Fixed point theorem on partial metric spaces involving rational expressions

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FIXED POINT THEOREM ON PARTIAL METRIC SPACES INVOLVING RATIONAL EXPRESSIONS

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Abstract. We establish a fixed point theorem involving a rational expression in a complete partial metric space. Our result generalizes a well-known result in (usual) metric spaces. Also, we introduce an example to illustrate the usability of our result.

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1. INTRODUCTION AND PRELIMINARIES

The notion of partial metric space was introduced by Matthews [10,11] in 1992. In fact, a partial metric space is a generalization of usual metric spaces in which \( d(x,x) \) are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see e.g. [1–9,12,13]).

The definition of partial metric space is given by Matthews (see e.g.[10, 11]) as follows:

**Definition 1.** Let \( X \) be a nonempty set and let \( p : X \times X \to [0,\infty) \) satisfy

\[
\begin{align*}
\text{(PM1)} & \quad x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y) \\
\text{(PM2)} & \quad p(x,x) \leq p(x,y) \\
\text{(PM3)} & \quad p(x,y) = p(y,x) \\
\text{(PM4)} & \quad p(x,y) \leq p(x,z) + p(z,y) - p(z,z)
\end{align*}
\]

for all \( x, y, z \in X \). Then the pair \((X,p)\) is called a partial metric space (in short PMS) and \( p \) is called a partial metric on \( X \).

Notice that for a partial metric \( p \) on \( X \), the function \( d_p : X \times X \to \mathbb{R}^+ \) given by

\[
d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)
\]

is a (usual) metric on \( X \). Observe that each partial metric \( p \) on \( X \) generates a \( T_0 \) topology \( \tau_p \) on \( X \) with a base of the family of open \( p \)-balls \( \{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\} \),
where $B_p(x, \varepsilon) = \{ y \in X : p(x, y) < p(x, x) + \varepsilon \}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, closed $p$-ball is defined as $B_p[x, \varepsilon] = \{ y \in X : p(x, y) \leq p(x, x) + \varepsilon \}$

Some basic concepts on partial metric spaces are defined as follows:

**Definition 2** (See e.g.[10, 11]).

(i) A sequence $\{x_n\}$ in a PMS $(X, p)$ converges to $x \in X$ if and only if $p(x, x_n) = \lim_{n \to \infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in a PMS $(X, p)$ is called Cauchy if and only if $\lim_{n, m \to \infty} p(x_n, x_m)$ exists (and finite).

(iii) A PMS $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{n, m \to \infty} p(x_n, x_m)$.

(iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon)$.

The following lemmas play important role in the proofs of our main result.

**Lemma 1** (See e.g.[10, 11]). (A) A sequence $\{x_n\}$ is Cauchy in a PMS $(X, p)$ if and only if $\{x_n\}$ is Cauchy in a metric space $(X, d_p)$.

(B) A PMS $(X, p)$ is complete if and only if a metric space $(X, d_p)$ is complete. Moreover,

$$\lim_{n \to \infty} d_p(x, x_n) = 0 \iff p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n, m \to \infty} p(x_n, x_m)$$

**Lemma 2** (See e.g.[4]). Let $(X, p)$ be a PMS. Then

(A) If $p(x, y) = 0$ then $x = y$.

(B) If $x \neq y$, then $p(x, y) > 0$.

**Lemma 3** (See e.g. [4]). Let $x_n \to z$ as $n \to \infty$ in a PMS $(X, p)$ where $p(z, z) = 0$. Then $\lim_{n \to \infty} p(x_n, y) = p(z, y)$ for every $y \in X$.

2. The Main Result

In this section we establish our main theorem which gives conditions for existence and uniqueness of a fixed point for a certain type operators on partial metric spaces.

