.$$

Remark 2. For $K(x)=x^2$ of Example 3 in Table 1, $\widehat{\underline{\theta}}_{sN}$ is the renowned $QMLE$ as debated by Guerbyenne and Kessira [10] and Bibi and Ghezal [3].

3. Asymptotic properties for $M-$estimators of $PTGARCH$ models

To study the strong consistency and the asymptotic normality of $\widehat{\underline{\theta}}_{sN}$, we first define the polynomials $\mathcal{A}_{0,v}(z)=\sum\limits_{i=1}^{q}a_{0,i}(v)z^i$, $\mathcal{B}_{0,v}(z)=\sum\limits_{i=1}^{q}b_{0,i}(v)z^i$ and $\mathcal{C}_{0,v}(z)=1-\sum\limits_{i=1}^{p}c_{0,i}(v)z^i$, by convention $\mathcal{A}_{0,v}(z)=0$, $\mathcal{B}_{0,v}(z)=0$ if $q=0$ and $\mathcal{C}_{0,v}(z)=1$ if $p=0$, for all $v\in\{1,...,s\}$. Now, consider the following regularities assumptions (for more details and discussions, see Iqbal [13]),

H.0 For the score function K, there exists a unique constant $c_K>0$ satisfying $E\left\{K\left(c_K^{-1/2}\eta_0\right)\right\}=1$.

H.1 $\underline{\theta}_0 \in \Theta$ and Θ is a compact subset of $]0,+\infty]^s \times [0,+\infty[^{s(2q+p)}$ and $\underline{\theta}_0$, $\underline{\theta}_0^{(K)} \in \mathring{\Theta}$, with $\mathring{\Theta}$ denotes the interior of Θ, where

$$\underline{\theta}^{(K)} := \left(\underline{a}'_{(K)}, \underline{b}'_{(K)}, \underline{c}'_{(K)}\right) = c_K\left(\underline{a}', \underline{b}', \underline{c}'\right).$$

H.2 The moment conditions:

$$E\left\{K^2\left(c_K^{-1/2}\eta_0\right)\right\} < \infty \; and 0 < E\left\{c_K^{-1/2}\eta_0 \nabla_{\underline{\theta}^{(K)}} K\left(c_K^{-1/2}\eta_0\right)\right\} < \infty.$$

H.3 $(\eta_t)_{t\in\mathbb{Z}}$ is non-degenerate and $P(\eta_t > 0) \in (0,1)$.

H.4 If $p > 0$, $\mathcal{A}_{0,v}(z)$ and $\mathcal{B}_{0,v}(z)$ have no common roots with $\mathcal{C}_{0,v}(z)$ for all v. Moreover, $\mathcal{A}_{0,v}(1) + \mathcal{B}_{0,v}(1) \neq 0$ and $a_{0,q}(v) + b_{0,q}(v) + c_{0,p}(v) \neq 0$ for all $v \in \{1,...,s\}$.

H.5 $\gamma_L^{(s)}(\Lambda_0) < 0$ and $\rho\left(\prod_{v=1}^{s} \Omega_v\right) < 1$ where $\gamma_L(\Lambda_0)$ is the Lyapunov exponent associated with the random matrix $\Lambda_0\left(\underline{\eta}_t\right)$, with $\Lambda_0\left(\underline{\eta}_t\right) := \prod_{v=0}^{s-1} \Gamma_{0,s-v}(\eta_t(s-v))$, where $\Gamma_{0,v}(\eta_t(v))$ is appropriate $(2q+p)\times(2q+p)$ matrice easily obtained and uniquely determined by

$$\left\{a_{0,i}(v), b_{0,i}(v), c_{0,j}(v), \eta_t^+(v), \eta_t^-(v), 1 \le i \le q, 1 \le j \le p\right\}$$

and Ω_v is appropriate $p \times p$ matrice easily obtained and uniquely determined by $\left\{c_j(v), 1 \le j \le p\right\}$.

H.6 There exist functions g, h and l satisfying

i. $|K(uv) - K(v)| \le g(v)\left|u^2 - 1\right|$, $v \in \mathbb{R}$, $u > 0$,
ii. $|\nabla K(uv) - \nabla K(v)| \le h(v)|u-1|$, $v \in \mathbb{R}$, $u > 0$, $uv, v \in \Theta$,
iii. $|h(v+uv) - h(v)| \le l(v)u$, $v \in \mathbb{R}$, $u > 0$,

where $E\left\{\max\left(\log\left(g\left(c_K^{-1/2}\eta_0\right)\right), 0\right)\right\} < \infty$, $E\left\{\left|c_K^{-1/2}\eta_0\right| h\left(c_K^{-1/2}\eta_0\right)\right\} < \infty$ and $E\left\{\max\left(\log\left(l\left(c_K^{-1/2}\eta_0\right)\right), 0\right)\right\} < \infty$.

We are now in a position to state the following result.

Theorem 1. *Under Assumptions* **H.0** $-$ **H.6**, *then* $\widehat{\underline{\theta}}_{Ns} \xrightarrow{P} \underline{\theta}_0^{(K)}$ *as* $N \to \infty$ *and* $\sqrt{Ns}\left(\widehat{\underline{\theta}}_{Ns} - \underline{\theta}_0^{(K)}\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, \sigma_K^2 J^{-1}\right)$ *as* $N \to \infty$ *where the matrix* J *given by*

$$J := \sum_{v=1}^{s} E_{\underline{\theta}_0^{(K)}}\left\{\sigma_{t,\underline{\theta}_0^{(K)}}^{-2}(v) \nabla_{\underline{\theta}^{(K)}} \sigma_{t,\underline{\theta}_0^{(K)}}(v) \nabla_{\underline{\theta}^{(K)\prime}} \sigma_{t,\underline{\theta}_0^{(K)}}(v)\right\},$$

and $\sigma_K^2 = 4Var_{\underline{\theta}_0^{(K)}}\left(K\left(c_K^{-1/2}\eta_0\right)\right)\left(E_{\underline{\theta}_0^{(K)}}\left\{c_K^{-1/2}\eta_0 \nabla_{\underline{\theta}^{(K)\prime}} K\left(c_K^{-1/2}\eta_0\right)\right\}\right)^{-2}$.

