Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 713–735
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4722

DYNAMICS OF POSITIVE SOLUTIONS OF A THREE DIMENSIONAL HIGHER ORDER SYSTEM OF DIFFERENCE EQUATIONS

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Received 04 May, 2023

Abstract. In this paper, we study the global behavior of positive solutions of the following system of difference equations

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{z_n}{z_{n-k}}, \quad z_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n \in \mathbb{N}_0,$$

where the parameter $A > 0$, the initial values $x_{-i}, y_{-i}, z_{-}i$, $i \in \{0, 1, 2, \ldots, k\}$, are arbitrary positive real number and $k \in \mathbb{N}$. Moreover, we provide semi-cycle analysis of positive solutions to the above system of difference equations. Finally, we also give some numerical examples which support our analytical results.

2010 *Mathematics Subject Classification:* 39A10; 39A23; 40A05

Keywords: system of difference equations, semi-cycle analysis, bounded solutions and periodic solutions, global asymptotic stability

1. Introduction and preliminaries

Let $\mathbb{N}$ be set of all natural numbers, $\mathbb{N}_0$ be set of non-negative integers, $\mathbb{Z}$ be set of all integers, $\mathbb{R}$ be set of all real numbers and for $k \in \mathbb{Z}$ the notation $\mathbb{N}_k$ represents the set of $\{n \in \mathbb{Z} : n \geq k\}$.

Discrete dynamical systems and difference equations have attracted attention of researchers in the last years, particularly these equations and systems, which arise in mathematical models describing the real world phenomenon, are used in natural sciences, engineering, operations research, social sciences and linguistics, etc. Recently, there has been a lot of study concerning the qualitative behaviour of nonlinear difference equation and system of difference equations [1–24]. Even though difference equations with higher-order and their systems have sometimes very simple in their form, they are actually difficult to understand exact the behaviour of their solutions. Hence, studying the qualitative behaviour of such difference equations and system of difference equations is worth further consideration. Later on, making a historical flash back for the system of difference equation we consider in this paper, we should mention that in paper [1], Abu-Saris and Devault investigated the global asymptotic stability results for the equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}},\ n \in \mathbb{N}_0, \tag{1.1}$$

where $A \in (0,\infty)$, $k \in \mathbb{N}_2$ and x_{-i}, for $i \in \{0,1,\ldots,k\}$, are positive real numbers, while the next difference equation

$$y_{n+1} = \alpha + \frac{y_{n-k}}{y_n},\ n \in \mathbb{N}_0, \tag{1.2}$$

where $\alpha \in (1,\infty)$, $k \in \mathbb{N}_1$ and x_{-i}, for $i \in \{0,1,\ldots,k\}$, are positive real numbers, was studied in [8]. Eqs. (1.1) and (1.2) were good prototypes for inspiring the investigation of their several extensions. A natural extension of (1.2) is the next system of difference equations

$$x_{n+1} = A + \frac{y_{n-k}}{y_n},\ y_{n+1} = A + \frac{x_{n-k}}{x_n},\ n \in \mathbb{N}_0, \tag{1.3}$$

with $A > 0$, and the initial conditions x_{-i}, y_{-i}, for $i \in \{0,1,\ldots,k\}$, are arbitrary positive real numbers. Zhang et al. [21] overcomed the asymptotic behavior of positive solutions to the system (1.3) in the cases $0 < A < 1$, $A = 1$ and $A > 1$. Additionally, in [9], Gümüş dealt with the global asymptotic stability of positive equilibrium, the rate of convergence of positive solutions and semi-cycle analysis of system (1.3). The other extensions of Eq. (1.2) can be found in references [2, 13, 15, 20, 22, 24]. Similarly, a natural extension of (1.1) is the following two-dimensional system of difference equation

$$x_{n+1} = A + \frac{y_n}{y_{n-k}},\ y_{n+1} = A + \frac{x_n}{x_{n-k}},\ n \in \mathbb{N}_0, \tag{1.4}$$

with $A > 0$, and the initial conditions x_{-i}, y_{-i}, for $i \in \{0,1,\ldots,k\}$, are arbitrary positive real numbers. Abualrub and Aloqeili performed semi-cycle analysis of positive solutions of system (1.3) and also studied the dynamical behavior of the solutions of system (1.3) in the cases $0 < A < 1$, $A = 1$ and $A > 1$. Some related to extensions of Eq. (1.1) have extensively studied. For more details see [4, 16, 20, 23] and the reference therein

Inspired with the previous valuable theoretical results, we consider the following three-dimensional symmetrical system of difference equation with high order

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{z_n}{z_{n-k}}, \quad z_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where the parameter $A > 0$, the initial conditions x_{-i}, y_{-i}, z_{-i}, $i \in \{0,1,2,\ldots,k\}$, are arbitrary positive number and $k \in \mathbb{N}$, which is natural extensions of Eq. (1.1) and system (1.4). The main purpose of this paper is to study global asymptotic stability of positive equilibrium, boundedness character of positive solutions, the rate of convergence of positive solutions and semi-cycle analysis of positive solutions to system (1.5). In addition, the theoretical results are verified by numerical examples. It is clearly seen that system (1.5) has a unique positive equilibrium $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$.

2. SEMI-CYCLE ANALYSIS OF SYSTEM (1.5)

In this section we investigate the behavior of positive solution to system (1.5) by means of semi-cycle analysis method. It is easy to see that system (1.5) has a unique positive equilibrium point $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$.

Theorem 1. *Assume that $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ is a solution to system (1.5). Then either this solution is non-oscillatory solution or it oscillates about the equilibrium $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$, with semi-cycles having $k+1$ terms.*

Proof. Let $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5) and for any integer $n_0 \in \mathbb{N}_0$, $(x_{n_0},y_{n_0},z_{n_0})$ be the last term of a semi-cycle which has at least k terms. Then, from this assumption, we can write either

$$\begin{cases} \dots,x_{n_0-k+1},\dots,x_{n_0-1},x_{n_0} < 1+A \leq x_{n_0+1}, \\ \dots,y_{n_0-k+1},\dots,y_{n_0-1},y_{n_0} < 1+A \leq y_{n_0+1}, \\ \dots,z_{n_0-k+1},\dots,z_{n_0-1},z_{n_0} < 1+A \leq z_{n_0+1}, \end{cases} \tag{2.1}$$

or

$$\begin{cases} \dots,x_{n_0-k+1},\dots,x_{n_0-1},x_{n_0} \geq 1+A > x_{n_0+1}, \\ \dots,y_{n_0-k+1},\dots,y_{n_0-1},y_{n_0} \geq 1+A > y_{n_0+1}, \\ \dots,z_{n_0-k+1},\dots,z_{n_0-1},z_{n_0} \geq 1+A > z_{n_0+1}. \end{cases} \tag{2.2}$$

Now, in here we will only present the first case since the other case can be done in a similar way. From (2.1) and (1.5), we have

$$\begin{cases} x_{n_0+2},x_{n_0+3},\dots,x_{n_0+k+1} > A+1, \\ y_{n_0+2},y_{n_0+3},\dots,y_{n_0+k+1} > A+1, \\ z_{n_0+2},z_{n_0+3},\dots,z_{n_0+k+1} > A+1, \end{cases} \tag{2.3}$$

from which it implies that the semi-cycle beginning with $(x_{n_0+1},y_{n_0+1},z_{n_0+1})$ has at least $k+1$ terms. Now, we may assume that the semi-cycle beginning with $(x_{n_0+1}, y_{n_0+1},z_{n_0+1})$ include exactly $k+1$ terms. Then, the next semi-cycle will start with $(x_{n_0+k+2},y_{n_0+k+2},z_{n_0+k+2})$ such that

$$\begin{cases} x_{n_0+1},x_{n_0+2},\dots,x_{n_0+k+1} \geq 1+A > x_{n_0+k+2}, \\ y_{n_0+1},y_{n_0+2},\dots,y_{n_0+k+1} \geq 1+A > y_{n_0+k+2}, \\ z_{n_0+1},z_{n_0+2},\dots,z_{n_0+k+1} \geq 1+A > z_{n_0+k+2}, \end{cases} \tag{2.4}$$

from which along with (1.5), for $i = 1, 2, \ldots, k$, it follows that

$$\begin{cases} x_{n_0+k+2+i} = A + \frac{y_{n_0+k+1+i}}{y_{n_0+1+i}} < A+1, \\ y_{n_0+k+2+i} = A + \frac{z_{n_0+k+1+i}}{z_{n_0+1+i}} < A+1, \\ z_{n_0+k+2+i} = A + \frac{x_{n_0+k+1+i}}{x_{n_0+1+i}} < A+1, \end{cases} \tag{2.5}$$

which is desired. □

Theorem 2. *System (1.5) has no nontrivial periodic solutions of period k (not necessarily prime period k).*

