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A REMARK ON MIDPOINT-TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. This paper proves an equality for the case of differentiable functions involving the conformable fractional integrals. By using this equality, we establish new midpoint-type inequalities via convex functions with the aid of the conformable fractional integrals. Some significant inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Moreover, we give example using graph in order to show that our main results are correct. Furthermore, our results generalize known results from the literature by special choices.

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1. Introduction

The convex theory is an available way to solve a large number of problems from different branches of mathematics. One of the most famous inequalities for the case of convex functions is the Hermite-Hadamard-type inequality. Thus, the Hermite-Hadamard-type inequalities have been considered seriously by many researchers in the last century. In addition, Riemann-Liouville fractional integrals, conformable fractional integrals, and many types of fractional integrals have been investigated to these type of inequalities.

Fractional calculus is a field of mathematics that expands the traditional derivative and integral ideas to non-integer orders. In recent decades, it has piqued the curiosity of mathematicians, physicists, and engineers [3, 24]. Fractional derivatives are also used to model a wide range of mathematical biology, as well as chemical processes and engineering problems [4, 9]. Fractional integral operators and their applicability in a variety of scientific disciplines have been concentrated. Using the derivative's

fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative is improved in paper [16]. Some important requirements that cannot be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. Nevertheless, in paper [1] the author proved that the conformable approach in [16] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by some extensions of the conformable approach [12,25].

A large number of mathematicians have focused on acquired trapezoid and midpoint type inequalities that give bounds via the right-hand side and left-hand side of the Hermite–Hadamard inequality, respectively. Dragomir and Agarwal first established trapezoid inequalities for convex functions in [8], while Kirmacı first proved inequalities of midpoint-type for convex functions in [18]. Moreover, Qaisar and Hussain in [21] investigated several generalized midpoint-type inequalities. Sarikaya et al. and Iqbal et al. presented some fractional midpoint and trapezoid-type inequalities for the case of convex mappings in papers [20] and [22], respectively. Furthermore, some Hermite–Hadamard-type inequalities for the fractional integrals are given in paper [13]. In addition to these, many mathematicians have considered twice differentiable functions to get many important inequalities. For instance, Barani et al. proved inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality in papers [5, 6]. Some generalized fractional integral inequalities of midpoint-type and trapezoid-type for the case of twice differentiable convex functions are obtained in paper [19]. In [23], authors obtained some new inequalities of the Simpson and the Hermite–Hadamard-type for functions whose absolute values of derivatives are convex. Some recent results connected with fractional integral inequalities, see [7, 10] and the references cited therein.

With the aid of the continuing studies and mentioned papers above, we will establish some midpoint-type inequalities for differentiable convex function involving conformable fractional integrals. The entire form of the study takes the form of four sections including introduction. In Section 2, the basic definitions of Riemann-Liouville integrals and conformable integrals will be explained for building our main results. In Section 3, an identity will be established for the case of differentiable convex functions including the conformable fractional integrals. By using this equality, we prove some midpoint-type inequalities by convex functions with the aid of the conformable fractional integrals. Moreover, we present example using graph in order to show that our main results are correct. Furthermore, we also give some corollaries and remarks.

2. PRELIMINARIES

For building our main results, this section discusses the fundamental definitions of Riemann-Liouville integrals and conformable integrals, which are well known in the literature, are given as follows:

Definition 1. For $0 < x, y < \infty$ and $x, y \in \mathbb{R}$, the gamma function, beta function, and incomplete beta function are described

$$\Gamma(x) := \int_0^\infty \mu^{x-1} e^{-\mu} d\mu, \qquad \mathcal{B}(x,y) := \int_0^1 \mu^{x-1} (1-\mu)^{y-1} d\mu,$$

and

$$\mathcal{B}(x,y,r) := \int_0^r \mu^{x-1} (1-\mu)^{y-1} d\mu,$$

respectively.