**Theorem 1.** Let $(X, p)$ be a complete partial metric space and $T : X \to X$ be a mapping satisfying

$$\psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \forall x, y \in X,$$

where $M(x, y)$ is given by

$$M(x, y) = \max \left\{ p(y, Ty) \frac{1 + p(x, Tx)}{1 + p(x, y)} , p(x, y) \right\}$$

and $\psi : [0, \infty) \to [0, \infty)$ is a continuous and monotone non-decreasing function with $\psi(t) = 0$ if and only if $t = 0$ and $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then $T$ has a unique fixed point.
Proof. Let $x_0$ be an arbitrary point in $X$. We construct the sequence $\{x_n\}$ in $X$ as follows:

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, 3, \ldots$$

If there exists $n$ such that $x_n = x_{n+1}$ then $x_n$ is a fixed point of $T$. Now, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Letting $x = x_{n-1}$ and $y = x_n$ in (2.1), we have

$$\psi(p(x_n, x_{n+1})) = \psi(p(Tx_{n-1}, Tx_n)) \leq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n))$$

where

$$M(x_{n-1}, x_n) = \max \left\{ p(x_n, Tx_n) \frac{1 + p(x_{n-1}, Tx_{n-1})}{1 + p(x_{n-1}, x_n)}, p(x_{n-1}, x_n) \right\}.$$ 

Hence, we obtain

$$\psi(p(x_n, x_{n+1})) \leq \psi(\max \{p(x_n, x_{n+1}), p(x_{n-1}, x_n)\})$$

$$- \phi(\max \{p(x_n, x_{n+1}), p(x_{n-1}, x_n)\}) \quad (2.2)$$

If $p(x_n, x_{n+1}) > p(x_{n-1}, x_n)$, then from (2.2), we have

$$\psi(p(x_n, x_{n+1})) \leq \psi(p(x_n, x_{n+1}) - \phi(p(x_n, x_{n+1})) < \psi(p(x_n, x_{n+1}))$$

which is a contradiction since $p(x_n, x_{n+1}) > 0$ by Lemma 2. So, we have $p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n)$, that is, $\{p(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Thus, there exists $L > 0$ such that

$$\lim_{n \to \infty} p(x_n, x_{n+1}) = L \quad (2.3)$$

Suppose that $L > 0$. Taking the upper limit in (2.2) as $n \to \infty$ and using (2.3) and the properties of $\psi$, $\phi$, we have

$$\psi(L) \leq \psi(L) - \lim_{n \to \infty} \inf \phi(p(x_{n-1}, x_n)) \leq \psi(L) - \phi(L) < \psi(L)$$

which is a contradiction. Therefore

$$\lim_{n \to \infty} p(x_n, x_{n+1}) = 0 \quad (2.4)$$

Due to (1.1), we have $d_p(x_n, x_{n+1}) \leq 2p(x_n, x_{n+1})$

$$\lim_{n \to \infty} d_p(x_n, x_{n+1}) = 0 \quad (2.5)$$

Now, we prove that

$$\lim_{n,m \to \infty} p(x_n, x_m) = 0.$$

Suppose the contrary, that is,

$$\lim_{n,m \to \infty} p(x_n, x_m) \neq 0.$$

Then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}, \{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, \quad p(x_{n(k)}, x_{m(k)}) \geq \epsilon. \quad (2.6)$$
This means that
\[ p(x_{n(k)-1}, x_{m(k)}) < \varepsilon. \]  
(2.7)

From (2.6) and (2.7), we have
\[
\varepsilon \leq p(x_{n(k)}, x_{m(k)}) \leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\
\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) \\
\leq \varepsilon + p(x_{n(k)}, x_{n(k)-1})
\]

Taking \( k \to \infty \) and using (2.4), we get
\[
\lim_{k \to \infty} p(x_{n(k)}, x_{m(k)}) = \varepsilon
\]
(2.8)