To prove Theorem 1, we make the following assertions gathered in the following proposition (see, [12], [18], for more details)

Proposition 1. *Under Assumptions* $\mathbf{H.0}-\mathbf{H.6}$*, we have*

(1) *Let* $\left\{A_t^{(i)}, i=1,2,3, t\geq 0\right\}$ *be a sequence of identically distributed random variables. If* $\sum_{i=1}^{3} E\left\{\max\left(\log A_0^{(i)}, 0\right)\right\} < +\infty$, *then for any* $|r|<1$, $\sum_{t\geq 0}\left(A_t^{(1)}+A_t^{(2)}A_t^{(3)}\right) r^t$ *converges with probability one.*

(2) *For all* $v\in\{1,\ldots,s\}$, $i\geq 0$, $t\in\mathbb{Z}$, *we have* $\sigma_{t,\underline{\theta}_0^{(K)}} = c_K\sigma_{t,\underline{\theta}_0}$, *and the coefficients* $\left\{\kappa_{i,\underline{\theta}}(v), \tau_{i,\underline{\theta}}(v)\right\}$ *and the functions* $\left\{\sigma_{t,\underline{\theta}}\right\}$ *are differentiable in the interior* $\mathring{\Theta}$ *of* Θ.

(3) *There is a number* $r\in(0,1)$ *such that* $\forall\theta\in\mathring{\Theta}$, $v\in\{1,\ldots,s\}$, $\forall i\geq 0$,

$$\kappa_1\varsigma^i < \kappa_{i,\underline{\theta}}(v) < \kappa_2 r^i,\quad \left|\nabla_{\underline{\theta}}\kappa_{i,\underline{\theta}}(v)\right| < \kappa_3 r^i,\quad \left|\nabla^2_{\underline{\theta}}\kappa_{i,\underline{\theta}}(v)\right| < \kappa_4 r^i,$$
$$\tau_1\varsigma^i < \tau_{i,\underline{\theta}}(v) < \tau_2 r^i,\quad \left|\nabla_{\underline{\theta}}\kappa_{i,\underline{\theta}}(v)\right| < \tau_3 r^i,\quad \left|\nabla^2_{\underline{\theta}}\kappa_{i,\underline{\theta}}(v)\right| < \tau_4 r^i,$$

where $\varsigma = \{a_0(v), a_1(v), \ldots, a_q(v), b_1(v), \ldots, b_q(v), c_1(v), \ldots, c_p(v);$ $v\in\{1,\ldots,s\}, \theta\in\Theta\}\in(0,1)$.

(4) *There exist random variables* e_0 *and* e_1, *both independent of* $\{\eta_t, t\geq 1\}$, *such that* $\forall\theta\in\mathring{\Theta}$, $0<\sigma_{t,\underline{\theta}_0}-\widetilde{\sigma}_{t,\underline{\theta}_0} < r^t e_0$ *and* $\left|\nabla_{\underline{\theta}}\left(\sigma_{t,\underline{\theta}_0}-\widetilde{\sigma}_{t,\underline{\theta}_0}\right)\right| < r^t e_1$.

(5) *For any* $d\geq 1$ *and* $v\in\{1,\ldots,s\}$,

$$E\left\{\sup\left\{\left|\sigma_{t,\underline{\theta}}(v)\right|^{-d}\left|\nabla_{\underline{\theta}}\sigma_{t,\underline{\theta}}(v)\right|^{d}, \theta\in\mathring{\Theta}\right\}\right\} < +\infty,$$
$$E\left\{\sup\left\{\left|\sigma_{t,\underline{\theta}}(v)\right|^{-d}\left|\nabla^2_{\underline{\theta}}\sigma_{t,\underline{\theta}}(v)\right|^{d}, \theta\in\mathring{\Theta}\right\}\right\} < +\infty.$$

(6) *Let* $R_1(x) = x\nabla_x K(x)$. *The following two inequalities hold,*

$$R_1(uv) - R_1(v) \leq |u-1|\,|v|\left(|\nabla_x K(v)| + uh(v)\right),$$
$$R_1(uv) - R_1(wv) \leq |u-w|\,|v|\left(|u-1|\,h(v) + |\nabla_x K(v)| + |h(wv)|\right).$$

Now, we use the modified result of [16] because we want to establish consistency and asymptotic properties of $\widehat{\underline{\theta}}_{Ns}$. Let function $G_n:\Theta\longrightarrow\mathbb{R}$ be twice differentiable with respect to a in a $\mu-$neighborhood $\{\underline{\vartheta}: \|\underline{\vartheta}-\underline{\vartheta}_0\|\leq\mu\}\subset\Theta$, $\underline{\vartheta}_0\in\Theta$, we define a sequence of estimators $\widehat{\underline{\vartheta}}_n\in\Theta$, as a solution to the equation

i. $\nabla_{\underline{\vartheta}} G_n\left(\widehat{\underline{\vartheta}}_n\right) = 0,$

ii. $G_n(\underline{\vartheta}) - G_n(\underline{\vartheta}_0) = (\underline{\vartheta}-\underline{\vartheta}_0)'\nabla_{\underline{\vartheta}} G_n(\underline{\vartheta}_0) + 2^{-1}(\underline{\vartheta}-\underline{\vartheta}_0)'\nabla^2_{\underline{\vartheta}} G_n(\underline{\vartheta}_0)(\underline{\vartheta}-\underline{\vartheta}_0)$
$+2^{-1}(\underline{\vartheta}-\underline{\vartheta}_0)'\nabla^2_{\underline{\vartheta}} F_n(\underline{\vartheta}^*)(\underline{\vartheta}-\underline{\vartheta}_0),$