Proof. Assume that system (1.5) has k-periodic solutions in the following form

$$\ldots, (\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2), \ldots, (\alpha_k, \beta_k, \gamma_k), \ldots. \tag{2.6}$$

Then, from (1.5), it is not hard to see that for all $n \geq 0$, $(x_{n-k}, y_{n-k}, z_{n-k}) = (x_n, y_n, z_n)$, and so

$$x_{n+1} = A + \frac{y_n}{y_{n-k}} = A+1,\ y_{n+1} = A + \frac{z_n}{z_{n-k}} = A+1,\ z_{n+1} = A + \frac{x_n}{x_{n-k}} = A+1, \tag{2.7}$$

for all $n \geq 0$, which means that the solution to system (1.5) is the equilibrium solution $(\bar{x}, \bar{y}, \bar{z}) = (A+1, A+1, A+1)$, which is desired. □

Theorem 3. *Any increasing solution to system (1.5) is non-oscillatory positive.*

Proof. Assume that $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$is an increasing non-oscillatory solution to system (1.5). Then, from (1.5), we can write either

$$\begin{cases} A+1 \leq x_1, \\ A+1 \leq y_1, \\ A+1 \leq z_1, \end{cases} \tag{2.8}$$

or

$$\begin{cases} x_1 < A+1, \\ y_1 < A+1, \\ z_1 < A+1. \end{cases} \tag{2.9}$$

From (1.5), it is easy to see that the inequalities in (2.8) hold. On the other hand, if the inequalities in (2.9) are satisfied, then we may assume that the semi-cycle involving (x_1, y_1, z_1) ends with (x_i, y_i, z_i) such that $i \in \{1, 2, \ldots, k+1\}$. If $i = k+2$, then

$$\begin{cases} x_{k+2} = A + \frac{y_{k+1}}{y_1} < A+1, \\ y_{k+2} = A + \frac{z_{k+1}}{z_1} < A+1, \\ z_{k+2} = A + \frac{x_{k+1}}{x_1} < A+1, \end{cases} \tag{2.10}$$

which means that $y_{k+1} < y_1$, $z_{k+1} < z_1$ and $x_{k+1} < x_1$, where $k+1 > 1$, which contradicts that the solution to system (1.5) is increasing. Hence, the proof is completed. □

Theorem 4. *System (1.5) has no non-oscillatory negative solutions (has no infinite negative semi-cycle)*

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5). We may suppose that system (1.5) has an infinite negative semi-cycle beginning with (x_N, y_N, z_N) for $N \geq -k$. Then, it is clearly to see that $x_n < A+1$, $y_n < A+1$ and $z_n < A+1$, for every $n \geq N$. From this and system (1.5), we get $y_n < y_{n-k}$, $z_n < z_{n-k}$ and $x_n < x_{n-k}$, for $n \geq \max\{1, N-1\}$, from which along with system (1.5) it follows that for every $n \geq \max\{1, N\}$

$$\begin{cases} A < \cdots < x_{n+k} < x_n < x_{n-k} < A+1, \\ A < \cdots < y_{n+k} < y_n < y_{n-k} < A+1, \\ A < \cdots < z_{n+k} < z_n < z_{n-k} < A+1. \end{cases} \tag{2.11}$$

From the inequalities in (2.11), we get that there exists α_i, β_i and γ_i for all $i \in \{0, 1, \ldots, k-1\}$ such that

$$\lim_{n\to\infty} x_{nk+i} = \alpha_i, \quad \lim_{n\to\infty} y_{nk+i} = \beta_i, \quad \lim_{n\to\infty} z_{nk+i} = \gamma_i. \tag{2.12}$$

Therefore,

$$(\alpha_0, \beta_0, \gamma_0), (\alpha_1, \beta_1, \gamma_1), \ldots, (\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}) \tag{2.13}$$

is a periodic solution (not necessarily prime period) of system (1.5) with period k, which contradicts with Theorem 2 in the case when the solution is not trivial solution. So, the solution to system (1.5) converge to the equilibrium, which contradicts that the solution to system (1.5) is diverging from the equilibrium. Thus, the proof is completed. □

Theorem 5. *System (1.5) has no decreasing non-oscillatory solutions.*

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5) which is a decreasing non-oscillatory solutions. Then, the solution to system (1.5) is either of the form

$$\begin{cases} \cdots \leq x_3 \leq x_2 \leq x_1 \leq A+1, \\ \cdots \leq y_3 \leq y_2 \leq y_1 \leq A+1, \\ \cdots \leq z_3 \leq z_2 \leq z_1 \leq A+1, \end{cases} \tag{2.14}$$

or there exists a positive integer $n_0 \geq k+1$ such that

$$\begin{cases} \cdots \leq x_{n_0+2} \leq x_{n_0+1} \leq A+1 \leq x_{n_0} \leq x_{n_0-1}, \\ \cdots \leq y_{n_0+2} \leq y_{n_0+1} \leq A+1 \leq y_{n_0} \leq y_{n_0-1}, \\ \cdots \leq z_{n_0+2} \leq z_{n_0+1} \leq A+1 \leq z_{n_0} \leq z_{n_0-1}. \end{cases} \tag{2.15}$$

In the second case, it is clearly seen that the positive semi-cycle ends with $(x_{n_0}, y_{n_0}, z_{n_0})$ and furthermore it contains at most $2k+2$ terms. From (2.14) and (2.15), the solution to system (1.5) has an infinite negative semi-cycle, which contradicts with Theorem 4. Thus, the proof is completed. □

Theorem 6. *Let k be even and $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5). Then the following statements hold.*

(a) *Every semi-cycle has length at most $3k+1$.*
(b) *The extreme term in a semi-cycle occurs in the first $k+3$ term of the semi-cycle.*
(c) *Every solution oscillates about $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$.*

Proof. Let $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5). In here we will just prove the case of a negative semi-cycle since the proof the other case can be done in a similar way. We may assume that system (1.5) has a negative semi-cycle beginning with (x_μ,y_μ,z_μ), for $\mu \geq -k$, and furthermore it involves $2k+1$ terms. Then, from definition of negative semi-cycle we can write the following inequalities

$$\begin{cases} x_\mu, x_{\mu+1}, \ldots, x_{\mu+2k} < A+1, \\ y_\mu, y_{\mu+1}, \ldots, y_{\mu+2k} < A+1, \\ z_\mu, z_{\mu+1}, \ldots, z_{\mu+2k} < A+1. \end{cases} \tag{2.16}$$

By considering system (1.5) and the inequalities in (2.16), we have

$$\begin{cases} x_{\mu+k+1+i} = A + \frac{y_{\mu+k+i}}{y_{\mu+i}} > A + \frac{y_{\mu+k+i}}{A+1} > y_{\mu+k+i}, \\ y_{\mu+k+i} = A + \frac{z_{\mu+k+i-1}}{z_{\mu+i-1}} > A + \frac{z_{\mu+k+i-1}}{A+1} > z_{\mu+k+i-1}, \\ z_{\mu+k+i-1} = A + \frac{x_{\mu+k+i-2}}{x_{\mu+i-2}} > A + \frac{x_{\mu+k+i-2}}{A+1} > x_{\mu+k+i-2}, \end{cases} \tag{2.17}$$

for $i \in \{2,3,\ldots,2k-1\}$, which means that

$$\begin{cases} x_{\mu+k} < z_{\mu+k+1} < y_{\mu+k+2} < x_{\mu+k+3} < z_{\mu+k+4} < y_{\mu+k+5} < x_{\mu+k+6} < \cdots < x_{\mu+3k}, \\ y_{\mu+k} < x_{\mu+k+1} < z_{\mu+k+2} < y_{\mu+k+3} < x_{\mu+k+4} < z_{\mu+k+5} < y_{\mu+k+6} < \cdots < y_{\mu+3k}, \\ z_{\mu+k} < y_{\mu+k+1} < x_{\mu+k+2} < z_{\mu+k+3} < y_{\mu+k+4} < x_{\mu+k+5} < z_{\mu+k+6} < \cdots < z_{\mu+3k}. \end{cases} \tag{2.18}$$

