Miskolc Mathematical Notes
Vol. 26 (2025), No. 2, pp. 813–823
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4745

BKN RINGS: A STUDY OF NONZERO HOMOMORPHISMS BETWEEN MODULES AND SOME APPLICATIONS TO HOMOLOGICAL ALGEBRA

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Received 26 May, 2023

Abstract. In this paper, we study BKN rings, which are rings that satisfy the property $\mathrm{Hom}_R(M,N) \neq 0$ for all nonzero left (right) R-modules M and N. Our aim is to provide a comprehensive overview of BKN rings and their connections to other types of rings, such as retractable and completely coretractable rings, commutative rings, and their connections to injective R-modules and non-trivial torsion theories. We prove several results, including the left-right symmetry of the BKN property and its preservation under various ring operations. Furthermore, we explore the relationship between BKN rings and the tensor product of modules. Our study provides insight into the comprehensive understanding of BKN rings, providing information about the relation between BKN rings and the Tor, Ext, and ext functors.

2010 *Mathematics Subject Classification:* 16S90; 16D90; 16E30

Keywords: BKN ring, left local, left semiartinian, left MAX

1. Introduction

In this article, all rings are assumed to be associative with identity. The category of left (right) unitary modules over the ring R is denoted by R-Mod (Mod-R), and when the side of an R-module is not specified, it will be considered as an R-bimodule.

The property $\mathrm{Hom}_R(M,N) \neq 0$ for certain R-modules M and N has been the subject of various studies. For example, in [12], *retractable rings* were examined. These are rings where nonzero homomorphisms exist between modules and their nonzero submodules. Dual to this concept, *coretractable rings* were investigated in [3]. In [9], it was proved that a fully right-bounded right Noetherian ring R (*FBN ring*) has the property that $\mathrm{Hom}_R(M,N) \neq 0$ for any nonzero submodule N of a nonzero right R-module M if and only if $\mathrm{Hom}_R(M,R/P) \neq 0$ for every associated prime ideal P of M. Additionally, nonzero homomorphisms between nonzero injective (projective) modules have been recently studied in [6] and [13]. The general case of nonzero homomorphisms between arbitrary nonzero modules M and N has been characterized in [5] by Bican, Kepka, and Němec. Inspired by this characterization, a ring R is referred to as a *left* (or *right*) *BKN ring* if $\mathrm{Hom}_R(M,N) \neq 0$ for all nonzero left (or right) R-modules M and N. A BKN ring is a ring that is both a left and right BKN ring. This term was first introduced in 2017 by Rincón-Mejía et al. in [2]. They established, among other interesting properties, that a ring R is a finite direct product of left BKN rings if and only if the lattices of R-tors (hereditary torsion classes), R-nat (natural classes), and R-conat (conatural classes) coincide. Moreover, it was shown in [1] that if R is a nonsemisimple BKN ring, then the operator $\phi_Z\colon R\text{-nat} \to R\text{-nat}$, defined by $\phi_Z(\mathcal{C}) = \{M \in R\text{-Mod} | Z(M) \in \mathcal{C}\}$ (where $Z(M)$ denotes the singular submodule of M), is a homomorphism of boolean algebras.

This paper is organized as follows: Section 2 provides a comprehensive overview of left BKN rings and their connections to injective left R-modules, retractable and completely coretractable rings, and other types of rings. In this section, several equivalent properties of left BKN rings are established. For instance, in Theorem 2, we prove that a ring R is a left BKN ring if and only if it is a left MAX ring and every nonzero injective left R-module is a cogenerator. This, in turn, is equivalent to there being no nontrivial torsion theories for the category of left R-modules. Furthermore, in Theorem 3, we establish that the property of being a BKN ring is left-right symmetric. In addition, we present some results regarding the preservation of the BKN property under various ring operations. For example, in Theorem 4, we prove that if a unitary ring homomorphism from a BKN ring R to another ring S is such that the Change of Rings functor is full, then S is also a BKN ring. We also examine the relationship between BKN rings and the tensor product of modules, showing in Proposition 2 that if R is a BKN ring, then the tensor product of any nonzero right R-module L with any nonzero R-module M is nonzero.