where $\underline{\vartheta}^*$ satisfies $\|\underline{\vartheta}^*-\underline{\vartheta}_0\|\leq\mu$ and $F_n(\underline{\vartheta}^*) = \nabla^2_{\underline{\vartheta}}\left(G_n(\underline{\vartheta}^*) - G_n(\underline{\vartheta}_0)\right)$. Moreover, the true parameter value $\underline{\vartheta}_0$ satisfies the following properties,

iii. $n^{-1}\nabla_{\underline{\vartheta}} G_n(\underline{\vartheta}_0) \xrightarrow{p} 0, n\longrightarrow+\infty,$

iv. $(2n)^{-1}\nabla^2_{\underline{\vartheta}} G_n(\underline{\vartheta}_0) \xrightarrow{p} \Sigma_G, n\longrightarrow+\infty,$

for some positive definite matrix Σ_G,

$$\textbf{v}.\quad \lim_{n\longrightarrow\infty}\lim_{\mu\overset{>}{\longrightarrow}0}\sup\left\{(2n)^{-1}\left|F_n\left(\underline{\vartheta}^*\right)\right|;\left\|\underline{\vartheta}^*-\underline{\vartheta}_0\right\|\leq\mu\right\}<+\infty,\text{ a.s.}$$

From the above mentioned we have the following theorem which is much similar to that of Klimko and Nelson [16]

Theorem 2. *Suppose that* **ii**. $-$ **v**. *hold, then the following results on asymptotics of* $\widehat{\underline{\vartheta}}_n$ *hold.*

(1) *For* $\widehat{\underline{\vartheta}}_n$ *satisfies the equation* **i**., *then,* $\widehat{\underline{\vartheta}}_n \xrightarrow{p} \underline{\vartheta}_0$.

(2) *In addition, if for some positive definite matrix* Γ_G, $\left(2n^{1/2}\right)^{-1}\nabla_{\underline{\vartheta}}G_n\left(\underline{\vartheta}_0\right)\sim N\left(\underline{O},\Gamma_G\right)$, *then* $n^{1/2}\left(\widehat{\underline{\vartheta}}_n-\underline{\vartheta}_0\right)\sim\mathcal{N}\left(\underline{O},\Sigma_G^{-1}\Gamma_G\Sigma_G^{-1}\right)$.

The proof of Theorem 1. Define a function ϕ by $\phi\left(x\right)=\int_0^{|x|}\varphi\left(t\right)dt,\forall x\in\mathbb{R}$ and define

$$m_t\left(\underline{\theta}\right)=\sum_{v=1}^{s}\left(\phi\left(\sigma_{t,\underline{\theta}}^{-1/2}\left(v\right)X_t\left(v\right)\right)+2^{-1}\log\left(\sigma_{t,\underline{\theta}}\left(v\right)\right)\right),t=0,...,N-1,$$

$$\widetilde{m}_t\left(\underline{\theta}\right)=\sum_{v=1}^{s}\left(\phi\left(\widetilde{\sigma}_{t,\underline{\theta}}^{-1/2}\left(v\right)X_t\left(v\right)\right)+2^{-1}\log\left(\widetilde{\sigma}_{t,\underline{\theta}}\left(v\right)\right)\right),t=0,...,N-1,$$

$$M_N\left(\underline{\theta}\right)=\sum_{t=0}^{N-1}m_t\left(\underline{\theta}\right),\ \widetilde{M}_N\left(\underline{\theta}\right)=\sum_{t=0}^{N-1}\widetilde{m}_t\left(\underline{\theta}\right).$$

Then, the first partial derivatives of $m_t\left(\underline{\theta}\right)$ and $\widetilde{m}_t\left(\underline{\theta}\right)$ (resp., $M_N\left(\underline{\theta}\right)$ and $\widetilde{M}_N\left(\underline{\theta}\right)$) are given by

$$\nabla_{\underline{\theta}}m_t\left(\underline{\theta}\right)=2^{-1}\sum_{v=1}^{s}\left(-K\left(\sigma_{t,\underline{\theta}}^{-1/2}\left(v\right)X_t\left(v\right)\right)+1\right)\sigma_{t,\underline{\theta}}^{-1}\left(v\right)\nabla_{\underline{\theta}}\sigma_{t,\underline{\theta}}\left(v\right),$$

$$\nabla_{\underline{\theta}}\widetilde{m}_t\left(\underline{\theta}\right)=2^{-1}\sum_{v=1}^{s}\left(-K\left(\widetilde{\sigma}_{t,\underline{\theta}}^{-1/2}\left(v\right)X_t\left(v\right)\right)+1\right)\widetilde{\sigma}_{t,\underline{\theta}}^{-1}\left(v\right)\nabla_{\underline{\theta}}\widetilde{\sigma}_{t,\underline{\theta}}\left(v\right),$$

$$\nabla_{\underline{\theta}}M_N\left(\underline{\theta}\right)=2^{-1}\sum_{t=0}^{N-1}\sum_{v=1}^{s}\left(-K\left(\sigma_{t,\underline{\theta}}^{-1/2}\left(v\right)X_t\left(v\right)\right)+1\right)\sigma_{t,\underline{\theta}}^{-1}\left(v\right)\nabla_{\underline{\theta}}\sigma_{t,\underline{\theta}}\left(v\right),$$

$$\nabla_{\underline{\theta}}\widetilde{M}_N\left(\underline{\theta}\right)=2^{-1}\sum_{t=0}^{N-1}\sum_{v=1}^{s}\left(-K\left(\widetilde{\sigma}_{t,\underline{\theta}}^{-1/2}\left(v\right)X_t\left(v\right)\right)+1\right)\widetilde{\sigma}_{t,\underline{\theta}}^{-1}\left(v\right)\nabla_{\underline{\theta}}\widetilde{\sigma}_{t,\underline{\theta}}\left(v\right).$$