By the triangle inequality, we have
\[
p(x_{n(k)}, x_{m(k)}) \leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\
\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) \\
\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) + p(x_{m(k)-1}, x_{m(k)}) \\
\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)})
\]
and
\[
p(x_{n(k)-1}, x_{m(k)-1}) \leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) - p(x_{n(k)-1}, x_{n(k)}) \\
\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) \\
\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) + p(x_{m(k)-1}, x_{m(k)-1}) \\
\leq p(x_{n(k)-1}, x_{m(k)}) + p(x_{n(k)-1}, x_{m(k)}) + p(x_{m(k)-1}, x_{m(k)-1})
\]

Taking \( k \to \infty \) in the above two inequalities and using (2.4),(2.8), we get
\[
\lim_{k \to \infty} p(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon
\]
(2.9)

Now from (2.1), we have
\[
\psi(p(x_{m(k)}, x_{n(k)})) = \psi(p(T x_{m(k)-1}, T x_{n(k)-1})) \\
\leq \psi(M(x_{m(k)-1}, x_{n(k)-1})) - \varphi(p(x_{m(k)-1}, x_{n(k)-1})
\]
(2.10)

where
\[
M(x_{m(k)-1}, x_{n(k)-1}) = \max \left\{ p(x_{n(k)-1}, x_{n(k)}) \frac{1 + p(x_{m(k)-1}, x_{m(k)})}{1 + p(x_{m(k)-1}, x_{n(k)-1})}, p(x_{m(k)-1}, x_{n(k)-1}) \right\}
\]
By (2.4), (2.8) and (2.9), we have

$$\lim_{k \to \infty} M(x_{m(k)-1}, x_{n(k)-1}) = \epsilon.$$  \hfill (2.11)

Now, passing to the upper limit when $k \to \infty$ in (2.10) and using (2.8), (2.11) and the properties of $\psi, \varphi$, we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \lim_{k \to \infty} \inf \psi(M(x_{m(k)-1}, x_{n(k)-1})) \leq \psi(\epsilon) - \psi(\epsilon) < \psi(\epsilon)$$

which is a contradiction. So, we have

$$\lim_{n,m \to \infty} p(x_n, x_m) = 0.$$  

Since $\lim_{n,m \to \infty} p(x_n, x_m)$ exists and finite, we conclude that $(x_n)$ is a Cauchy sequence in $(X, p)$.

Due to (1.1), we have $d_p(x_n, x_m) \leq 2p(x_n, x_m)$. Therefore

$$\lim_{n,m \to \infty} d_p(x_n, x_m) = 0.$$  \hfill (2.12)

Thus, by Lemma 1, $\{x_n\}$ is a Cauchy sequence in both $(X, d_p)$ and $(X, p)$.

Since $(X, p)$ is a complete partial metric space, then there exists $x \in X$ such that $\lim_{n \to \infty} p(x_n, x) = p(x, x)$. Since $\lim_{n,m \to \infty} p(x_n, x_m) = 0$, then again by Lemma 1, we have $p(x, x) = 0$. Let us now prove that $x$ is a fixed point of $T$. Suppose that $Tx \neq x$. From (2.1) and Lemma 3, we have

$$\psi(p(x_n, Tx)) = \psi(p(Tx, x))$$

$$\leq \psi(\max \left\{ \frac{p(x, Tx) 1 + p(x_{n-1}, Tx_{n-1})}{1 + p(x_{n-1}, x)}, p(x_{n-1}, x) \right\})$$

$$- \varphi(\max \left\{ \frac{p(x, Tx) 1 + p(x_{n-1}, Tx_{n-1})}{1 + p(x_{n-1}, x)}, p(x_{n-1}, x) \right\})$$  \hfill (2.13)

Letting $n \to \infty$ in the above inequality and regarding the property of $\phi, \psi$, we obtain

$$\psi(p(x, Tx)) \leq \psi(p(x, Tx)) - \varphi(p(x, Tx)) < \psi(p(x, Tx))$$

which is a contradiction. Thus $Tx = x$, that is, $x$ is fixed point of $T$. Finally, we shall prove the uniqueness. Suppose that $y$ is another fixed point of $T$ such that $x \neq y$. From (2.1),

$$\psi(p(x, y)) = \psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y))$$

$$\Rightarrow \psi(p(x, y)) - \varphi(p(x, y)) < \psi(p(x, y))$$

which is a contradiction since $p(x, y) > 0$. Hence $x = y$.