Finally, in Section 3, our focus is to determine conditions under which $\mathrm{Tor}_n^R(M,N)$, $\mathrm{Ext}_R^n(N,M)$, and $\mathrm{ext}_R^n(M,N)$ are zero. Here, N represents a fixed nonzero R-module, while M is a nonzero right (or left) R-module, and R is a BKN ring. Also in Theorem 7, we establish that $\_\otimes_R N$ reflects epimorphisms if and only if $M \otimes_R N \neq 0$ for each nonzero right R-module M. Dual results are obtained for the functors $\mathrm{Hom}_R(N,\_)$ and $\mathrm{Hom}_R(\_,N)$.

2. BKN RINGS

2.1. *Preliminary results*

Our main goal is to investigate the properties and implications of left and right BKN rings. As mentioned earlier, a characterization of this kind of ring appears in [5]. For the sake of clarity, we include a proof of this result, which can be also be found in [1].

Recall that a ring R is *left local* if there is a unique (up to isomorphism) nonzero simple left module in R-**Mod**; *left semiartinian* if each nonzero left R-module M contains an essential socle, or equivalently, contains a left simple module; *left MAX* if every left module has a simple quotient.

Theorem 1. *The ring R is left BKN if and only if R is left local, left semiartinian, and left MAX.*

Proof. Suppose that R is a left BKN ring. In this case, any nonzero homomorphism between two left simple modules must be an isomorphism. Therefore, R is left local. Furthermore, if M is a nonzero left R-module and S is a simple left R-module, then any nonzero homomorphisms $S \longrightarrow M$ and $M \longrightarrow S$ must be a monomorphism and an epimorphism, respectively. It follows that R is left semiartinian and left MAX.

Conversely, let M and N be nonzero left R-modules. If S is a left simple R-module, then there exists an epimorphism $M \xrightarrow{\alpha} S$ and a monomorphism $S \xrightarrow{\beta} N$. Hence $0 \neq \beta\alpha \in \mathrm{Hom}_R(M,N)$. □

The following result is a consequence of the definitions and other known results on ring and module theory.

Proposition 1. *Over a left BKN ring R, every nonzero injective left R-module is a cogenerator.*

2.2. *Retractable and completely coretractable rings*

Recall that a ring R is *left mod-retractable* if $\mathrm{Hom}_R(M,N) \neq 0$ for each nonzero left R-module M and each nonzero left R-submodule N of M.

Theorem 2. *The following properties of a ring R are equivalent:*

(i) *R is left mod-retractable, and each nonzero injective left R-module is a cogenerator,*
(ii) *R is left MAX, and each nonzero injective left R-module is a cogenerator,*
(iii) *R is a left BKN ring,*
(iv) *There are no nontrivial torsion theories for the category of left R-modules.*

Proof. (i)⇒(ii). It follows from [11, Theorem 3.3].

(ii)⇒(iii) Let M,N be nonzero left R-modules. Since R is left MAX there exists a simple left R-module S and a nonzero epimorphism $f\colon M \to S$. Since $E(N)$ is a cogenerator, $\mathrm{Hom}_R(S,E(N)) \neq 0$. This entails S embeds in N. The composition of f with this embedding yields a nonzero homomorphism from M to N, as required.

(iii)⇒(iv). Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for R-**Mod** and suppose that $0 \neq M \in \mathcal{T}$ and $N \in \mathcal{F}$. Then $\mathrm{Hom}_R(M,N) = 0$, and since R is left BKN, this implies that $N = 0$. Therefore, $\mathcal{F} = 0$ and $\mathcal{T}$ is the class of all left R-modules.