In paper [17], Kilbas et al. defined fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 2 ([17, Page 71])**.** For $F \in L_1[\sigma, \delta]$, the Riemann-Liouville integrals $J^{\beta}_{\sigma+}F(x)$ and $J^{\beta}_{\delta-}F(x)$ of order $\beta > 0$ are given by

$$J^{\beta}_{\sigma+}F(x) = \frac{1}{\Gamma(\beta)} \int_\sigma^x (x-\mu)^{\beta-1} F(\mu) d\mu, \qquad x > \sigma \tag{2.1}$$

and

$$J^{\beta}_{\delta-}F(x) = \frac{1}{\Gamma(\beta)} \int_x^\delta (\mu-x)^{\beta-1} F(\mu) d\mu, \qquad x < \delta, \tag{2.2}$$

respectively. The Riemann-Liouville integrals reduce to the classical integrals for the case of $\beta = 1$.

In paper [15], Jarad et al. presented the fractional conformable integral operators. They also provided certain characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined as follows:

Definition 3 ([15, Page 5])**.** For $F \in L_1[\sigma, \delta]$, the fractional conformable integral operators ${}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(x)$ and ${}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(x)$ of order $\beta \in C$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are given by

$${}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(x) = \frac{1}{\Gamma(\beta)} \int_\sigma^x \left(\frac{(x-\sigma)^{\alpha} - (\mu-\sigma)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{F(\mu)}{(\mu-\sigma)^{1-\alpha}} d\mu, \qquad \mu > \sigma \tag{2.3}$$

and

$${}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(x) = \frac{1}{\Gamma(\beta)} \int_x^\delta \left(\frac{(\delta-x)^{\alpha} - (\delta-\mu)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{F(\mu)}{(\delta-\mu)^{1-\alpha}} d\mu, \qquad \mu < \delta, \tag{2.4}$$

respectively.

If we consider $\sigma = 0$ and $\alpha = 1$, then the fractional integral in (2.3) reduces to the Riemann-Liouville fractional integral in (2.1). Furthermore, the fractional integral in (2.4) coincides with the Riemann-Liouville fractional integral in (2.2) when $\delta = 0$ and $\alpha = 1$. Some recent results connected with fractional integral inequalities, see [2, 14] and the references cited therein.

3. Main Results

In this section, midpoint-type inequalities are created for differentiable convex functions by using the conformable fractional integrals. Let us first set up the following identity for obtaining midpoint-type inequalities.

Lemma 1. *Note that* $F\colon [\sigma,\delta] \to \mathbb{R}$ *is a differentiable function on* (σ,δ) *such that* $F' \in L_1[\sigma,\delta]$. *Then, the following equality holds:*

$$F\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta_-}F(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma_+}F(\delta)\right] \tag{3.1}$$
$$= \frac{(\delta-\sigma)\alpha^{\beta}}{2}\left\{A_1 - A_2 - A_3 + A_4\right\}.$$

Here,

$$\begin{cases} A_1 = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} F'(\mu\delta + (1-\mu)\sigma)\,d\mu, \\ A_2 = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} F'(\mu\sigma + (1-\mu)\delta)\,d\mu, \\ A_3 = \int_{\frac{1}{2}}^{1} \left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right] F'(\mu\delta + (1-\mu)\sigma)\,d\mu, \\ A_4 = \int_{\frac{1}{2}}^{1} \left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right] F'(\mu\sigma + (1-\mu)\delta)\,d\mu. \end{cases}$$

Proof. From the fact of the integrating by parts, we obtain

$$A_1 = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} F'(\mu\delta + (1-\mu)\sigma)\,d\mu \tag{3.2}$$
$$= \frac{1}{\delta-\sigma}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} F(\mu\delta + (1-\mu)\sigma)\Big|_0^{\frac{1}{2}}$$
$$- \frac{\beta}{\delta-\sigma}\int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1} (1-\mu)^{\alpha-1} F(\mu\delta + (1-\mu)\sigma)\,d\mu$$

$$= \frac{1}{\delta-\sigma}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta} \mathrm{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$- \frac{\beta}{\delta-\sigma}\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\delta+(1-\mu)\sigma)\,\mathrm{d}\mu.$$