Thus, our proof depends on checking the conditions of the Theorem 1 for the criterion functions $\widetilde{M}_N$. In the first step, we verify the above-mentioned conditions of Theorem 2 for the criterion functions M_N. The second partial derivatives of $m_t\left(\underline{\theta}\right)$ and $\widetilde{m}_t\left(\underline{\theta}\right)$

are given by

$$\begin{aligned}
\nabla^2_{\underline{\theta}} m_t(\underline{\theta}) &= 2^{-2}\sum_{v=1}^{s} R_1\left(\sigma^{-1/2}_{t,\underline{\theta}}(v)X_t(v)\right)\sigma^{-2}_{t,\underline{\theta}}(v)\nabla_{\underline{\theta}}\sigma_{t,\underline{\theta}}(v)\nabla_{\underline{\theta}'}\sigma_{t,\underline{\theta}}(v)\\
&\quad+\sum_{v=1}^{s} R_2\left(\sigma^{-1/2}_{t,\underline{\theta}}(v)X_t(v)\right)\sigma^{-2}_{t,\underline{\theta}}(v)\left(\sigma_{t,\underline{\theta}}(v)\nabla^2_{\underline{\theta}}\sigma_{t,\underline{\theta}}(v)\right.\\
&\quad\left.-\nabla_{\underline{\theta}}\sigma_{t,\underline{\theta}}(v)\nabla_{\underline{\theta}'}\sigma_{t,\underline{\theta}}(v)\right),\\
\nabla^2_{\underline{\theta}} \widetilde{m}_t(\underline{\theta}) &= 2^{-2}\sum_{v=1}^{s} R_1\left(\widetilde{\sigma}^{-1/2}_{t,\underline{\theta}}(v)X_t(v)\right)\widetilde{\sigma}^{-2}_{t,\underline{\theta}}(v)\nabla_{\underline{\theta}}\widetilde{\sigma}_{t,\underline{\theta}}(v)\nabla_{\underline{\theta}'}\widetilde{\sigma}_{t,\underline{\theta}}(v)\\
&\quad+\sum_{v=1}^{s} R_2\left(\widetilde{\sigma}^{-1/2}_{t,\underline{\theta}}(v)X_t(v)\right)\widetilde{\sigma}^{-2}_{t,\underline{\theta}}(v)\left(\widetilde{\sigma}_{t,\underline{\theta}}(v)\nabla^2_{\underline{\theta}}\widetilde{\sigma}_{t,\underline{\theta}}(v)\right.\\
&\quad\left.-\nabla_{\underline{\theta}}\widetilde{\sigma}_{t,\underline{\theta}}(v)\nabla_{\underline{\theta}'}\widetilde{\sigma}_{t,\underline{\theta}}(v)\right),
\end{aligned}$$

where $R_2(x) = 2^{-1}(1-K(x))$. Hence, **ii**. holds. For the rest of the proofs are very similar to those of Fan et al. [20]. □

The following result gives an estimator of the asymptotic variance

Theorem 3. *Assume that Assumptions* **H.0** $-$ **H.6**, $E\left\{g\left(c_K^{-1/2}\eta_0\right)\right\} < +\infty$ *and* $E\left\{X^2_{st+v}\right\} < +\infty$, $1 \le v \le s$ *hold, then*

$$\widehat{\sigma}^2_{K,N}\widehat{J}_N^{-1} \xrightarrow{p} \sigma^2_K J^{-1}, N \longrightarrow +\infty,$$

where

$$\begin{aligned}
\widehat{\sigma}^2_{K,N} = 4&\left((Ns)^{-1}\sum_{t=0}^{N-1}\sum_{v=1}^{s}K^2\left(X_t(v)\widehat{\sigma}^{-1/2}_{t,\widehat{\underline{\theta}}_{Ns}}(v)\right)\right.\\
&\left.-\left((Ns)^{-1}\sum_{t=0}^{N-1}\sum_{v=1}^{s}K\left(X_t(v)\widehat{\sigma}^{-1/2}_{t,\widehat{\underline{\theta}}_{Ns}}(v)\right)\right)^2\right)\\
&\times\left((Ns)^{-1}\sum_{t=0}^{N-1}\sum_{v=1}^{s}X_t(v)\widehat{\sigma}^{-1/2}_{t,\widehat{\underline{\theta}}_{Ns}}(v)\nabla K\left(X_t(v)\widehat{\sigma}^{-1/2}_{t,\widehat{\underline{\theta}}_{Ns}}(v)\right)\right)^{-2}
\end{aligned}$$

and $\widehat{J}_N := N^{-1}\sum_{t=0}^{N-1}\sum_{v=1}^{s}\widehat{\sigma}^{-2}_{t,\widehat{\underline{\theta}}_{Ns}}(v)\nabla\widehat{\sigma}_{t,\widehat{\underline{\theta}}_{Ns}}(v)\left(\nabla\sigma_{t,\widehat{\underline{\theta}}_{Ns}}(v)\right)'$.

Proof. The proof is very similar to those of Mukherjee [18]. It suffices to replace the stationarity and ergodicity arguments by the *SPS* and *PE* ones, respectively. For this we omit the proof. □

Remarks 1. (1) The large class of $M-$estimators for estimating the parameters of the *PTGARCH* model includes the *QMLE* as a special case and *LAD* estimator as well as many other useful estimators. It is worth noting that $M-$estimators also contain less-known estimators (see, Examples 2, 6 and 7 in Table 1), which can be considered an attractive alternative to the *QMLE*.