(iv)⇒(i). Let E be a nonzero injective left R-module. Then E cogenerates the torsion theory $(0,\mathcal{F})$, where $\mathcal{F}$ is the class of all left R-modules. This shows that E

is certainly an injective cogenerator. Finally, note that the only two torsion theories for R-**Mod** are hereditary. It follows from [11, Theorem 3.5] that R is left mod-retractable. □

Following [3], a ring R is *left completely coretractable* (*left CC* for short) if, for each nonzero left module M and each proper left R-submodule N, there exists a nonzero homomorphism $M/N \to M$. Hence, we have the following theorem:

Theorem 3. *The following statements are equivalent for a ring R:*

(i) *R is a left BKN ring,*
(ii) *R is a left CC and a left local ring,*
(iii) *R is a right BKN ring,*
(iv) *R is a right CC and a right local ring.*

Proof. (i)⇒(ii) and (iii)⇒(iv) are clear.

(iv)⇒ (iii). According to [3, Theorem 3.10], R is a two-sided perfect ring. Consequently, it is right semiartinian as stated in [10, Proposition 5.1, pag. 189], and it is also a right MAX ring based on [4, Theorem 28.4]. Therefore, applying Theorem 1, we conclude that R is a right BKN ring. Similarly (ii)⇒(i).

(i)⇒(iii). At this stage, we can suppose that R is a two-sided perfect ring and consequently R is a right MAX and semiartinian ring. Also, R/J is a semisimple ring with only one isomorphism class of left simple R/J-modules, with J being the Jacobson radical. By the Wedderburn-Artin Theorem, we conclude that there is only one isomorphism class of right simple R/J-modules. As every right simple R-module is annihilated by J, we obtain the desired result. In a similar manner, we obtain (iii)⇒(i). □

By conditions (i) and (iii) in the above theorem, left BKN rings are also right BKN rings, and vice versa. Therefore, we can simply refer to such rings as BKN rings. Consequently, Theorem 2 will hold when the phrase "left BKN" is replaced by "right BKN" in condition (iii) of that theorem. Moreover, Theorem 2 will hold if the word "left" is randomly replaced by "right" in conditions (i) through (iv).

2.3. *Morita equivalent BKN rings*

Given two unitary rings R and S, a ring homomorphism $\rho\colon R \to S$ is called *unitary* if ρ maps the unit element of R to the unit element of S. To every unitary ring homomorphism $\rho\colon R \to S$, the well-known *Change of Rings functor* (see [7]) $\mathbf{F}_\rho\colon S\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ is defined as follows: For a left S-module M, let the additive group of $\mathbf{F}_\rho(M)$ be equal to the underlying abelian group M, and the structure of a left R-module is defined by

$$rm = \rho(r)m, \qquad m \in M, r \in R.$$

Let $f\colon M \to N$ be a homomorphism of left S-modules. Then $\mathbf{F}_\rho(f)(rm) = f(\rho(r)m) = \rho(r)f(m) = rf(m)$. Thus, every S-homomorphism is an R-homomorphism. It follows as a consequence that $\mathrm{Hom}_S(M,N) \leq \mathrm{Hom}_R(M,N)$.

Finally, if ρ is surjective, then we have that $\mathrm{Hom}_S(M,N) = \mathrm{Hom}_R(M,N)$.

Theorem 4. *If $\rho\colon R \to S$ is a unitary ring homomorphism with $\mathbf{F}_\rho$ being a full functor and R being a BKN ring, then S is also a BKN ring.*

Proof. If M and N are nonzero left S-modules, then $\mathbf{F}_\rho(M)$ and $\mathbf{F}_\rho(N)$ are nonzero left R-modules. Since R is a left BKN ring, there exists a nonzero homomorphism $f\colon \mathbf{F}_\rho(M) \to \mathbf{F}_\rho(N)$. By hypothesis, there exists $g \in \mathrm{Hom}_S(M,N)$ such that $\mathbf{F}_\rho(g) = f$. Since $f \neq 0$, then g must be a nonzero homomorphism. □

Corollary 1. *If $\rho\colon R \to S$ is a surjective unitary ring homomorphism with R being a BKN ring, then S is also a BKN ring.*