Similar to foregoing process, we have

$$A_2 = -\frac{1}{\delta-\sigma}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta} \mathrm{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$+ \frac{\beta}{\delta-\sigma}\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\sigma+(1-\mu)\delta)\,\mathrm{d}\mu,$$

$$A_3 = \frac{1}{\delta-\sigma}\left[\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta} - \frac{1}{\alpha^{\beta}}\right]\mathrm{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$+ \frac{\beta}{\delta-\sigma}\int_{\frac{1}{2}}^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\delta+(1-\mu)\sigma)\,\mathrm{d}\mu,$$

and

$$A_4 = \frac{1}{\delta-\sigma}\left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}\right]\mathrm{F}\left(\frac{\sigma+\delta}{2}\right) \tag{3.3}$$
$$- \frac{\beta}{\delta-\sigma}\int_{\frac{1}{2}}^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\sigma+(1-\mu)\delta)\,\mathrm{d}\mu.$$

If we collect from (3.2) to (3.3), then we get

$$A_1 - A_2 - A_3 + A_4 = \frac{2}{(\delta-\sigma)\,\alpha^{\beta}}\mathrm{F}\left(\frac{\sigma+\delta}{2}\right) \tag{3.4}$$
$$- \frac{\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\delta+(1-\mu)\sigma)\,\mathrm{d}\mu$$
$$- \frac{\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\sigma+(1-\mu)\delta)\,\mathrm{d}\mu.$$

If we use change of variables in (3.4), then the equality (3.4) is converted as follows:

$$\begin{aligned}
&A_1 - A_2 - A_3 + A_4 \\
&= \frac{2}{(\delta-\sigma)\alpha^{\beta}} \mathrm{F}\left(\frac{\sigma+\delta}{2}\right) \\
&\quad - \left(\frac{1}{\delta-\sigma}\right)^{\alpha\beta+1} \frac{\Gamma(\beta+1)}{\Gamma(\beta)} \int_{\sigma}^{\delta} \left(\frac{(\delta-\sigma)^{\alpha} - (\delta-x)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{\mathrm{F}(x)}{(\delta-x)^{1-\alpha}} dx \\
&\quad - \left(\frac{1}{\delta-\sigma}\right)^{\alpha\beta+1} \frac{\Gamma(\beta+1)}{\Gamma(\beta)} \int_{\sigma}^{\delta} \left(\frac{(\delta-\sigma)^{\alpha} - (x-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{\mathrm{F}(x)}{(x-\sigma)^{1-\alpha}} dx \\
&= \frac{2}{(\delta-\sigma)\alpha^{\beta}} \mathrm{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta+1}} \left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right].
\end{aligned} \tag{3.5}$$

If the equality (3.5) is multiplied by $\frac{(\delta-\sigma)\alpha^{\beta}}{2}$, then the proof of Lemma 1 is completed. □

Theorem 1. *Let us note that* $\mathrm{F}\colon [\sigma,\delta] \to \mathbb{R}$ *is a differentiable function on* (σ,δ) *such that* $\mathrm{F}' \in L_1[\sigma,\delta]$ *and* $|\mathrm{F}'|$ *is convex on* $[\sigma,\delta]$. *Then, the following inequality holds:*

$$\begin{aligned}
&\left|\mathrm{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right]\right| \\
&\le \frac{(\delta-\sigma)\alpha^{\beta}}{2}\left(\varphi_1(\alpha,\beta) + \varphi_2(\alpha,\beta)\right)\left(\left|\mathrm{F}'(\sigma)\right| + \left|\mathrm{F}'(\delta)\right|\right).
\end{aligned} \tag{3.6}$$

Here,

$$\begin{cases}
\varphi_1(\alpha,\beta) = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} \mathrm{d}\mu \\
= \frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right) - \mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right], \\
\varphi_2(\alpha,\beta) = \int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\mathrm{d}\mu = \frac{1}{2\alpha^{\beta}} - \frac{1}{\alpha^{\beta+1}}\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right),
\end{cases} \tag{3.7}$$

where $\mathcal{B}$ *and* $\mathscr{B}$ *denote the beta function and incomplete beta function, respectively.*

Proof. If we take the absolute value of both sides of (3.1), then we obtain

$$\left|\mathrm{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right]\right| \tag{3.8}$$

$$\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2}\left\{\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left|F'(\mu\delta+(1-\mu)\sigma)\right|d\mu\right.$$
$$+\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left|F'(\mu\sigma+(1-\mu)\delta)\right|d\mu$$
$$+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left|F'(\mu\delta+(1-\mu)\sigma)\right|d\mu$$
$$\left.+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left|F'(\mu\sigma+(1-\mu)\delta)\right|d\mu\right\}.$$