From (2.18), we have

$$x_{\mu+k} < x_{\mu+3k}, \quad y_{\mu+k} < y_{\mu+3k}, \quad z_{\mu+k} < z_{\mu+3k}, \tag{2.19}$$

from which along with system (1.5), it follows that

$$\begin{cases} x_{\mu+3k+1} = A + \frac{y_{\mu+3k}}{y_{\mu+2k}} > A+1, \\ y_{\mu+3k+1} = A + \frac{z_{\mu+3k}}{z_{\mu+2k}} > A+1, \\ z_{\mu+3k+1} = A + \frac{x_{\mu+3k}}{x_{\mu+2k}} > A+1. \end{cases} \tag{2.20}$$

Thus, from (2.20), we conclude that a negative semi-cycle has at most $3k+1$ terms. Furthermore, the first $k+3$ terms of a semi-cycle are formed as the extreme term in this semi-cycle. Finally, every solution to system (1.5) oscillates about the equilibrium $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$. Therefore the proof is completed. □

3. THE ASYMPTOTIC BEHAVIOR OF SYSTEM (1.5)

In this section, we study the asymptotic behavior of system (1.5) according to values of A.

Theorem 7. *Assume that $A = 1$. Then every positive solutions of system (1.5) is bounded and persists.*

Proof. Assume that $A = 1$ and $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ is a positive solutions of system (1.5). It is easily seen that for every $n \in \mathbb{N}_0$, $x_n, y_n, z_n > A = 1$. Thus, we can write

$$x_i, y_i, z_i \in \left[K, \frac{K}{K-1}\right], \; i = 1, 2, \ldots, k+1, \tag{3.1}$$

where $K = min\{\alpha, \frac{\beta}{\beta-1}\} > 1$, $\alpha = \min_{1\le i\le k+1}\{x_i, y_i, z_i\}$, $\beta = \max_{1\le i\le k+1}\{x_i, y_i, z_i\}$. From system (1.5), we have

$$K = 1 + \frac{K}{\frac{K}{K-1}} \le x_{k+2} = 1 + \frac{y_{k+1}}{y_1} \le 1 + \frac{\frac{K}{K-1}}{K} = \frac{K}{K-1}, \tag{3.2}$$

$$K = 1 + \frac{K}{\frac{K}{K-1}} \le y_{k+2} = 1 + \frac{z_{k+1}}{z_1} \le 1 + \frac{\frac{K}{K-1}}{K} = \frac{K}{K-1}, \tag{3.3}$$

$$K = 1 + \frac{K}{\frac{K}{K-1}} \le z_{k+2} = 1 + \frac{x_{k+1}}{x_1} \le 1 + \frac{\frac{K}{K-1}}{K} = \frac{K}{K-1}. \tag{3.4}$$

By using the method of induction, it follows that

$$x_i, y_i, z_i \in \left[K, \frac{K}{K-1}\right], \text{ for all } i = 1, 2, \ldots, \tag{3.5}$$

which is desired. □

Theorem 8. *Assume that $A = 1$ and $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ is a positive solutions of system (1.5). Then the following result holds true.*

$$\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \inf y_n = \lim_{n\to\infty} \inf z_n$$

$$\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \sup y_n = \lim_{n\to\infty} \sup z_n$$

Proof. Assume that

$$\begin{cases} l_1 = \lim_{n\to\infty} \inf x_n, \; u_1 = \lim_{n\to\infty} \sup x_n, \\ l_2 = \lim_{n\to\infty} \inf y_n, \; u_2 = \lim_{n\to\infty} \sup y_n, \\ l_3 = \lim_{n\to\infty} \inf z_n, \; u_3 = \lim_{n\to\infty} \sup z_n. \end{cases} \tag{3.6}$$

From (3.6) and system (1.5), we get the following inequalities

$$1 < l_1 \le u_1, \; 1 < l_2 \le u_2, \; 1 < l_3 \le u_3 \tag{3.7}$$

and

$$\begin{cases} 1+\frac{l_2}{u_2}\le l_1,\ u_1\le 1+\frac{u_2}{l_2}, \\ 1+\frac{l_3}{u_3}\le l_2,\ u_2\le 1+\frac{u_3}{l_3}, \\ 1+\frac{l_1}{u_1}\le l_3,\ u_3\le 1+\frac{u_1}{l_1}, \end{cases} \tag{3.8}$$

from which it follows that

$$\begin{cases} u_1 l_2\le l_2+u_2\le l_1 u_2, \\ u_2 l_3\le l_3+u_3\le l_2 u_3, \\ u_3 l_1\le u_1+l_1\le u_1 l_3. \end{cases} \tag{3.9}$$

Also, from the inequalities in (3.9), without loss of generality assume that we continue only with the following equality cases

$$l_2 u_1 = l_1 u_2, \tag{3.10}$$

$$l_2 u_1 = l_2+u_2, \tag{3.11}$$

$$l_3 u_2 = l_2 u_3, \tag{3.12}$$

$$l_2 u_3 = l_3+u_3, \tag{3.13}$$

$$l_1 u_3 = u_1 l_3, \tag{3.14}$$

$$l_3 u_1 = l_1+u_1. \tag{3.15}$$

From (3.10), (3.12) and (3.14), it follows that

$$\frac{l_1}{l_2}=\frac{u_1}{u_2}, \tag{3.16}$$

$$\frac{l_2}{l_3}=\frac{u_2}{u_3}, \tag{3.17}$$

$$\frac{l_1}{l_3}=\frac{u_1}{u_3}. \tag{3.18}$$

Now, upon dividing both side of (3.13) by l_2 and both side of (3.15) by l_3, we obtain

$$u_3=\frac{l_3}{l_2}+\frac{u_3}{l_2}, \tag{3.19}$$

$$u_1=\frac{l_1}{l_3}+\frac{u_1}{l_3}, \tag{3.20}$$

from which along with (3.17), (3.12) and after some basic calculations, it follows that

$$l_2=\frac{u_2}{u_2-1}, \tag{3.21}$$

$$l_3=\frac{u_3}{u_1-1}. \tag{3.22}$$

Replacing l_2 by $\frac{u_2}{u_2-1}$ in (3.11) and l_3 by $\frac{u_3}{u_1-1}$ in (3.13), we get

$$u_1\frac{u_2}{u_2-1}=\frac{u_2}{u_2-1}+u_2, \tag{3.23}$$

$$u_2\frac{u_3}{u_3-1}=\frac{u_3}{u_3-1}+u_3, \tag{3.24}$$

which can be written as follows

$$\frac{u_2}{u_2-1}=\frac{u_1}{u_2-1}, \tag{3.25}$$

$$\frac{u_3}{u_3-1}=\frac{u_2}{u_3-1}. \tag{3.26}$$

From this and the equalities (3.16) and (3.17), consequently

$$u_2=u_1=u_3, \tag{3.27}$$

$$l_2=l_1=l_3 \tag{3.28}$$

and this complete the proof. □

Theorem 9. *Assume that $A=1$. Then, system (1.5) has no positive nontrivial solution prime period-two.*