Proof. Since for every left S-modules M and N, we have $\mathrm{Hom}_S(M,N) = \mathrm{Hom}_R(M,N)$, the Change of Rings functor $\mathbf{F}_\rho$ is full and, by Theorem 4, we are done. □

Recall that a left R-module M is a *progenerator* if it is finitely generated, projective, and a generator. With this in mind, we have the following theorem:

Theorem 5. *In the situation of Corollary 1, if $Q = \mathbf{F}_\rho(S)$ is a projective left R-module, then R and S are Morita equivalent BKN rings.*

Proof. Suppose that Q is a projective left R-module. Since R is a BKN ring, $\mathrm{Hom}_R(Q,C) \neq 0$ for each nonzero left R-module C. It follows from [10, Proposition 6.3. Pag. 93] that Q is a generator module. After noting that Q is finitely generated by 1_S, we obtain that Q is actually a progenerator module. Now, we know that $\mathrm{End}_R(Q)$ and $\mathrm{End}_S(S)$ are the same as abelian groups. Moreover, we claim that they are isomorphic rings. Indeed, the assignment $\alpha \mapsto \mathbf{F}_\rho(\alpha)$ is a ring isomorphism from $\mathrm{End}_S(S)$ onto $\mathrm{End}_R(Q)$. But $\mathrm{End}_S(S) \cong S$ by [7, Lemma 3.7.3]. The result now follows from [4, Corollary 22.4]. □

Corollary 2. *In the situation of Corollary 1 if $Q = \mathbf{F}_\rho(S)$ and there exists a R-homomorphism $\varphi\colon Q \to R$ with $\varphi(1_S) = 1_R$, then R and S are Morita equivalent BKN rings.*

Proof. Each element of Q has the form $x = r \cdot 1_S$ for some $r \in R$. Hence,

$$\varphi(x) \cdot 1_S = \varphi(r \cdot 1_S) \cdot 1_S = (r\varphi(1_S)) \cdot 1_S = (r \cdot 1_R) \cdot 1_S = r \cdot 1_S = x.$$

According to [10, Proposition 6.3, p.p. 20], Q is a projective left R-module, and the proof is concluded by virtue of the preceding theorem. □

2.4. *BKN rings and Tensor product*

Given two rings R and S and modules L_R, ${}_RM_S$, N_S, from the well-known natural isomorphism

$$\mathrm{Hom}_S(L \otimes_R M, N) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_S(M, N)), \tag{2.1}$$

the following proposition is obtained:

Proposition 2. *If R is a BKN ring, then for any nonzero modules L_R and ${}_RM_R$, $L \otimes_R M \neq 0$.*

Proof. This can be proven by taking $N = M$ and $R = S$ in formula (2.1). □

Definition 1. A nonzero right S-module N is *a right anti-terminal* S-module whenever $\mathrm{Hom}_S(M, N) \neq 0$ for all nonzero right S-modules M.

Note that every right injective cogenerator S-module is an anti-terminal right S-module. Also, if $L \otimes_R M$ is nonzero for all nonzero modules L_R, and ${}_RM_S$, then the following proposition shows that $\mathrm{Hom}_S(M, N)$ must be an anti-terminal right S-module.

Proposition 3. *If R and S are rings such that $L \otimes_R M \neq 0$ for all nonzero modules L_R and ${}_RM_S$, and if N is an anti-terminal right S-module, then $\mathrm{Hom}_S(M, N)$ is an anti-terminal right R-module for every nonzero $(R$-$S)$-bimodule M.*

Proof. Let L be a nonzero right R-module. Since $L \otimes_R M \neq 0$ and N is anti-terminal, we obtain $\mathrm{Hom}_S(L \otimes_R M, N) \neq 0$. Hence, the right-hand side of (2.1) is nonzero, and this establishes the result. □

2.5. *Commutative BKN rings*

Since the notions semiartinian and mod-retractable coincide for commutative rings ([11, Theorem 3.10]), together with Theorem 2, we obtain the following:

Proposition 4. *The following conditions are equivalent for a commutative ring R:*

(i) *R is mod-retractable and each nonzero injective R-module is a cogenerator,*
(ii) *R is semiartinian and each nonzero injective R-module is a cogenerator,*
(iii) *R is MAX and each nonzero injective R-module is a cogenerator,*
(iv) *R is a BKN ring,*
(v) *There are no nontrivial torsion theories.*

Theorem 6. *The following sentences are equivalent for a commutative ring R:*

(i) *If L and M are nonzero R-modules, then $L \otimes_R M \neq 0$,*
(ii) *R is a local ring with a unique maximal ideal J, such that for any R-module M, $M = JM$ implies $M = 0$,*
(iii) *R is a BKN ring.*

Proof. (i)$\Rightarrow$(ii). Suppose that $L \otimes_R M \neq 0$ whenever L and M are nonzero R-modules. Since $R/I_1 \otimes_R R/I_2 \cong R/(I_1 + I_2)$, the sum of two proper ideals is a proper ideal, and therefore R is a local ring with a unique maximal ideal J. Let M be an R-module such that $M = JM$. Since $JM \otimes_R R/J = 0$, we must have that $M = 0$.

(ii)$\Rightarrow$(iii). Notice that as R is a local ring, by virtue of Theorem 1, it suffices to show that R is semiartinian and MAX. By [4, Lemma 28.3], J is a T-nilpotent ideal. Since R/J is semisimple (by [4, Theorem 28.4]), we can conclude that R is a perfect ring. Furthermore, by part (c) of [4, Theorem 28.4], R is a MAX ring. Finally, [10, Proposition 5.1, p.p. 189] implies that R is semiartinian.

(iii)$\Rightarrow$(i). Let R be a BKN ring, and let L and M be nonzero R-modules. Using Proposition 2 we see that $L \otimes_R M \neq 0$. □

3. BKN RINGS AND THE FUNCTORS TOR, EXT, AND EXT

In this section, we establish a relationship between BKN rings and the functors Tor, Ext and ext. Our goal is to determine when $\mathrm{Tor}_n^R(M,N)$, $\mathrm{Ext}_R^n(N,M)$, and $\mathrm{ext}_R^n(M,N)$ are nonzero for an arbitrary (but fixed) nonzero R-module N, provided M is a nonzero right (left) R-module and R is a BKN ring.

First, let us recall that an additive covariant (respectively, contravariant) functor T between abelian categories *reflects epimorphisms* if the exactness of $TM \longrightarrow TK \longrightarrow 0$ (respectively, $TK \longrightarrow TM \longrightarrow 0$) implies the exactness of $M \longrightarrow K \longrightarrow 0$ (respectively, $0 \longrightarrow M \longrightarrow K$). A similar definition is valid if we replace "epimorphisms" with "monomorphisms".

Definition 2. An additive covariant (contravariant) functor T between abelian categories is *anti projective* if for every exact nonsplit short sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$, the exactness of $0 \longrightarrow TK \longrightarrow TL \longrightarrow TM \longrightarrow 0$ (respectively, $0 \longrightarrow TM \longrightarrow TL \longrightarrow TK \longrightarrow 0$) implies that $TK = 0$. The functor is *anti injective* if for every nonsplit exact sequence $0 \longrightarrow M \longrightarrow L \longrightarrow K \longrightarrow 0$, the exactness of $0 \longrightarrow TM \longrightarrow TL \longrightarrow TK \longrightarrow 0$ (respectively, $0 \longrightarrow TK \longrightarrow TL \longrightarrow TM \longrightarrow 0$) implies that $TK = 0$.

Notice that if **C** is an abelian category where every short exact sequence splits, then any covariant (contravariant) additive functor having **C** as the source is anti projective (anti injective). Examples of these sorts of categories are *spectral categories* (see [10, Pag. 128]). A Grothendieck category whose objects are all projectives (and consequently injectives) is a spectral category (*e.g.*, the category R-**Mod**, for R a semisimple ring).