It is known that $|F'|$ is convex on $[\sigma,\delta]$. It follows

$$\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(\delta)\right]\right|$$
$$\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2}$$
$$\times\left\{\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right.$$
$$\times\left(\mu\left|F'(\delta)\right|+(1-\mu)\left|F'(\sigma)\right|+\mu\left|F'(\sigma)\right|+(1-\mu)\left|F'(\delta)\right|\right)d\mu$$
$$+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]$$
$$\left.\times\left(\mu\left|F'(\delta)\right|+(1-\mu)\left|F'(\sigma)\right|+\mu\left|F'(\sigma)\right|+(1-\mu)\left|F'(\delta)\right|\right)d\mu\right\}$$
$$=\frac{(\delta-\sigma)\alpha^{\beta}}{2}\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}d\mu+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]d\mu\right)$$
$$\times\left(\left|F'(\sigma)\right|+\left|F'(\delta)\right|\right).$$

This completes the proof of Theorem 1. □

Example 1. Let $F\colon[0,1]\to\mathbb{R}$ be defined by $F(x)=x^2$. From Theorem 1 with $\beta\in(0,10)$ and $\alpha\in(0,1)$, the left side of (3.6) becomes

$$\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(\delta)\right]\right| \tag{3.9}$$
$$=\left|\beta\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right]-\frac{1}{4}\right|$$

and the right side of (3.6) becomes

$$\frac{(\delta-\sigma)\alpha^{\beta}}{2}\left(\varphi_1(\alpha,\beta)+\varphi_2(\alpha,\beta)\right)\left(\left|F'(\sigma)\right|+\left|F'(\delta)\right|\right) \tag{3.10}$$
$$=\frac{1}{2}+\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right)-2\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right].$$

After calculating (3.9) and (3.10), the graph is shown as in Figure 1 using the MATLAB software.

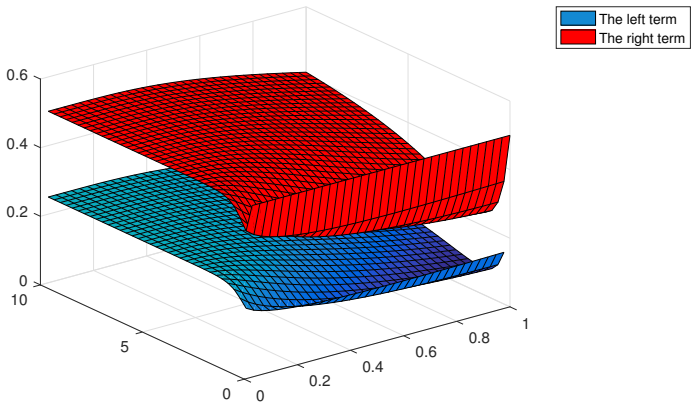

FIGURE 1. Plot illustration for Theorem 1.

Hence, we can see that the inequality (3.6) is valid in Figure 1.

Remark 1. If we set $\alpha=1$ in Theorem 1, then Theorem 1 reduces to [11, Theorem 2.1].

Remark 2. For $\alpha=1$ and $\beta=1$ in Theorem 1, Theorem 1 becomes to [18, Theorem 2.2].

Theorem 2. *Note that* $F:[\sigma,\delta]\to\mathbb{R}$ *is a differentiable function on* (σ,δ) *so that* $F'\in L_1[\sigma,\delta]$. *Note also that* $|F'|^q$ *is convex on* $[\sigma,\delta]$ *with* $q>1$. *Then, the following*

inequalities hold:

$$\begin{aligned}
&\left|\mathrm{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right]\right| \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2^{1+\frac{1}{q}}}\left(\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}+\left(\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right) \\
&\quad\times\left[\left(\frac{|\mathrm{F}'(\delta)|^q+3|\mathrm{F}'(\sigma)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathrm{F}'(\sigma)|^q+3|\mathrm{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2^{1+\frac{1}{q}}}\left(\left(4\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}+\left(4\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right)\left[\left|\mathrm{F}'(\sigma)\right|+\left|\mathrm{F}'(\delta)\right|\right].
\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\begin{cases}
\psi_1^{\alpha,\beta}(p)=\int\limits_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta p}\mathrm{d}\mu, \\
\psi_2^{\alpha,\beta}(p)=\int\limits_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]^{p}\mathrm{d}\mu.
\end{cases}$$