(2) Regarding the closed-form expression of the coefficients c_K in Assumption $H.0$ are $c_K^{1/2} = E\{|\eta_0|\}$ for the *LAD* score and $c_K = E\{\eta_0^2\}$ for the *QMLE* score, $c_K = 1$ for the *MLE* score and $c_K = \left(aE\left\{|\eta_0|^b\right\}\right)^{2/b}$ for the exponential pseudo *MLE* score. For the other scores, c_K does not have a closed-form expression but the corresponding numerical values can be computed from $H.0$ and $H.3$ for various error distributions (for further discussion see [11]).

(3) It is worth noting that the condition $\gamma_L^{(s)}(\Lambda_0) < 0$ in $H.5$ provide the existence of a *SPS* solution to the *PTGARCH* models. Moreover, the solution is unique and *PE*. Now, the non stationarity condition stems from the fact that the *SPS* condition is not met, i.e., $\gamma_L^{(s)}(\Lambda_0) > 0$. Thus, it is useful and fruitful the study the asymptotic properties of $M-$estimators for non-stationary *PTGARCH* models (cf., [19], [14], [15], when $K(x) = x^2$).

(4) Francq and Zakoïan [7] established the asymptotic distribution of the *QMLE* when some coefficients are equal to zero, thus, it would be beneficial to generalize the asymptotic properties of $M-$estimaors for *PTGARCH* when $\underline{\theta}_0$ is on the boundary.

(5) Various authors have supposed that the period s is known and fixed (see Ghezal et al. [3], [6], [5], [4], [8], [9] and references therein), if otherwise, the period s is obviously an important parameter, hence one can estimate the parameter s using the periodogram, as we can consider it the most important method to determine the presence of periodicities (see Martin and Kedem [17] for more details).

4. Simulation Study

Now, we illustrate the $M-$estimators described in previous sections (at least for a moderate periodicity coefficients $s = 2$ say), we provide some numerical results from Monte Carlo experiment. We simulate 500 independent trajectories via some specifications of *PTGARCH* $(1,1)$ models with length $n \in \{1000, 3000\}$ with standard $\mathcal{N}(0,1)$ and Student$-t_{(10)}$ innovation distributions and computed the *QMLE*, *LAD*, and Huber's estimates with $k = 1.5$ for each replication with vector $\underline{\theta}$ of parameters described in the bottom of each table below which is chosen to satisfy the *SPS* condition. For each trajectory, the vector $\underline{\theta}$ has been estimated with these three $M-$estimators noted as $\widehat{\underline{\theta}}_n$. In Table below, the columns are correspond to the average of the parameters estimates over the 500 simulations. In order to show the performance of three $M-$estimators, the roots mean square error $(RMSE)$ of the each $\widehat{\underline{\theta}}_n(v)$, $v = 1, \ldots s$, (results between bracket), are reported in Table 2 below.

		$\mathcal{N}(0,1)$						$t_{(10)}$					
		QMLE		*LAD*		Huber,	$k=1.5$	*QMLE*		*LAD*		Huber,	$k=1.5$
	c_K	1		$2/\pi$		0.825		1		0.55		0.725	
Parameters	TV \ n	1000	3000	1000	3000	1000	3000	1000	3000	1000	3000	1000	3000
$a_0^{(K)}(1)$	c_K	0.9964	1.0037	0.6329	0.6389	0.8211	0.8278	0.9949	1.0039	0.5453	0.5529	0.7198	0.7292
		(0.0826)	(0.0679)	(0.0490)	(0.0450)	(0.0688)	(0.0392)	(0.0749)	(0.0612)	(0.0610)	(0.0511)	(0.0804)	(0.0594)
$a_0^{(K)}(2)$	$0.5c_K$	0.5032	0.4984	0.3216	0.3167	0.4157	0.4108	0.5099	0.4971	0.2846	0.2722	0.3650	0.3601
		(0.0559)	(0.0456)	(0.0844)	(0.0723)	(0.0908)	(0.0551)	(0.0607)	(0.0573)	(0.0932)	(0.0820)	(0.0921)	(0.0604)
$a_1^{(K)}(1)$	$0.5c_K$	0.4995	0.4996	0.3177	0.3180	0.4119	0.4121	0.5039	0.4994	0.2788	0.2796	0.3672	0.3620
		(0.0643)	(0.0397)	(0.0364)	(0.0223)	(0.0600)	(0.0343)	(0.0648)	(0.0407)	(0.0372)	(0.0230)	(0.0613)	(0.0360)
$a_1^{(K)}(2)$	$0.25c_K$	0.2459	0.2501	0.1552	0.1594	0.2018	0.2065	0.2419	0.2502	0.1301	0.1379	0.1767	0.1816
		(0.0678)	(0.0402)	(0.0372)	(0.0220)	(0.0642)	(0.0391)	(0.0686)	(0.0403)	(0.0388)	(0.0228)	(0.0646)	(0.0395)
$b_1^{(K)}(1)$	$0.25c_K$	0.2479	0.2493	0.1569	0.1584	0.2042	0.2056	0.2462	0.2481	0.1338	0.1355	0.1794	0.1806
		(0.0501)	(0.0303)	(0.0281)	(0.0169)	(0.0445)	(0.0247)	(0.0504)	(0.0305)	(0.0289)	(0.0174)	(0.0472)	(0.0275)
$b_1^{(K)}(2)$	$0.45c_K$	0.4429	0.4466	0.2802	0.2853	0.3682	0.3702	0.4424	0.4468	0.2400	0.2461	0.3239	0.3283
		(0.0799)	(0.0467)	(0.0440)	(0.0259)	(0.0754)	(0.0452)	(0.0799)	(0.0470)	(0.0453)	(0.0265)	(0.0760)	(0.0461)
$c_1^{(K)}(1)$	$0.15c_K$	0.1391	0.1426	0.0885	0.0904	0.1129	0.1284	0.1377	0.1424	0.0722	0.0769	0.0993	0.1016
		(0.0269)	(0.0266)	(0.0562)	(0.0529)	(0.0225)	(0.0131)	(0.0290)	(0.0268)	(0.0675)	(0.0608)	(0.0441)	(0.0437)
$c_1^{(K)}(2)$	$0.55c_K$	0.5525	0.5518	0.3532	0.3524	0.4553	0.4556	0.5527	0.5622	0.3054	0.3047	0.4011	0.3996
		(0.0297)	(0.0283)	(0.0416)	(0.0379)	(0.0365)	(0.0192)	(0.0355)	(0.0290)	(0.0354)	(0.0299)	(0.0387)	(0.0238)