Proof. Let $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be a positive solution of system (1.5). Assume that $a,b,c,d,e,f>1$ and for all $n\in\mathbb{N}_0$, $x_{2n-k}=a$, $y_{2n-k}=b$, $z_{2n-k}=c$, $x_{2n+1-k}=d$, $y_{2n+1-k}=e$ and $z_{2n+1-k}=f$. Then, it is clearly seen that the positive solutions of system (1.5) has the following form

$$\ldots,(a,b,c),(d,e,f),(a,b,c),(d,e,f),\ldots. \tag{3.29}$$

If it was $a=d$, $b=e$ and $c=f$, then we would have that the solution to system (1.5) is not two-periodic. So, it must be $a\neq d$, $b\neq e$ and $c\neq f$. By considering Theorem (8), we can write the following equalities

$$\min\{a,d\}=\liminf_{n\to\infty}x_n=\liminf_{n\to\infty}y_n=\min\{b,e\}=\liminf_{n\to\infty}z_n=\min\{c,f\} \tag{3.30}$$

and

$$\max\{a,d\}=\limsup_{n\to\infty}x_n=\limsup_{n\to\infty}y_n=\max\{b,e\}=\limsup_{n\to\infty}z_n=\max\{c,f\}. \tag{3.31}$$

There are eight cases to be considered.

a) If $a<d$, $b<e$ and $c<f$, then $a=b=c$ and $d=e=f$ and the solution to system (1.5) can be written in the following form

$$\ldots,(a,a,a),(d,d,d),(a,a,a),(d,d,d),\ldots. \tag{3.32}$$

b) If $a<d$, $b<e$ and $c>f$, then $a=b=f$ and $d=c=e$ and the solution to system (1.5) is expressed in the following form

$$\ldots,(a,a,d),(d,d,a),(a,a,d),(d,d,a),\ldots. \tag{3.33}$$

c) If $a<d$, $b>e$ and $c>f$ then $a=e=f$ and $d=b=c$ and the solution to system (1.5) is written as in the form

$$\ldots,(a,d,d),(d,a,a),(a,d,d),(d,a,a),\ldots. \tag{3.34}$$

d) If $d > a$, $b > e$, $f > c$ then $a = e = c$ and $d = b = f$ and the solution to system (1.5) can be written in the following form

$$\ldots,(a,d,a),(d,a,d),(a,d,a),(d,a,d),\ldots. \tag{3.35}$$

e) If $a > d$, $e > b$, $f > c$ then $d = b = c$ and $a = e = f$ and we obtain the case (c).
f) If $a > d$, $e > b$, $c > f$ then $d = b = f$ and $a = e = c$ and we get the case (d).
g) If $a > d$, $b > e$, $f > c$ then $d = e = c$ and $a = b = f$ and we have the case (b).
h) If $a > d$, $b > e$, $c > f$ then $d = e = f$ and $a = b = c$ and we reach the case (a).

In here, we will only prove in the case when k is odd, since the proof of other case can be done in a similar way. Then, from our above assumptions and (3.32), we can write $x_{2n-k} = a$, $y_{2n-k} = a$, $z_{2n-k} = a$, $x_{2n+1-k} = d$ $y_{2n+1-k} = d$, $z_{2n+1-k} = d$ for the case (a), from which it follows that $x_1 = 1+\frac{y_0}{y_{-k}}$, so $a = 1+\frac{d}{a}$ that also means $a^2 = a+d$. Further, $x_2 = 1+\frac{y_1}{y_{-k+1}}$, so $d = 1+\frac{a}{d}$ which implies that $d^2 = a+d$. Then $d^2 = a^2$, but $d \neq a$, so $d = -a$. Therefore, we conclude that in this case system (1.5) has no positive prime period-two solutions. The other cases can be obtained similar to the case (a). So we will omit proofs of other cases.

□

Theorem 10. *Assume that $A > 1$. Then every positive solutions of system (1.5) is bounded and persists.*

Proof. Let $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be a positive solutions of system (1.5). Since $A > 1$ and the initial conditions x_{-i},y_{-i},z_{-i}, for all $i \in \{0,1,2,\ldots,k\}$, are arbitrary positive numbers, then we can write the following inequalities

$$x_n > A,\ y_n > A,\ z_n > A, \quad \text{for all } n \geq 1. \tag{3.36}$$

By considering system (1.5) and (3.36), we get for all $n \geq k+2$

$$x_n = A+\frac{y_{n-1}}{y_{n-k-1}} < A+\frac{1}{A}y_{n-1}, \tag{3.37}$$

$$y_n = A+\frac{z_{n-1}}{z_{n-k-1}} < A+\frac{1}{A}z_{n-1}, \tag{3.38}$$

$$z_n = A+\frac{x_{n-1}}{x_{n-k-1}} < A+\frac{1}{A}x_{n-1}. \tag{3.39}$$

Now assume that $\{(u_n,v_n,w_n)\}$ is solutions of the following difference equations systems

$$u_n = A+\frac{1}{A}v_{n-1},\ v_n = A+\frac{1}{A}w_{n-1},\ w_n = A+\frac{1}{A}u_{n-1},\ n \geq k+2, \tag{3.40}$$

such that $u_i = x_i,\ v_i = y_i,\ w_i = z_i$, where $i \in \{1,2,\dots,k+1\}$. Now we will prove the following inequalities using the induction method for $n \geq k+2$

$$x_n < u_n, \quad y_n < v_n, \quad z_n < w_n. \tag{3.41}$$

It is easy to see that, for $n = k+2$, from system (1.5) and (3.41), we have

$$x_{k+2} = A + \frac{y_{k+1}}{y_1} < A + \frac{y_{k+1}}{A} = A + \frac{1}{A} v_{k+1} = u_{k+2}. \tag{3.42}$$

Assume that the inequalities in (3.41) hold for $m > k+2$. Then, from this assumption, system (1.5) and (3.40), we obtain

$$\begin{cases} x_{m+1} < A + \frac{1}{A} y_m < A + \frac{1}{A} v_m = u_{m+1}, \\ y_{m+1} < A + \frac{1}{A} z_m < A + \frac{1}{A} w_m = v_{m+1}, \\ z_{m+1} < A + \frac{1}{A} x_m < A + \frac{1}{A} u_m = w_{m+1}, \end{cases} \tag{3.43}$$

which ends up the induction. Hence, considering (3.40), the following equations are obtained for $m \geq k$

$$\begin{cases} u_{m+3} &= \frac{1}{A^3} u_m + \frac{1}{A} + A + 1, \\ v_{m+3} &= \frac{1}{A^3} w_m + \frac{1}{A} + A + 1, \\ w_{m+3} &= \frac{1}{A^3} v_m + \frac{1}{A} + A + 1. \end{cases} \tag{3.44}$$

From (3.44), we have that the sequences $(u_{k+3l+i})_{l\geq 0}$, $(v_{k+3l+i})_{l\geq 0}$ and $(w_{k+3l+i})_{l\geq 0}$, for $i \in \{0,1,2\}$, provide the following recurrence relation

$$r_{n+1} = \frac{1}{A^3} r_n + \frac{1}{A} + A + 1, \ n \geq k, \tag{3.45}$$

from which it follows that for all $l \geq 0$,

$$\begin{cases} u_{k+3l+i} &= \left(x_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \\ v_{k+3l+i} &= \left(y_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \\ w_{k+3l+i} &= \left(z_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}. \end{cases} \tag{3.46}$$

By considering (3.36), (3.41) and (3.46), we can write the boundaries of the solutions to system (1.5) for all $l \geq 0$ and $i \in \{0,1,2\}$ as follows

$$\begin{cases} A < x_{k+3l+i} \leq \left(x_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \\ A < y_{k+3l+i} \leq \left(y_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \\ A < z_{k+3l+i} \leq \left(z_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \end{cases} \tag{3.47}$$

which is desired.