Proof. If we use Hölder inequality in (3.8), then we have

$$\begin{aligned}
&\left|\mathrm{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right]\right| \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2} \\
&\quad\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta p}\mathrm{d}\mu\right)^{\frac{1}{p}}\right. \\
&\quad\times\left[\left(\int_0^{\frac{1}{2}}\left|\mathrm{F}'(\mu\delta+(1-\mu)\sigma)\right|^q\mathrm{d}\mu\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}\left|\mathrm{F}'(\mu\sigma+(1-\mu)\delta)\right|^q\mathrm{d}\mu\right)^{\frac{1}{q}}\right] \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]^{p}\mathrm{d}\mu\right)^{\frac{1}{p}}\right.
\end{aligned}$$

$$\times \left[\left(\int_{\frac{1}{2}}^{1} \left| F'\left(\mu\delta + (1-\mu)\sigma\right)\right|^q d\mu \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{1} \left| F'\left(\mu\sigma + (1-\mu)\delta\right)\right|^q d\mu \right)^{\frac{1}{q}} \right] \Bigg\}.$$

From the fact that $|F'|^q$ is convex on $[\sigma,\delta]$, then it yields

$$\begin{aligned}
&\left| F\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^\beta \Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}} \left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-} F(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+} F(\delta) \right] \right| \\
&\leq \frac{(\delta-\sigma)\alpha^\beta}{2} \Bigg\{ \left(\int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^{\beta p} d\mu \right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_0^{\frac{1}{2}} \left(\mu \left| F'(\delta)\right|^q + (1-\mu)\left| F'(\sigma)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(\int_0^{\frac{1}{2}} \left(\mu \left| F'(\sigma)\right|^q + (1-\mu)\left| F'(\delta)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right] \\
&\quad + \left(\int_{\frac{1}{2}}^{1} \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^{\beta} \right]^p d\mu \right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_{\frac{1}{2}}^{1} \left(\mu \left| F'(\delta)\right|^q + (1-\mu)\left| F'(\sigma)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(\int_{\frac{1}{2}}^{1} \left(\mu \left| F'(\sigma)\right|^q + (1-\mu)\left| F'(\delta)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right] \Bigg\} \\
&= \frac{(\delta-\sigma)\alpha^\beta}{2^{1+\frac{1}{q}}} \left(\left(\psi_1^{\alpha,\beta}(p) \right)^{\frac{1}{p}} + \left(\psi_2^{\alpha,\beta}(p) \right)^{\frac{1}{p}} \right) \\
&\quad \times \left[\left(\frac{\left| F'(\delta)\right|^q + 3\left| F'(\sigma)\right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left| F'(\sigma)\right|^q + 3\left| F'(\delta)\right|^q}{4} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

The second inequality of Theorem 2 can be obtained immediately by letting $\varpi_1 = 3\left|F'(\sigma)\right|^q$, $\rho_1 = \left|F'(\delta)\right|^q$, $\varpi_2 = \left|F'(\sigma)\right|^q$ and $\rho_2 = 3\left|F'(\delta)\right|^q$ and applying the inequality:

$$\sum_{k=1}^{n}(\varpi_k+\rho_k)^s \le \sum_{k=1}^{n}\varpi_k^s + \sum_{k=1}^{n}\rho_k^s, \quad 0 \le s < 1.$$