The proof is completed. \qed
Example 1. Let $X = [0, 1]$ and $p(x, y) = \max \{x, y\}$ then $(X, p)$ is a PMS. Suppose $T : X \to X$ such that $Tx = \frac{x^2}{1+x}$ for all $x \in X$ and $\phi(t) : [0, \infty) \to [0, \infty)$ such that $\psi(t) = \frac{t}{1+t}$ and $\phi(t) = t$. Without loss of generality assume $x \geq y$. Then, we have

\[
p(Tx, Ty) = \max \left\{ \frac{x^2}{1+x}, \frac{y^2}{1+y} \right\} = \frac{x^2}{1+x}\]

On the other hand,

\[
\max \left\{ p(y, Ty) \frac{1 + p(x, Tx)}{1 + p(x, y)}, p(x, y) \right\} = \max \left\{ y \frac{1+x}{1+x}, x \right\} = x
\]

Combining the observations above, we get

\[
p(Tx, Ty) = \max \left\{ \frac{x^2}{1+x}, \frac{y^2}{1+y} \right\} = \frac{x^2}{1+x} \leq x - \frac{x}{1+x} = \frac{x^2}{1+x}.
\]

Thus, it satisfies all conditions of Theorem 1. Hence, $T$ has a unique fixed point, indeed $x = 0$ is the required point.

In Theorem 1, taking $\psi(t) = t$ for all $t \in [0, \infty)$ and $\phi(t) = (1-k)t$ for all $t \in [0, \infty)$ with $k \in (0, 1)$, we get the following result.

Corollary 1. Let $(X, d)$ be a complete partial metric space and $T : X \to X$ be a mapping satisfying

\[
p(Tx, Ty) \leq k \max \left\{ p(y, Ty) \frac{1 + p(x, Tx)}{1 + p(x, y)}, p(x, y) \right\}
\]

where $k \in (0, 1)$. Then $T$ has a unique fixed point.

Example 2. Let $X = [0, 1]$. Define $T : X \times X \to X$ by $Tx = \frac{x}{2}$. Also, define $p : X \times X \to \mathbb{R}^+$ by $p(x, y) = \max \{x, y\}$, then $(X, p)$ is a complete partial metric space and

\[
p(Tx, Ty) \leq \frac{1}{2} \max \left\{ p(y, Ty) \frac{1 + p(x, Tx)}{1 + p(x, y)}, p(x, y) \right\}.
\]

Thus by Corollary 1, $T$ has a unique fixed point. Here 0 is the unique fixed point of $T$.

Example 3. Let $X = [0, \infty)$. Define $T : X \times X \to X$ by $Tx = 2x$. Also, define $p : X \times X \to \mathbb{R}^+$ by $p(x, y) = \max \{x, y\}$, then $(X, p)$ is a complete partial metric space. It is clear that Matthew’s Theorem (analog of Banach Contraction Mapping principle) does not work. Indeed, without loss of generality, we may assume that $x \leq y$. 
Then
\[ p(Tx, Ty) = 2y > ky = kp(x, y) \]
for any \( k \in [0, 1) \).
However, for \( k = \frac{1}{3} \), we have
\[ 2y = p(Tx, Ty) \leq \frac{1}{3} \frac{2y + 2x}{1 + x} = k \max \left\{ p(y, Ty), \frac{1 + p(x, Tx)}{1 + p(x, y)}, p(x, y) \right\}. \]
Thus by Corollary 1, \( T \) has a unique fixed point. Here 0 is the unique fixed point of \( T \).

REFERENCES


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