□

Theorem 11. *Assume that $A \geq 1$ and $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ is a solution to system (1.5). Then, system (1.5) has no non-oscillatory positive solutions (has no infinite positive semi-cycle).*

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5). Conversely, we suppose that system (1.5) has a non-oscillatory positive solutions and also this solution has an infinite positive semi-cycle beginning with (x_μ, y_μ, z_μ) for $\mu \geq -k$. By considering definition of positive semi-cycle and system (1.5), we have

$$\begin{cases} x_{n+1} = A + \frac{y_n}{y_{n-k}} \geq A+1, \\ y_{n+1} = A + \frac{z_n}{z_{n-k}} \geq A+1, \\ z_{n+1} = A + \frac{x_n}{x_{n-k}} \geq A+1, \end{cases} \tag{3.48}$$

for $n \geq \mu - 1$. From Theorem (10), there exist three real numbers K, L and M such that for every $n \geq k+2$, $x_n \leq K$, $y_n \leq L$ and $z_n \leq M$. From this and (3.48), we can write

$$\begin{cases} A+1 \leq x_{n-k} \leq x_n \leq x_{n+k} \leq \cdots \leq K, \\ A+1 \leq y_{n-k} \leq y_n \leq y_{n+k} \leq \cdots \leq L, \\ A+1 \leq z_{n-k} \leq z_n \leq z_{n+k} \leq \cdots \leq M, \end{cases} \tag{3.49}$$

for all $n \geq max\{\mu - 1, 2k+1\}$. From the inequalities in (3.49), we have that there exists β_i, γ_i and δ_i for all $i \in \{0, 1, \ldots, k-1\}$ such that

$$\lim_{n\to\infty} x_{nk+i} = \beta_i, \quad \lim_{n\to\infty} y_{nk+i} = \gamma_i, \quad \lim_{n\to\infty} z_{nk+i} = \delta_i. \tag{3.50}$$

Hence,

$$(\beta_0, \gamma_0, \delta_0), (\beta_1, \gamma_1, \delta_1), \ldots, (\beta_{k-1}, \gamma_{k-1}, \delta_{k-1}) \tag{3.51}$$

is a periodic solution of (not necessarily prime period) system (1.5) with period k, which contradicts with Theorem 2 in the case when the solution is not trivial solution. So, the solution to system (1.5) converge to the equilibrium, which contradicts that the solution to system (1.5) is diverging from the equilibrium. We complete the proof. □

Lemma 1. *[3] Assume that $A > 1$ and $0 < \varepsilon < \frac{A-1}{(A+1)(k+1)}$, for $k \in \mathbb{N}$. Then $\frac{2}{(1-(k+1))\varepsilon(A+1)} < 1$.*

The following theorem gives us the local stability of the unique equilibrium point of the system (1.5) when $A > 1$.

Theorem 12. *If $A > 1$ then the unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z}) = (A+1, A+1, A+1)$ of system (1.5) is locally asymptotically stable.*

Proof. For analysing the local stability of the unique equilibrium point of the system (1.5), firstly, the system (1.5) is transformed into the following equivalent system as

$$X_{n+1} = F(X_n), \; n \in \mathbb{N}_0, \tag{3.52}$$

where $X_n = \left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k+1)}, y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(k+1)}, z_n^{(1)}, z_n^{(2)}, \ldots, z_n^{(k+1)}\right)^T$, where

$$\begin{cases} x_n^{(1)} = x_n, x_n^{(2)} = x_{n-1}, \ldots, x_n^{(k+1)} = x_{n-k}, \\ y_n^{(1)} = y_n, y_n^{(2)} = y_{n-1}, \ldots, y_n^{(k+1)} = y_{n-k}, \\ z_n^{(1)} = z_n, z_n^{(2)} = z_{n-1}, \ldots, z_n^{(k+1)} = z_{n-k}, \end{cases} \tag{3.53}$$

and $F : [0,\infty)^{3k+3} \to [0,\infty)^{3k+3}$ such that for all

$$X = \left(x^{(1)}, x^{(2)}, \ldots, x^{(k+1)}, y^{(1)}, y^{(2)}, \ldots, y^{(k+1)}, z^{(1)}, z^{(2)}, \ldots, z^{(k+1)}\right) \in [0,\infty)^{3k+3},$$

$$F(X) = \left(f_1(X), x^{(2)}, \ldots, x^{(k+1)}, f_2(X), y^{(2)}, \ldots, y^{(k+1)}, f_3(X), z^{(2)}, \ldots, z^{(k+1)}\right),$$

where

$$f_1(X) = A + \frac{y^{(1)}}{y^{(k+1)}},\ f_2(X) = A + \frac{z^{(1)}}{z^{(k+1)}},\ f_3(X) = A + \frac{x^{(1)}}{x^{(k+1)}}.$$

Then, we have

$$\begin{cases} \frac{\partial f_1}{\partial y^{(1)}}(X) = \frac{1}{y^{(k+1)}}, & \frac{\partial f_1}{\partial y^{(k+1)}}(X) = -\frac{y^{(1)}}{(y^{(k+1)})^2}, \\ \frac{\partial f_2}{\partial z^{(1)}}(X) = \frac{1}{z^{(k+1)}}, & \frac{\partial f_2}{\partial z^{(k+1)}}(X) = -\frac{z^{(1)}}{(z^{(k+1)})^2}, \\ \frac{\partial f_3}{\partial x^{(1)}}(X) = \frac{1}{x^{(k+1)}}, & \frac{\partial f_3}{\partial x^{(k+1)}}(X) = -\frac{x^{(1)}}{(x^{(k+1)})^2}. \end{cases} \tag{3.54}$$

$\mathcal{J}_{\mathcal{F}}$ is the Jacobian matrix of F at the equilibrium point $(\bar{x}, \bar{y}, \bar{z}) = (A+1, A+1, A+1)$, which is given by

$$\mathcal{J}_{\mathcal{F}} = \begin{pmatrix} 0 & 0\cdots0 & 0 & \frac{1}{A+1} & 0\cdots0 & \frac{-1}{A+1} & 0 & 0\cdots0 & 0 \\ 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & \frac{1}{A+1} & 0\cdots0 & \frac{-1}{A+1} \\ 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 \\ \frac{1}{A+1} & 0\cdots0 & \frac{-1}{A+1} & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 \end{pmatrix} \tag{3.55}$$

Assume that $\lambda_1, \lambda_2, \ldots, \lambda_{3k+3}$ are the eigenvalues of matrix $\mathcal{J}_{\mathcal{F}}$ and that $\mathcal{D} = diag(d_1, d_2, \ldots, d_{3k+3})$ is a diagonal matrix, where

$$d_1 = d_{k+2} = d_{2k+3} = 1, \quad d_m = d_{k+1+m} = d_{2k+2+m} = 1 - m\varepsilon,$$

for $m \in \{2,3,\dots,k+1\}$ and $0 < \varepsilon < \frac{A-1}{(k+1)(A+1)}$. From this and taking into account the fact that $1 - m\varepsilon > 0$, for all $m \in \{2,3,\dots,k+1\}$, we conclude that the matrix $\mathcal{D}$ is invertible. The matrix DBD^{-1} is given by

$$\begin{pmatrix}
0 & 0\cdots 0 & 0 & \frac{1}{A+1}\frac{d_1}{d_{k+2}} & 0\cdots 0 & \frac{-1}{A+1}\frac{d_1}{d_{2k+2}} & 0 & 0\cdots 0 & 0\\
\frac{d_2}{d_1} & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\\
 & & & & & & & & \\
0 & 0\cdots \frac{d_{k+1}}{d_k} & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\\
0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & \frac{1}{A+1}\frac{d_{k+2}}{d_{2k+3}} & 0\cdots 0 & \frac{-1}{A+1}\frac{d_{k+2}}{d_{3k+3}}\\
0 & 0\cdots 0 & 0 & \frac{d_{k+3}}{d_{k+2}} & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\\
 & & & & & & & & \\
0 & 0\cdots 0 & 0 & 0 & 0\cdots \frac{d_{2k+2}}{d_{2k+1}} & 0 & 0 & 0\cdots 0 & 0\\
\frac{1}{A+1}\frac{d_{2k+3}}{d_1} & 0\cdots 0 & \frac{-1}{A+1}\frac{d_{2k+3}}{d_{k+1}} & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\\
0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & \frac{d_{2k+4}}{d_{2k+3}} & 0\cdots 0 & 0\\
 & & & & & & & & \\
0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots \frac{d_{3k+3}}{d_{3k+2}} & 0
\end{pmatrix} \tag{3.56}$$