Thus, the proof of Theorem 2 is finished. □

Corollary 1. *If we set* $\alpha = 1$ *in Theorem 2, then we derive*

$$\begin{aligned}
&\left|F\left(\frac{\sigma+\delta}{2}\right) - \frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\beta}}\left[J^{\beta}_{\delta-}F(\sigma) + J^{\beta}_{\sigma+}F(\delta)\right]\right| \\
&\le \frac{\delta-\sigma}{2^{1+\frac{1}{q}}} \\
&\quad\times\left(\left(\frac{1}{(\beta p+1)2^{\beta p+1}}\right)^{\frac{1}{p}} + \left(\frac{1}{\beta}\left[\mathcal{B}\left(\frac{1}{\beta},p+1\right) - \mathscr{B}\left(\frac{1}{\beta},p+1,\left(\frac{1}{2}\right)^{\beta}\right)\right]\right)^{\frac{1}{p}}\right) \\
&\quad\times\left[\left(\frac{\left|F'(\delta)\right|^q + 3\left|F'(\sigma)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|F'(\sigma)\right|^q + 3\left|F'(\delta)\right|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\le \frac{\delta-\sigma}{2^{1+\frac{1}{q}}}\left(\left(\frac{4}{(\beta p+1)2^{\beta p+1}}\right)^{\frac{1}{p}}\right. \\
&\quad\left.+\left(\frac{4}{\beta}\left[\mathcal{B}\left(\frac{1}{\beta},p+1\right) - \mathscr{B}\left(\frac{1}{\beta},p+1,\left(\frac{1}{2}\right)^{\beta}\right)\right]\right)^{\frac{1}{p}}\right)\left[\left|F'(\sigma)\right| + \left|F'(\delta)\right|\right].
\end{aligned}$$

Remark 3. If we take $\alpha = 1$ and $\beta = 1$ in Theorem 2, we have

$$\begin{aligned}
&\left|F\left(\frac{\sigma+\delta}{2}\right) - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}F(x)\,dx\right| \\
&\le \frac{\delta-\sigma}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|F'(\delta)\right|^q + 3\left|F'(\sigma)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|F'(\sigma)\right|^q + 3\left|F'(\delta)\right|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\le \frac{\delta-\sigma}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left|F'(\sigma)\right| + \left|F'(\delta)\right|\right],
\end{aligned}$$

which is given in [18, Theorem 2.4].

Theorem 3. *Suppose that* $F\colon [\sigma,\delta] \to \mathbb{R}$ *is a differentiable function on* (σ,δ) *such that* $F' \in L_1([\sigma,\delta])$. *Suppose also that* $\left|F'\right|^q$ *is convex on* $[\sigma,\delta]$ *with* $q \ge 1$. *Then, the*

following inequality holds:

$$\begin{aligned}
&\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2} \\
&\quad\times\left\{(\varphi_1(\alpha,\beta))^{1-\frac{1}{q}}\right. \\
&\qquad\times\left[(\varphi_3(\alpha,\beta)\left|F'(\delta)\right|^q+(\varphi_1(\alpha,\beta)-\varphi_3(\alpha,\beta))\left|F'(\sigma)\right|^q)^{\frac{1}{q}}\right. \\
&\qquad\quad\left.+(\varphi_3(\alpha,\beta)\left|F'(\sigma)\right|^q+(\varphi_1(\alpha,\beta)-\varphi_3(\alpha,\beta))\left|F'(\delta)\right|^q)^{\frac{1}{q}}\right] \\
&\qquad+(\varphi_2(\alpha,\beta))^{1-\frac{1}{q}} \\
&\qquad\times\left[(\varphi_4(\alpha,\beta)\left|F'(\delta)\right|^q+(\varphi_2(\alpha,\beta)-\varphi_4(\alpha,\beta))\left|F'(\sigma)\right|^q)^{\frac{1}{q}}\right. \\
&\qquad\quad\left.\left.+(\varphi_4(\alpha,\beta)\left|F'(\sigma)\right|^q+(\varphi_2(\alpha,\beta)-\varphi_4(\alpha,\beta))\left|F'(\delta)\right|^q)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

Here, $\varphi_1(\alpha,\beta)$, $\varphi_2(\alpha,\beta)$ *are defined in* (3.7) *and*

$$\begin{cases}
\varphi_3(\alpha,\beta)=\int_0^{\frac{1}{2}}\mu\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}d\mu=\frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right. \\
\left.-\mathcal{B}\left(\frac{2}{\alpha},\beta+1\right)+\mathscr{B}\left(\frac{2}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right], \\
\varphi_4(\alpha,\beta)=\int_{\frac{1}{2}}^1\mu\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]d\mu \\
=\frac{3}{8\alpha^{\beta}}+\frac{1}{\alpha^{\beta+1}}\left[\mathscr{B}\left(\frac{2}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right],
\end{cases}$$

where $\mathcal{B}$ *and* $\mathscr{B}$ *denote the beta function and incomplete beta function, respectively.*