3.1. *The functor Tor*

Here, we study the relationship between BKN rings and the functor Tor. We begin with the following theorem:

Theorem 7. *Let R be a ring and N be a nonzero left R-module. The functor $\_\otimes_R N$ reflects epimorphisms if and only if $M \otimes_R N \neq 0$ for each nonzero right R-module M.*

Proof. Let M be a right R-module, and consider any short exact sequence

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} M \longrightarrow 0$$

in Mod-R of the form shown (such short exact sequences clearly exist). Applying the right exact functor $\_\otimes_R N$, yields the right exact sequence

$$X \otimes_R N \xrightarrow{\alpha\otimes 1_N} Y \otimes_R N \xrightarrow{\beta\otimes 1_N} M \otimes_R N \longrightarrow 0.$$

Suppose $M \otimes_R N = 0$. Then $\alpha \otimes 1_N$ is an epimorphism. Since $\_\otimes_R N$ reflects epimorphisms, α is an epimorphism, whence $\beta = 0$ and $M = 0$.

For the converse, let $\alpha\colon A \to B$ be an arbitrary homomorphism of right R-modules and suppose that $\alpha \otimes 1_N : A \otimes_R N \to B \otimes_R N$ is an epimorphism. Applying $\_\otimes_R N$ to the exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\pi} \operatorname{Coker}\alpha \longrightarrow 0$$

we obtain:

$$A \otimes_R N \xrightarrow{\alpha\otimes 1_N} B \otimes_R N \xrightarrow{\pi\otimes 1_N} \operatorname{Coker}\alpha \otimes_R N \longrightarrow 0.$$

Since $\alpha \otimes 1_N$ is an epimorphism, $\pi \otimes 1_N = 0$, whence $\operatorname{Coker}\alpha \otimes_R N = 0$. By hypothesis, this entails $\operatorname{Coker}\alpha = 0$ which is to say α is an epimorphism. We conclude that $\_\otimes_R N$ reflects epimorphisms. □

Note that there are plenty of modules satisfying the hypothesis of the above theorem. For example, any free nonzero R-module N satisfies the property that the functor $\_\otimes_R N$ reflects epimorphisms. Additionally, as we will see in the following theorem, every nonzero module over a BKN ring also satisfies this property.

For the next result, we need to introduce the notion of an "*anti flat module*".

Definition 3. An R-module N is *anti flat* if the functor $\_\otimes_R N$ is anti projective.

Theorem 8. *Let R be a BKN ring and N a nonzero anti flat R-module. Then $\operatorname{Tor}_1^R(M,N) = 0$ if and only if M is a projective right R-module. Moreover, for a projective resolution of M:*

$$\boldsymbol{P} = \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

define $K_0 = \operatorname{Ker}\varepsilon$, and define $K_n = \operatorname{Ker} d_n$ for all $n \geq 1$. Then $\operatorname{Tor}_{n+1}^R(M,N) = 0$ if and only if K_{n-1} is projective.

Proof. Let M be a nonzero right R-module, and consider any short exact sequence in Mod-R of the form

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} M \longrightarrow 0. \tag{3.1}$$

Applying the functor $\_\otimes_R N$, yields the (right exact) sequence:

$$0 \dashrightarrow X\otimes_R N \xrightarrow{\alpha\otimes 1_N} Y\otimes_R N \xrightarrow{\beta\otimes 1_N} M\otimes_R N \longrightarrow 0. \tag{3.2}$$

Now suppose that M is projective. This implies that Sequence 3.1 splits, so that Sequence 3.2 is exact (all additive functors are exact on split sequences). It follows that $\mathrm{Ker}(\alpha\otimes 1_N) = \mathrm{Tor}_1^R(M,N) = 0$, as required.