To calculate the infinity norm of $\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}$, we will show that inequality $\max\limits_{1\le i\le 3k+3} \sum_{j=1}^{3k+3} |a_{ij}| < 1$ holds. For this, by considering the fact that $d_1 > d_2 > \cdots > d_{k+1}$, $d_{k+2} > d_{k+3} > \cdots > d_{2k+2}$ and $d_{2k+3} > d_{2k+4} > \cdots > d_{3k+3}$, we can write the following inequalities for every $j \in \{1,2,\dots,3k+2\}$

$$\frac{d_{j+1}}{d_j} < 1 \tag{3.57}$$

from which along with $A > 1$ and lemma (1), it yields

$$\begin{aligned}
\frac{1}{A+1}\frac{d_1}{d_{k+2}} + \frac{1}{A+1}\frac{d_1}{d_{2k+2}} &= \frac{1}{A+1} + \frac{1}{(1-(k+1)\varepsilon)(A+1)}\\
&< \frac{1}{(1-(k+1)\varepsilon)(A+1)} + \frac{1}{(1-(k+1)\varepsilon)(A+1)}\\
&< \frac{2}{(1-(k+1)\varepsilon)(A+1)}\\
&< 1,
\end{aligned} \tag{3.58}$$

$$\begin{aligned}
\frac{1}{A+1}\frac{d_{k+2}}{d_{2k+3}} + \frac{1}{A+1}\frac{d_{k+2}}{d_{3k+3}} &= \frac{1}{A+1} + \frac{1}{(1-(k+1)\varepsilon)(A+1)}\\
&< \frac{1}{(1-(k+1)\varepsilon)(A+1)} + \frac{1}{(1-(k+1)\varepsilon)(A+1)}\\
&< \frac{2}{(1-(k+1)\varepsilon)(A+1)}\\
&< 1
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}\frac{1}{A+1}\frac{d_{2k+3}}{d_1}+\frac{1}{A+1}\frac{d_{2k+3}}{d_{k+1}} &= \frac{1}{A+1}+\frac{1}{(1-(k+1)\varepsilon)(A+1)}\\ &< \frac{1}{(1-(k+1)\varepsilon)(A+1)}+\frac{1}{(1-(k+1)\varepsilon)(A+1)}\\ &< \frac{2}{(1-(k+1)\varepsilon)(A+1)}\\ &< 1. \end{aligned} \tag{3.60}$$

Since $\mathcal{J}_{\mathcal{F}}$ and $\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}$ similar matrices, they have the same eigenvalues. Consequently, it can easily seen that the following inequality holds

$$\begin{aligned}\rho\left(\mathcal{J}_{\mathcal{F}}\right) &= \max|\lambda_i|\\ &\leq ||\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}||_\infty\\ &= \max\{d_2d_1^{-1},\dots,d_{k+1}d_k^{-1},\dots,d_{k+3}d_{k+2}^{-1},\dots,d_{3k+3}d_{3k+2}^{-1} < 1.\end{aligned} \tag{3.61}$$

This means that all eigenvalues of $\mathcal{J}_{\mathcal{F}}$ lie inside the unit disk. Therefore, the unique equilibrium point $(\bar{x},\bar{y},\bar{z})=(A+1,A+1,A+1)$ of system (1.5) is locally asymptotically stable. □

The following theorem gives us the global attractor to the unique positive equilibrium point of system (1.5) when $A>1$.

Theorem 13. *Assume that $A>1$. Then the unique equilibrium point $(A+1,A+1,A+1)$ of system (1.5) is globally asymptotically stable.*

Proof. Evidently, since the unique positive equilibrium $(\bar{x},\bar{y},\bar{z})=(A+1,A+1,A+1)$ of system (1.5) is locally asymptotically stable when $A>1$, we only prove that the unique positive equilibrium $(\bar{x},\bar{y},\bar{z})=(A+1,A+1,A+1)$ of system (1.5) is a global attractor. For this, firstly, assume that

$$\begin{cases} l_1=\lim_{n\to\infty}\inf x_n,\ u_1=\lim_{n\to\infty}\sup x_n,\\ l_2=\lim_{n\to\infty}\inf y_n,\ u_2=\lim_{n\to\infty}\sup y_n,\\ l_3=\ lim_{n\to\infty}\inf z_n,\ u_3=\ lim_{n\to\infty}\sup z_n.\end{cases} \tag{3.62}$$

From (3.62) and system (1.5), it can be easily seen that the inequalities

$$\begin{cases} 1<l_1\leq u_1,\\ 1<l_2\leq u_2,\\ 1<l_3\leq u_3,\end{cases} \tag{3.63}$$

and

$$\begin{cases} A+\frac{l_2}{u_2}\leq l_1,\ u_1\leq A+\frac{u_2}{l_2},\\ A+\frac{l_3}{u_3}\leq l_2,\ u_2\leq A+\frac{u_3}{l_3},\\ A+\frac{l_1}{u_1}\leq l_3,\ u_3\leq A+\frac{u_1}{l_1},\end{cases} \tag{3.64}$$

hold. From (3.63), we can write the following inequalities

$$\begin{cases} Au_2 + l_2 \leq l_1 u_2, \\ l_2 u_1 \leq A l_2 + u_2, \\ Au_3 + l_3 \leq l_2 u_3, \\ l_3 u_2 \leq A l_3 + u_3, \\ Au_1 + l_1 \leq l_3 u_1, \\ l_1 u_3 \leq A l_1 + u_1. \end{cases} \tag{3.65}$$

Now, by multiplying both sides of the first inequality in (3.65) by u_3 and both sides of the sixth one in (3.65) by u_2, then we can write

$$Au_2u_3 + l_2u_3 \leq l_1u_2u_3, \; u_2u_3l_1 \leq Al_1u_2 + u_1u_2, \tag{3.66}$$

from which it follows that

$$Au_2u_3 + l_2u_3 \leq Al_1u_2 + u_1u_2. \tag{3.67}$$

Similarly, upon multiplying both sides of the second inequality in (3.65) by u_3 and both sides of the third one in (3.65) by u_1, then we have

$$l_2u_1u_3 \leq Al_2u_3 + u_2u_3, \; Au_1u_3 + l_3u_1 \leq l_2u_1u_3, \tag{3.68}$$

from which it follows that

$$Au_1u_3 + l_3u_1 \leq Al_2u_3 + u_2u_3 \tag{3.69}$$

Likewise, upon multiplying both sides of the fourth inequality (3.65) by u_1 and both sides of the fifth one in (3.65) by u_2, we reach the following inequality

$$l_3u_1u_2 \leq Al_3u_1 + u_1u_3, \; Au_1u_2 + l_1u_2 \leq l_3u_1u_2, \tag{3.70}$$

from which it follows that

$$Au_1u_2 + l_1u_2 \leq Al_3u_1 + u_1u_3. \tag{3.71}$$

From (3.67), (3.69) and (3.71), we have

$$\begin{aligned} &Au_2u_3 + l_2u_3 + Au_1u_3 + l_3u_1 + Au_1u_2 + l_1u_2 \\ &\quad \leq Al_1u_2 + u_1u_2 + Al_2u_3 + u_2u_3 + Al_3u_1 + u_1u_3 \end{aligned} \tag{3.72}$$

which implies that

$$Au_3(u_2 - l_2) + Au_1(u_3 - l_3) + Au_2(u_1 - l_1) - u_3(u_2 - l_2) - u_1(u_3 - l_3) - u_2(u_1 - l_1) \leq 0 \tag{3.73}$$

and, consequently

$$(A-1)u_3(u_2 - l_2) + (A-1)u_1(u_3 - l_3) + (A-1)u_2(u_1 - l_1) \leq 0. \tag{3.74}$$

Since $A > 1$, from (3.74), we can reach the following results

$$u_1 = l_1, \; u_2 = l_2, \; u_3 = l_3, \tag{3.75}$$

from which along with (3.62), it follows that

$$\lim_{n\to\infty} x_n = A+1,\ \lim_{n\to\infty} y_n = A+1,\ \lim_{n\to\infty} z_n = A+1, \tag{3.76}$$

which is desired. □

4. Rate of convergence

In this section, we investigate the rate of convergence of a solutions which converges to the equilibrium point $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$ of the system (1.5) in the region of parameters described by $A > 1$. The following result gives the rate of convergence of solutions of the system of difference equations

$$U_{n+1} = (A+B_n)U_n \tag{4.1}$$

where U_n is a $(3k+3)$-dimensional vector, $A \in C^{(3k+3)\times(3k+3)}$ is a constant matrix and $B : \mathbb{Z}^+ \to C^{(3k+3)\times(3k+3)}$ is a matrix function providing

$$\|B_n\| \to 0, \quad \text{as} \quad n\to\infty, \tag{4.2}$$

where $\|.\|$ denotes any matrix norm.