Proof. Let us apply power-mean inequality in (3.8). Then, we have

$$\begin{aligned}
&\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(\delta)\right]\right| \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2} \\
&\quad\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}d\mu\right)^{1-\frac{1}{q}}\right.
\end{aligned}$$

$$\times\left[\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left|\mathrm{F}'\left(\mu\delta+(1-\mu)\sigma\right)\right|^{q}\mathrm{d}\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left|\mathrm{F}'\left(\mu\sigma+(1-\mu)\delta\right)\right|^{q}\mathrm{d}\mu\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\mathrm{d}\mu\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left|\mathrm{F}'\left(\mu\delta+(1-\mu)\sigma\right)\right|^{q}\mathrm{d}\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left|\mathrm{F}'\left(\mu\sigma+(1-\mu)\delta\right)\right|^{q}\mathrm{d}\mu\right)^{\frac{1}{q}}\right]\Bigg\}.$$

Since $|\mathrm{F}'|^{q}$ is convex on $[\sigma,\delta]$, we obtain

$$\left|\mathrm{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right]\right|$$
$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{2}$$
$$\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\mathrm{d}\mu\right)^{1-\frac{1}{q}}\right.$$
$$\times\left[\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left(\mu\left|\mathrm{F}'(\delta)\right|^{q}+(1-\mu)\left|\mathrm{F}'(\sigma)\right|^{q}\right)\mathrm{d}\mu\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left(\mu\left|\mathrm{F}'(\sigma)\right|^{q}+(1-\mu)\left|\mathrm{F}'(\delta)\right|^{q}\right)\mathrm{d}\mu\right)^{\frac{1}{q}}\right]\right.$$

$$+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]d\mu\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\mu\left|F'(\delta)\right|^{q}+(1-\mu)\left|F'(\sigma)\right|^{q}\right)d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\mu\left|F'(\sigma)\right|^{q}+(1-\mu)\left|F'(\delta)\right|^{q}\right)d\mu\right)^{\frac{1}{q}}\right]\Bigg\},$$

which completes the proof of Theorem 3. □

Corollary 2. *If we set* $\alpha=1$ *in Theorem 3, then we acquire*

$$\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}\left[J^{\beta}_{\delta-}F(\sigma)+J^{\beta}_{\sigma+}F(\delta)\right]\right|$$
$$\leq\frac{\delta-\sigma}{2}$$
$$\times\left\{\left(\frac{1}{(\beta+1)2^{\beta+1}}\right)\right.$$
$$\times\left[\left(\frac{(\beta+1)|F'(\delta)|^{q}}{2(\beta+2)}+\frac{(\beta+3)|F'(\sigma)|^{q}}{2(\beta+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\beta+1)|F'(\sigma)|^{q}}{2(\beta+2)}+\frac{(\beta+3)|F'(\delta)|^{q}}{2(\beta+2)}\right)^{\frac{1}{q}}\right]+\left(\frac{2^{\beta}(\beta-1)+1}{(\beta+1)2^{\beta+1}}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{2^{\beta-1}(3\beta-2)+1}{(\beta+2)2^{\beta+2}}\left|F'(\delta)\right|^{q}+\frac{2^{\beta-1}\left(\beta^{2}+3\beta-6\right)+\beta+3}{(\beta+1)(\beta+2)2^{\beta+2}}\left|F'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{2^{\beta-1}(3\beta-2)+1}{(\beta+2)2^{\beta+2}}\left|F'(\sigma)\right|^{q}\right.$$
$$\left.\left.\left.+\frac{2^{\beta-1}\left(\beta^{2}+3\beta-6\right)+\beta+3}{(\beta+1)(\beta+2)2^{\beta+2}}\left|F'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\right\}.$$

Remark 4. If we take $\alpha = 1$ and $\beta = 1$ in Theorem 3, then we obtain

$$\left| F\left(\frac{\sigma+\delta}{2}\right) - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta} F(x)\,dx \right|$$
$$\leq \frac{\delta-\sigma}{8}\left[\left(\frac{|F'(\delta)|^q + 2|F'(\sigma)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{|F'(\sigma)|^q + 2|F'(\delta)|^q}{3}\right)^{\frac{1}{q}}\right],$$

which is given in [13].

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