Table 2: Average and *RMSE* of 500 simulations of three $M-$estimators for $PTGARCH\,(1,1)$

Now, a few comments are in order. Table 2 compares the asymptotic parameters estimates and their *RMSE* over 500 independent simulations of the periodic $TGARCH(1,1)$ for sample sizes $n=1000$ and $n=3000$. As expected the estimates of the periodic $TGARCH(1,1)$ coefficients based on *QMLE*, *LAD*, and Huber's displayed *RMSE* decrease as the sample size increases. We can also see that the values of the estimates of the *QMLE*, *LAD*, and Huber's corresponding to $\mathcal{N}(0,1)$ are generally found to be more efficient than t_{10} ones. Furthermore, *LAD*$-$estimator performs even better compared to all its competitors in terms of even smaller *RMSE*.

5. An Application to Exchange Rates

In this section, we apply our model for modeling the two daily time series $(X_{1t}, X_{2t})_{t\geq 1}$ of exchange rates: Euro/Algerian dinar (EUR/DZD) and U.S. dollar/ Algerian dinar (USD/DZD) provided by the Bank of Algeria. We first removed all the days where market was closed (i.e., holidays and weekends). The observations cover the period from January 3, 2000 to September 29, 2011. The final sample consists of 3055 observations for each series. In order to do so, we collect the two exchange rates at time t,

$$X_{1t} = EUR/DZD_t,\ X_{2t} = USD/DZD_t,$$

and their corresponding log $-$return series r_{1t} and r_{2t}, where $r_{kt} = 100\log\left(X_{kt}/X_{kt-1}\right)$, for $k=1,2$. The graphics of prices X_{1t} and X_{2t}, the daily returns series of prices r_{1t}, r_{2t}, squad and absolute returns are plotted in Fig 1

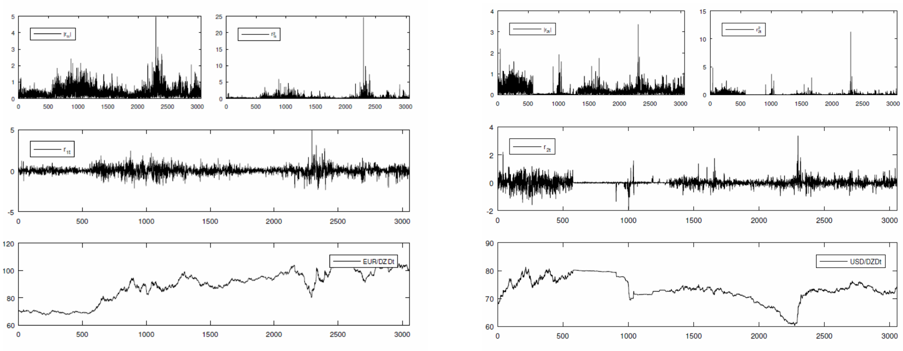

FIGURE 1. The plots of the price series $(X_{kt}), (r_{kt}), (r_{kt}^2)$ and $(|r_{kt}|), k = 1, 2$.

Some elementary descriptive statistics are provided for the two log $-$return series in Table 3.

Series	mean	Std. Dev	Median	Skewness	Kurtosis	Min	Max
r_{1t}	0.1000	5.0000	0.1000	0.4000	9.0000	-23.300	49.700
r_{2t}	0.0000	3.0000	0.0000	1.0000	13.000	-19.000	34.000

Table 3: Summary statistics for returns series $(r_{1t}, r_{2t})_{t\geq 1}$.

Continuation of Table 3

Series	Arch(300)	J. Bera	LBtest
r_{1t}	100%	4.597×10^3	87.33%
r_{2t}	100%	1.301×10^4	100%

The findings indicate that for the EUR/DZD (resp. USD/DZD), the lowest returns (-23.300) (resp. -19.000) and the highest returns (49.700) (resp. 34.000). The skewness for two log $-$return series is positive. Moreover, one of the features which prominently stands out most from Table 3 is that the kurtosis for two log $-$returns is much larger than 3, which indicates that the models based on the Gaussian assumption may not well describe the data.