Theorem 14. *(Perron's Theorem, see, [14]) Assume that condition* (4.2) *holds. If U_n is a solution of* (4.1)*, then either $U_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|U_{n+1}\|}{\|U_n\|}$$

or

$$\rho = \lim_{n\to\infty} \left(\|U_n\|\right)^{\frac{1}{n}}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 15. *Assume that $\{(x_n,y_n,z_n)\}_{n\geq -k}$ is a solution of system* (1.5) *such that the equalities in* (3.76) *are provided. Then, the error vector*

$$e_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ \vdots \\ e_{n-k}^1 \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-k}^2 \\ e_n^3 \\ e_{n-1}^3 \\ \vdots \\ e_{n-k}^3 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ x_{n-1} - \bar{x} \\ \vdots \\ x_{n-k} - \bar{x} \\ y_n - \bar{y} \\ y_{n-1} - \bar{y} \\ \vdots \\ y_{n-k} - \bar{y} \\ z_n - \bar{z} \\ z_{n-1} - \bar{z} \\ \vdots \\ z_{n-k} - \bar{x} \end{pmatrix}$$

of every solution of system (1.5) *satisfies both of the asymptotic relations for some* $i \in \{1,2,\dots,k\}$,

$$\rho = \lim_{n\to\infty}(||U_n||)^{\frac{1}{n}} = |\lambda_i J_F(\bar{x},\bar{y},\bar{z})|, \quad \rho = \lim_{n\to\infty}\frac{||U_{n+1}||}{||U_n||} = |\lambda_i J_F(\bar{x},\bar{y},\bar{z})|, \tag{4.3}$$

where ρ *is equal to the modulus of one of the eigenvalues of* $\mathcal{J}_{\mathcal{F}}$ *about* $(\bar{x},\bar{y},\bar{z})$.

Proof. Let $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be any solution to system (1.5) that satisfies the following conditions

$$\lim_{n\to\infty} x_n = \bar{x} = A+1, \quad \lim_{n\to\infty} y_n = \bar{y} = A+1, \quad \lim_{n\to\infty} z_n = \bar{z} = A+1. \tag{4.4}$$

To obtain the error terms, one gets

$$x_{n+1} - \bar{x} = A + \frac{y_n}{y_{n-k}} - A - 1 = \frac{y_n - y_{n-k}}{y_{n-k}} = \frac{y_n - \bar{y}}{y_{n-k}} - \frac{y_{n-k} - \bar{y}}{y_{n-k}}, \tag{4.5}$$

$$y_{n+1} - \bar{y} = A + \frac{z_n}{z_{n-k}} - A - 1 = \frac{z_n - z_{n-k}}{z_{n-k}} = \frac{z_n - \bar{z}}{z_{n-k}} - \frac{z_{n-k} - \bar{z}}{z_{n-k}}, \tag{4.6}$$

$$z_{n+1} - \bar{z} = A + \frac{x_n}{x_{n-k}} - A - 1 = \frac{x_n - x_{n-k}}{x_{n-k}} = \frac{x_n - \bar{x}}{x_{n-k}} - \frac{x_{n-k} - \bar{x}}{x_{n-k}}, \tag{4.7}$$

for $i \in \{1,2,\dots,k\}$. Now, let

$$e_n^1 = x_n - \bar{x}, \quad e_n^2 = y_n - \bar{y}, \quad e_n^3 = z_n - \bar{z}, \tag{4.8}$$

and then the equations in (4.5)-(4.7) become

$$e_{n+1}^1 = \frac{e_n^2}{y_{n-k}} - \frac{e_{n-k}^2}{y_{n-k}}, \tag{4.9}$$

$$e_{n+1}^2 = \frac{e_n^3}{z_{n-k}} - \frac{e_{n-k}^3}{z_{n-k}}, \tag{4.10}$$

$$e_{n+1}^3 = \frac{e_n^1}{x_{n-k}} - \frac{e_{n-k}^1}{x_{n-k}}. \tag{4.11}$$

Now, we let $\mathcal{A}_{1i} = \mathcal{C}_{1i} = \mathcal{A}_{2i} = \mathcal{B}_{2i} = \mathcal{B}_{3i} = \mathcal{C}_{3i} = 0$, for $i \in \{0,1,\dots,k\}$, $\mathcal{B}_{10} = \frac{1}{y_{n-k}}$, $\mathcal{B}_{1j} = 0$, for $j \in \{1,2,\dots,k-1\}$, $\mathcal{B}_{1k} = -\frac{1}{y_{n-k}}$, $\mathcal{C}_{20} = \frac{1}{z_{n-k}}$, $\mathcal{C}_{2j} = 0$, for $j \in \{1,2,\dots,k-1\}$, $\mathcal{C}_{2k} = -\frac{1}{z_{n-k}}$, $\mathcal{A}_{30} = \frac{1}{x_{n-k}}$, $\mathcal{A}_{3j} = 0$, for $j \in \{1,2,\dots,k-1\}$ and $\mathcal{A}_{3k} = -\frac{1}{x_{n-k}}$. Then, the equations in (4.9)-(4.11) can be written the next form of

$$e_{n+1}^1 = \sum_{i=0}^{k} \mathcal{A}_{1i} e_{n-i}^1 + \sum_{i=0}^{k} \mathcal{B}_{1i} e_{n-i}^2 + \sum_{i=0}^{k} \mathcal{C}_{1i} e_{n-i}^3, \tag{4.12}$$

$$e_{n+1}^2 = \sum_{i=0}^{k} \mathcal{A}_{2i} e_{n-i}^1 + \sum_{i=0}^{k} \mathcal{B}_{2i} e_{n-i}^2 + \sum_{i=0}^{k} \mathcal{C}_{2i} e_{n-i}^3, \tag{4.13}$$

$$e_{n+1}^3 = \sum_{i=0}^{k} \mathcal{A}_{3i} e_{n-i}^1 + \sum_{i=0}^{k} \mathcal{B}_{3i} e_{n-i}^2 + \sum_{i=0}^{k} \mathcal{C}_{3i} e_{n-i}^3. \tag{4.14}$$

Then we get

$$\lim_{n\to\infty} \mathcal{A}_{1i} = \lim_{n\to\infty} \mathcal{C}_{1i} = \lim_{n\to\infty} \mathcal{A}_{2i} = \lim_{n\to\infty} \mathcal{B}_{2i} = \lim_{n\to\infty} \mathcal{B}_{3i} = \lim_{n\to\infty} \mathcal{C}_{3i} = 0, \tag{4.15}$$

for $i \in \{0,1,\dots,k\}$,

$$\lim_{n\to\infty} \mathcal{B}_{10} = \frac{1}{\bar{y}}, \quad \lim_{n\to\infty} \mathcal{C}_{20} = \frac{1}{\bar{z}}, \quad \lim_{n\to\infty} \mathcal{A}_{30} = \frac{1}{\bar{x}}, \tag{4.16}$$

$$\lim_{n\to\infty} \mathcal{B}_{1j} = \lim_{n\to\infty} \mathcal{C}_{2j} = \lim_{n\to\infty} \mathcal{A}_{3j} = 0, \tag{4.17}$$

for $j \in \{1,2,\dots,k-1\}$, and

$$\lim_{n\to\infty} \mathcal{B}_{1k} = -\frac{1}{\bar{y}}, \quad \lim_{n\to\infty} \mathcal{C}_{2k} = -\frac{1}{\bar{z}}, \quad \lim_{n\to\infty} \mathcal{A}_{3k} = -\frac{1}{\bar{x}}, \tag{4.18}$$