Conversely, let us suppose that $\mathrm{Tor}_1^R(M,N) = \mathrm{Ker}(\alpha\otimes 1_N) = 0$. It follows that Sequence 3.2 is exact. If Sequence 3.1 is non-splitting, then it follows from the anti-flat property for N, that $X\otimes_R N = 0$. This, in turn, implies $X = 0$ by the BKN property (Proposition 2). This contradicts the fact that Sequence 3.1 is non-splitting. Since every short exact sequence of the form shown in 3.1 splits, we may conclude that M is projective, as required.

For the second part, the first part gives $\mathrm{Tor}_1^R(K_{n-1},N) = 0$ if and only if K_{n-1} is projective. On the other hand, [8, Corollary 6.23], establishes that $\mathrm{Tor}_{n+1}^R(M,N) \cong \mathrm{Tor}_1^R(K_{n-1},N)$ for all $n \geq 1$. The result now follows easily. □

3.2. *BKN rings and functors Ext and ext*

The proofs of the following theorems concerning the functor Ext are essentially duals of the proofs we have given for the functor Tor, and so they will be omitted (see [8, Sections 6.2.3] for required prior results). In this regard, we note that BKN rings possess the necessary symmetry insofar as $\mathrm{Hom}_R(M,N)$ and $M\otimes_R N$ are nonzero for all nonzero modules ${}_RM_R$ and ${}_RN$.

Theorem 9. *Let R be a ring and N a left nonzero R-module. Then the functor $\mathrm{Hom}_R(N,\_)$ reflects monomorphisms if and only if $\mathrm{Hom}_R(N,M) \neq 0$ for each nonzero left R-module M.*

Definition 4. A left R-module N is *domain anti injective* whenever the functor $\mathrm{Hom}_R(N,\_)$ is anti injective.

Theorem 10. *Let R be a BKN ring and N a nonzero domain anti injective left R-module. Then $\mathrm{Ext}_R^1(N,M) = 0$ if and only if M is an injective left R-module. Moreover, for an injective resolution of M:*

$$\boldsymbol{E} = 0 \longrightarrow M \xrightarrow{\eta} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} E_3 \longrightarrow,$$

define $V_0 = \mathrm{Im}\,\eta$ and for all $n \geq 1$ define $V_n = \mathrm{Im}\,d_{n-1}$. Then $\mathrm{Ext}_R^{n+1}(N,M) = 0$ if and only if V_{n-1} is injective.

Likewise, the proof of Theorems 11 to 12 concerning the functor ext are similar to those of Section 3.1, so they will also be omitted (see [8, Section 6.2.4] for the necessary details).

Theorem 11. *Let R be a ring and N a nonzero left R-module. Then the contravariant functor $\mathrm{Hom}_R(\_,N)$ reflects monomorphisms if and only if $\mathrm{Hom}_R(M,N) \neq 0$ for each nonzero left R-module M.*

Definition 5. The left R-module N is *codomain anti projective* if the functor $\mathrm{Hom}_R(\_,N)$ is anti projective.

Theorem 12. *Let R be a BKN ring and N a codomain anti projective left R-module. Then, $\mathrm{ext}^1_R(M,N) \neq 0$ if and only if M is a nonprojective left R-module. Moreover, for a projective resolution of M:*

$$\boldsymbol{P} = \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

define $K_0 = \mathrm{Ker}\,\varepsilon$ and $K_n = \mathrm{Ker}\,d_n$ for all $n \geq 1$. Then, $\mathrm{ext}^{n+1}_R(M,N) \neq 0$ if and only if K_{n-1} is nonprojective.

Question 1. Does there exist a nonsemisimple (BKN) ring R and a nonzero anti flat (domain anti injective, codomain anti projective) R-module N?

Question 2. What are the consequences and implications for the ring R or in the category R-**Mod** if there were no such modules?

Question 3. Does $L \otimes_R M \neq 0$ for all nonzero L and M imply R is BKN for the noncommutative case?

4. Acknowledgements

We thank the referee for the careful reading, valuable comments and suggestions that substantially improve the present work.

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