From Fig 2, that $(r_{kt})_{t\geq 1, k=1,2}$ presents a Taylor-effect (characterized by $\widehat{\rho}_{|r_{kt}|}(h) > \widehat{\rho}_{r_{kt}^2}(h)$ for some lag $h \geq 1$). Now, we will propose the 5$-$periodic $TGARCH(1,1)$ model that allows for the description of intraweek effect in the daily exchange rate, where the parameters are allowed to vary with the day of the week ($v = 1, 2, 3, 4$ and 5)

$$v = \begin{cases} 1 \text{ if the day corresponding is a Monday} \\ \vdots \\ 5 \text{ if the day corresponding is a Friday} \end{cases}.$$

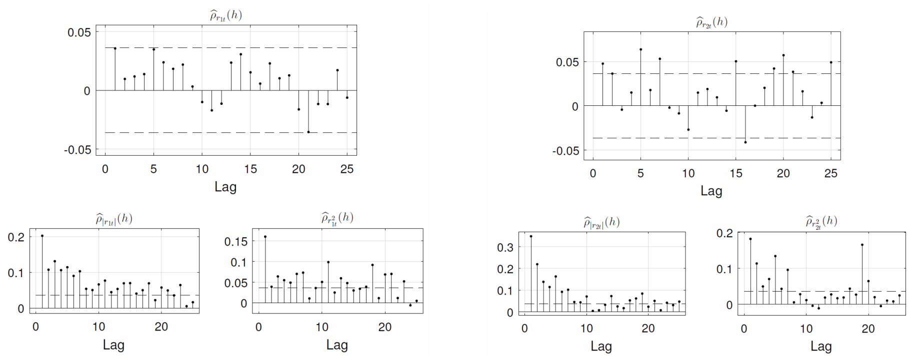

FIGURE 2. Sample autocorrelations of returns associated to Euro and Dollar.

The estimated parameters of the 5−periodic $TGARCH\,(1,1)$ model and their $RMSE$ are reported in Tables $4-5$.

	QMLE				*LAD*			
Days	$\widehat{a}_0^K$	$\widehat{a}_1^K$	$\widehat{b}_1^K$	$\widehat{c}_1^K$	$\widehat{a}_0^K$	$\widehat{a}_1^K$	$\widehat{b}_1^K$	$\widehat{c}_1^K$
Monday	0.0308	0.0239	0.0259	0.0986	0.0013	0.0089	0.0067	0.3464
	(0.0403)	(0.0165)	(0.0177)	(0.1130)	(0.0008)	(0.0018)	(0.0016)	(0.1043)
Tuesday	0.0434	0.0277	0.0363	0.1154	0.0051	0.0031	0.0028	0.2154
	(0.0370)	(0.0135)	(0.0145)	(0.0893)	(0.0004)	(0.0007)	(0.0008)	(0.0560)
Wednesday	0.0489	0.0255	0.0232	0.1348	0.0048	0.0095	0.0091	3.3459
	(0.0311)	(0.0145)	(0.0127)	(0.0713)	(0.0006)	(0.0009)	(0.0008)	(0.1415)
Thursday	0.0413	0.0196	0.0180	0.1105	0.0117	0.0106	0.0096	5.0442
	(0.0316)	(0.0146)	(0.0127)	(0.0825)	(0.0012)	(0.0013)	(0.0014)	(0.1880)
Friday	0.0291	0.0183	0.0234	0.0884	0.0048	0.0031	0.0086	0.5705
	(0.0331)	(0.0148)	(0.0179)	(0.0964)	(0.0009)	(0.0014)	(0.0020)	(0.1103)

Table 4. *QMLE*, *LAD*, Huber's estimates from the Euro data and their *RMSE*.

	QMLE				*LAD*			
Days	$\widehat{a}_0^K$	$\widehat{a}_1^K$	$\widehat{b}_1^K$	$\widehat{c}_1^K$	$\widehat{a}_0^K$	$\widehat{a}_1^K$	$\widehat{b}_1^K$	$\widehat{c}_1^K$
Monday	0.0480	0.0749	0.0870	0.2028	0.0332	0.0933	0.0792	0.0952
	(0.0062)	(0.0295)	(0.0224)	(0.0606)	(0.0153)	(0.0301)	(0.0377)	(0.0183)
Tuesday	0.0301	0.0800	0.0453	0.1269	0.0051	0.0697	0.2578	0.1409
	(0.0058)	(0.0168)	(0.0128)	(0.0412)	(0.0006)	(0.0327)	(0.0670)	(0.0217)
Wednesday	0.0220	0.1063	0.0526	0.0792	0.0037	0.1128	0.0259	0.0458
	(0.0051)	(0.0173)	(0.0174)	(0.0381)	(0.0018)	(0.0508)	(0.0097)	(0.0195)
Thursday	0.0219	0.0755	0.0529	0.1016	0.0100	0.1159	0.0903	0.3014
	(0.0045)	(0.0193)	(0.0127)	(0.0440)	(0.0086)	(0.0991)	(0.0341)	(0.1665)
Friday	0.0308	0.0968	0.0562	0.1458	0.0222	0.0468	0.0521	0.0560
	(0.0055)	(0.0159)	(0.0191)	(0.0485)	(0.0119)	(0.0249)	(0.0139)	(0.0069)

Table 5. *QMLE*, *LAD*, Huber's estimates from the Dollar data and their *RMSE*.

	Huber,	$k = 1.5$		
Days	$\widehat{a}_0^K$	$\widehat{a}_1^K$	$\widehat{b}_1^K$	$\widehat{c}_1^K$
Monday	0.1808	0.1061	0.6063	0.3197
	(0.0333)	(0.0155)	(0.0835)	(0.1352)
Tuesday	0.1849	0.6622	4.0544	0.1452
	(0.0368)	(0.0271)	(0.0672)	(0.0934)
Wednesday	0.8388	5.0880	0.5673	0.5964
	(0.0146)	(0.0493)	(0.0230)	(0.0350)
Thursday	0.4958	0.2560	0.1840	1.9462
	(0.0416)	(0.0725)	(0.0558)	(0.1735)
Friday	0.1753	0.2997	0.2316	0.3591
	(0.0216)	(0.0373)	(0.0758)	(0.0608)

Continuous of Table 4.