that is,

$$\begin{cases} \mathcal{B}_{10} = \frac{1}{\bar{y}} + a_n, & \mathcal{B}_{1k} = -\frac{1}{\bar{y}} + b_n, \\ \mathcal{C}_{20} = \frac{1}{\bar{z}} + \alpha_n, & \mathcal{C}_{2k} = -\frac{1}{\bar{z}} + \beta_n, \\ \mathcal{A}_{30} = \frac{1}{\bar{x}} + \gamma_n, & \mathcal{A}_{3k} = -\frac{1}{\bar{x}} + \delta_n, \end{cases} \tag{4.19}$$

where $a_n \to 0,\ b_n \to 0,\ \alpha_n \to 0,\ \beta_n \to 0,\ \gamma_n \to 0,\ \delta_n \to 0$, for $n \to \infty$. Then, we obtain the system

$$\mathcal{E}_{n+1} = (A + B_n)\mathcal{E}_n, \tag{4.20}$$

where $\mathcal{E}_n = (e_n^1, e_{n-1}^1, \dots, e_{n-k}^1, e_n^2, e_{n-1}^2, \dots, e_{n-k}^2, e_n^3, e_{n-1}^3, \dots, e_{n-k}^3)^T$ and, the constant matrix A is of the form

$$A = \begin{pmatrix}
0 & 0\cdots0 & 0 & a_n & 0\cdots0 & b_n & 0 & 0\cdots0 & 0 \\
1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
 & & & & & & & & \\
0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & \alpha_n\cdots0 & \beta_n \\
0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
 & & & & & & & & \\
0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 \\
\gamma_n & 0\cdots0 & \delta_n & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\
0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 \\
0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 \\
 & & & & & & & & \\
0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0
\end{pmatrix} \tag{4.21}$$

and

$$B_n = \begin{pmatrix} 0 & 0\cdots 0 & 0 & \frac{1}{A+1} & 0\cdots 0 & \frac{-1}{A+1} & 0 & 0\cdots 0 & 0 \\ 1 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ 0 & 1\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots 1 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & \frac{1}{A+1} & 0\cdots 0 & \frac{-1}{A+1} \\ 0 & 0\cdots 0 & 0 & 1 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ 0 & 0\cdots 0 & 0 & 0 & 1\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 1 & 0 & 0 & 0\cdots 0 & 0 \\ \frac{1}{A+1} & 0\cdots 0 & \frac{-1}{A+1} & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 1 & 0\cdots 0 & 0 \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 1\cdots 0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 1 & 0 \end{pmatrix} \tag{4.22}$$

where $\|B_n\| \to 0$ as $n \to \infty$. The matrix A is equal to the $\mathcal{J}_{\mathcal{F}}$. So, by applying Theorem 14 to system (1.5), the result holds. □

5. NUMERICAL EXAMPLE

To support our theoretical results, we will present some interesting numerical examples. These examples present different types of qualitative behavior of solutions to system (1.5).

Example 1. Consider the system (1.5) with the initial values $x_{-2} = 2.1$, $x_{-1} = 0.67$, $x_0 = 10.21$, $y_{-2} = 6.45$, $y_{-1} = 9.38$, $y_0 = 4.45$, $z_{-2} = 0.47$, $z_{-1} = 4.41$ and $z_0 = 0.67$. Further, we take the parameters $k = 2$ and $A = 1.5$, i.e.,

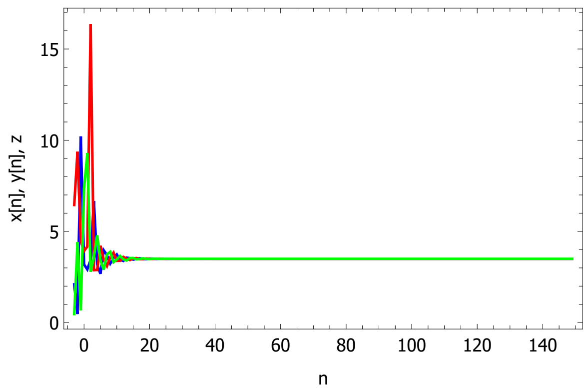

FIGURE 1. The plot of system (1.5) when $A > 1$ and $k = 2$.

This figure shows the global attractivity of the equilibrium point

$$(\bar{x}, \bar{y}, \bar{z}) = (2.5, 2.5, 2.5)$$

of system (1.5).

Example 2. Consider the system (1.5) with the initial values $x_{-3} = 3.7$, $x_{-2} = 0.1$, $x_{-1} = 5.47$, $x_0 = 1.17$, $y_{-3} = 0.45$, $y_{-2} = 1.1$, $y_{-1} = 4.38$, $y_0 = 0.87$, $z_{-3} = 1.8$, $z_{-2} = 0.67$, $z_{-1} = 0.41$ and $z_0 = 11.28$. Further, we take the parameters $k = 3$ and $A = 1$, i.e.,

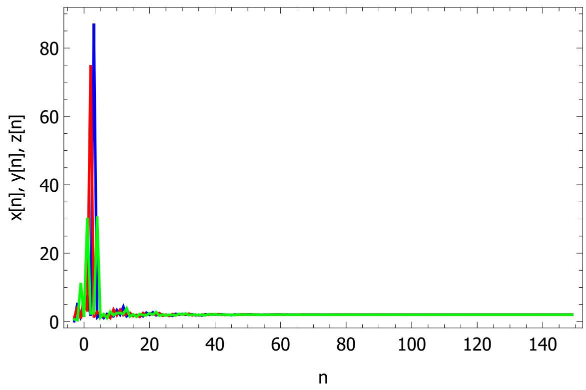

FIGURE 2. The plot of system (1.5) when $A = 1$ and $k = 3$.

Example 3. Consider the system (1.5) with the initial values $x_{-4} = 2.41$, $x_{-3} = 3.39$, $x_{-2} = 0.27$, $x_{-1} = 1.91$, $x_0 = 0.19$, $y_{-4} = 7.82$, $y_{-3} = 2.71$, $y_{-2} = 4.33$, $y_{-1} = 0.38$, $y_0 = 0.01$, $z_{-4} = 0.82$, $z_{-3} = 2.73$, $z_{-2} = 0.08$, $z_{-1} = 0.31$ and $z_0 = 5.28$. Further, we take the parameters $k = 4$ and $A = 0.19$, i.e.,

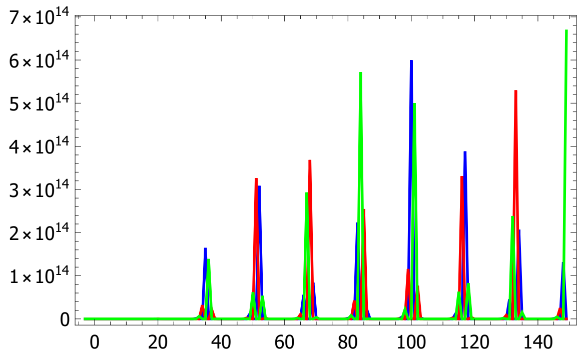

FIGURE 3. The plot of system (1.5) when $A < 1$ and $k = 4$.

This figure shows that the equilibrium point $(\bar{x},\bar{y},\bar{z}) = (1.19, 1.19, 1.19)$ of system (1.5) is not global asymptotic stable. Further, system (1.5) has unbounded solutions.

6. Conclusion

This study represents a contribution to the analysis of three-dimensional concrete nonlinear system of difference equations. This paper mainly discusses the dynamic properties of a class of higher-order system of difference equations by utilizing semi-cycle analysis, stability theory and rate of convergence. The main results are as follows.

i) From semi-cycle analysis of system (1.5), it is obtained that system (1.5) has no non-oscillatory negative solutions, no decreasing non-oscillatory solutions, no nontrivial periodic solutions of period k. It is also obtained that the solution of system (1.5) is either non-oscillatory solution or it oscillates about the equilibrium point of system (1.5), with semi-cycles having $k+1$ terms.

ii) When $A \geq 1$, the positive solution of system (1.5) is bounded and persists. Further system (1.5) has no positive nontrivial solution prime period-two in this case $A = 1$.

iii) When $A \geq 1$, system (1.5) has no non-oscillatory positive solutions.

iv) When $A > 1$, the unique equilibrium point of system (1.5) is globally asymptotically stable.

Acknowledgements

This study is a part of the first author's Ph.D. Thesis.

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