	Huber,	$k = 1.5$		
Days	$\widehat{a}_0^K$	$\widehat{a}_1^K$	$\widehat{b}_1^K$	$\widehat{c}_1^K$
Monday	0.0390	0.0999	0.0670	0.1465
	(0.0270)	(0.0369)	(0.0437)	(0.0323)
Tuesday	0.0198	0.2140	0.2896	0.0510
	(0.0137)	(0.0739)	(0.0804)	(0.0304)
Wednesday	0.0038	0.0920	0.0114	0.0039
	(0.0004)	(0.0215)	(0.0016)	(0.0018)
Thursday	0.0088	0.1271	0.1389	0.2626
	(0.0062)	(0.0731)	(0.0615)	(0.1380)
Friday	0.0439	0.7237	0.1481	0.0658
	(0.0140)	(0.0301)	(0.0916)	(0.0363)

Continuous of Table 5.

6. CONCLUSION

In this paper, we consider estimating the parameters of the periodic asymmetric $GARCH(p,q)$ model using $M-$estimators. Regardless of the usually-used $QMLE$, alternative estimators based on the LAD and Huber's score functions are used. Simulation results expose that some of these estimators can perform better than the $QMLE$. Finally, the methodology is illustrated through a simulation study and an empirical application of two daily time series of exchange rates: Euro/Algerian dinar (EUR/DZD) and $U.S.$ dollar/ Algerian dinar (USD/DZD) provided by the Bank of Algeria.

REFERENCES

[1] A. A, Aknouche, "Quasi-maximum likelihood estimation of periodic *garch* and periodic *arma* − *garch* processes." vol. 29, no. 1, pp. 19–45, 2008, doi: 10.1111/j.1467-9892.2008.00598.x.

[2] A. A. A Bibi, "On periodic *garch* processes: stationarity, existence of moments and geometric ergodicity." vol. 17, no. 4, pp. 305–316, 2008, doi: 10.3103/S1066530708040029.

[3] A. G. A Bibi, "Asymptotic properties of *qmle* for periodic asymmetric strong and semi-strong *garch* models." 2017.

[4] G. A. Bibi, A., "On periodic time-varying bilinear processes: structure and asymptotic inference," vol. 25, no. 3, pp. 395–420, 2016, doi: 10.1007/s10260-015-0344-5.

[5] G. A. Bibi, A., "*qmle* of periodic bilinear models and of *parma* models with periodic bilinear innovations," vol. 54, no. 2, pp. 375–399, 2018, doi: 10.14736/kyb-2018-2-0375.

[6] G. A. Bibi, A., "*qmle* of periodic time-varying bilinear$-$*garch* models." vol. 48, no. 13, pp. 3291–3310, 2019, doi: 10.1080/03610926.2018.1476703.
[7] Z. J. Francq, C., "Quasi-maximum likelihood estimation in *garch* processes when some coefficients are equal to zero," vol. 117, no. 9, pp. 1265–1284, 2004, doi: 10.1016/j.spa.2007.01.001.
[8] A. Ghezal, "*qmle* for periodic time-varying asymmetric log *garch* models." vol. 9, no. 1, pp. 273–297, 2021, doi: 10.1007/s40304-019-00193-4.
[9] A. Ghezal, "Asymptotic inference for periodic time-varying bivariate poisson $ingarch(1,1)$ processes," vol. 10, no. 1, pp. 77–82, 2023, doi: 10.18576/jsapl/100106.
[10] K. A. Guerbyenne, H., "Power periodic threshold *garch* model: Structure and estimation." vol. 48, no. 19, pp. 4834–4860, 2019, doi: 10.1080/03610926.2018.1496258.
[11] L. H. Hallin, M. and K. Mukherjee, "$m-$estimation in *garch* models in the absence of higher-order moments," pp. 1–23, 2020.
[12] L. H. I, Berkes and P. Kokoszka., "Garch processes: structure and estimation." vol. 9, no. 2, pp. 201–207, 2003, doi: 10.3150/bj/1068128975.
[13] M. K. Iqbal, F., "$m-$estimators of some $garch-$type models: computation and application," vol. 20, pp. 435–445, 2010, doi: 10.1007/s11222-009-9135-x.
[14] R. A. Jensen, S.T., "Asymptotic normality of the *qmle* estimator of *arch* in the nonstationary case," vol. 72, no. 2, pp. 641–646, 2004a, doi: 10.1111/j.1468-0262.2004.00504.x.
[15] R. A. Jensen, S.T., "Asymptotic inference for nonstationary *garch*," vol. 20, no. 6, pp. 1203–1226, 2004b, doi: 10.1017/S0266466604206065.
[16] N. P. I. Klimko, L.A., "On conditional least squares estimation for stochastic processes," vol. 6, no. 3, pp. 629–642, 1978, doi: 10.1214/aos/1176344207.
[17] K. B. Martin, D.E.K., "Estimation of the period of periodically correlated sequences," vol. 14, no. 2, pp. 193–205, 1993, doi: 10.1111/j.1467-9892.1993.tb00137.x.
[18] K. Mukherjee, "$m-$estimation in *garch* models," vol. 24, no. 6, pp. 1530–1553, 2008, doi: 10.1017/S0266466608080602.
[19] C. T. N.H.Chan, "Statistical inference for nonstationary $garch(p,q)$ models." vol. 3, pp. 956–992, 2009, doi: 10.1214/09-EJS452.
[20] Z.-L. P-YFan, S-X.Wu and MChen, "$m-$estimation for periodic *garch* model with high-frequency data." vol. 33, no. 3, pp. 717–730, 2017, doi: 10.1007/s10255-017-0694-x.

Authors' addresses

Ahmed Ghezal
(**Corresponding author**) University Center of Mila, Department of Mathematics and Computer Sciences, Algeria.
E-mail address: a.ghezal@centre-univ-mila.dz

Imane Zemmouri
University of Annaba, Department of Mathematics, University of Annaba, Elhadjar 23, Annaba, Algeria.
E-mail address: imanezemmouri25@gmail.com