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AN EXISTENCE RESULT FOR PARABOLIC EQUATION OF KIRCHHOFF TYPE BY TOPOLOGICAL DEGREE METHOD

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Abstract. In the present paper, we will study the existence of at least one weak solution for the nonlinear parabolic initial boundary value problem associated with the following equation of Kirchoff type

$$\frac{\partial u}{\partial t} - \mathcal{M}\left(\int_\Omega (\mathcal{B}(x,t,\nabla u) + \frac{1}{\theta}|\nabla u|^\theta)\mathrm{d}x\right)\operatorname{div}\left(b(x,t,\nabla u) + |\nabla u|^{\theta-2}\nabla u\right) = f(x,t) - g(x,t,u,\nabla u).$$

By using the Topological degree theory for operators of the type $\mathtt{L}+\mathtt{S}+\mathtt{C}$, where $\mathtt{L}$ is a maximal monotone map, $\mathtt{S}$ is bounded demicontinuous map of class (S_+) and $\mathtt{C}$ be compact and belongs to Γ_σ^τ (i.e there exist $\tau,\sigma \geq 0$ such that $\|\mathtt{C}x\| \leq \tau\|x\| + \sigma$). Our focus of the study is centered on this problem in space $L^\theta(0,T,W^{1,\theta}(\Omega))$, where $\theta \geq 2$ and Ω is a bounded open domain in $\mathbb{R}^N$.

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1. Introduction

Throughout the paper, $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ designantes a bounded open subset with Lipschitz boundary $\partial\Omega$, and we denote by Q the cylinder $\Omega \times (0,T)$ that $\Gamma = \partial\Omega \times (0,T)$ is its lateral surface, where $T > 0$ is a fixing time.

The objective of this article, is to study the existence of at least one weak solution of the the following parabolic equations of Kirchhoff type,

$$(\mathcal{P}) \begin{cases} \frac{\partial u}{\partial t} - \mathcal{M}\left(\mathcal{E}(u)\right)\operatorname{div}\left(b(x,t,\nabla u) + |\nabla u|^{\theta-2}\nabla u\right) = f(x,t) - g(x,t,u,\nabla u) & \text{in } Q; \\ u(x,0) = u_0(x) & \text{in } \Omega; \\ u(x,t) = 0 & \text{on } \Gamma; \end{cases}$$

where

$$\mathcal{E}(u) = \int_\Omega (\mathcal{B}(x,t,\nabla u) + \frac{1}{\theta}|\nabla u|^\theta)\mathrm{d}x$$

the terme $-\text{div}b(x,t,\nabla u)$ is a Leray-Lions operator acting from $\mathcal{V}$ to its dual $\mathcal{V}^*$. which is coercive, such that

$$\mathcal{V} = L^{\theta}(0,T,W_0^{1,\theta}(\Omega)) \text{ and } \mathcal{V}^* = L^{\theta'}(0,T,W^{-1,\theta}(\Omega)) \quad (\theta \geq 2),$$

where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$, the Kirchhoff type function $\mathcal{M}\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, the right-hand side f is assumed to belong to $\mathcal{V}^*$, and the Carathéodory function $g\colon Q \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfy some conditions which will be stated later. Problem $(\mathcal{P})$ is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

introduced in 1883 by Kirchhoff see [14] for mor details, where the parameters h, E, ρ, L, ρ_0, are all constants which respectively have some physical meaning, this equation is an extension of the classical d'Alembert's wave equation. In recent years, much interest has grown on Kirchhoff type problems see([4, 10–12, 20, 22, 23, 25]). This interest arises from their contributions to the modeling of many physical and biological phenomena. We refer the readers to [2, 5, 13, 16–18], for some interesting results and further references.

In [25] S. Zediri, R. Guefaifia and S. Boulaaras studied the existence of positive solutions for $p(x)$-Kirchhoff parabolic systems of the form

$$\begin{cases} \frac{\partial u}{\partial t} - M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x\right) \Delta_{p(x)} u = \lambda^{p(x)} [\lambda_1 f(v) + m u_1 h(u)] & \text{in } Q; \\ \frac{\partial u}{\partial t} - M\left(\int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)} \mathrm{d}x\right) \Delta_{p(x)} v = \lambda^{p(x)} [\lambda_2 g(v) + \mu_2 \tau(u)] & \text{in } Q; \\ u(x,0) = \phi(x) & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

$M(t)$ is a continuous function, the technical approach is bassed on the sub-super-solutions method.

In [12] Y. Fu, and M. Xiang, proved the d the local existence of weak solutions for the following parabolic initial boundary problem for nonlocal parabolic equations

$$\begin{cases} \frac{\partial u}{\partial t} - \left[a + b \left(\int_\Omega \frac{|\nabla u|^{p(x,t)}}{p(x,t)} \mathrm{d}x \right)^{r(t)} \right] \operatorname{div}\left(|\nabla u|^{p(x,t)-2} \nabla u \right) & \text{in } Q, \\ + |u|^{p(x,t)-2} u = f(x,t,u), & \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

where a, b are tow positive constants. The authors proved the local existence of the weak solution of the above problem, using the Galerkin approximation method.

Motivated by the previous results, and the work of Assfaw to show the existence of at least one weak solutions of $(\mathcal{P})$ in $\mathcal{V}$ by using the Topological degree theory for operators of the type type $\mathtt{L} + \mathtt{S} + \mathtt{C}$, where $\mathtt{L}\colon D(\mathtt{L}) \subset \mathtt{V} \longrightarrow \mathcal{V}^*$ is a linear densely defined maximal monotone map, $\mathtt{S}$ defined from $\mathcal{V}$ to its dual $\mathcal{V}^*$ is a bounded map of type (S_+) and $\mathtt{C}\colon D(\mathtt{C}) \subset \mathcal{V} \longrightarrow \mathcal{V}^*$ is compact with $D(\mathtt{L}) \subset D(\mathtt{C})$ such that $\mathtt{C}$ belongs to Γ_σ^τ. For more information on the history of this theory and it applications, see for example [1, 6–9, 15, 19, 24].

The paper is organized as follows. In Section 2, we state some mathematical preliminaries about the functional framework where we will treat our problem and the degree theory for operator of the type $\mathtt{L}+\mathtt{S}+\mathtt{C}$. In Section 3, we give some hypotheses and we will prove our main result.

2. Preliminaries

In this part, we briefly review some basic knowledge of the functional framework required to investigate the problem $(\mathcal{P})$, as well as the fundamental definitions and theorems of topological degree theory that are relevant to our goal.

Let $\mathcal{D} \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary. Let $\theta \geq 2$ and $\theta' = \frac{\theta}{\theta-1}$. We will denote by $L^\theta(\mathcal{D})$ the Banach space of all measurable functions ϕ defined in $\mathcal{D}$ such that

$$\|\phi\|_{L^\theta(\mathcal{D})} = \left(\int_{\mathcal{D}} |\phi(x)|^\theta \mathrm{d}x\right)^{1/\theta} < \infty.$$

We define the functional space $W_0^{1,\theta}(\mathcal{D})$ as the closure of $C_0^\infty(\mathcal{D})$ in the Sobolev space

$$W^{1,\theta}(\mathcal{D}) = \left\{\phi \in L^\theta(\mathcal{D})\colon \nabla\phi \in L^\theta(\mathcal{D})\right\},$$

with respect to the norm

$$\|\phi\|_{1,\theta} = \left(\|\phi\|^\theta_{L^\theta(\mathcal{D})} + \|\nabla\phi\|^\theta_{L^\theta(\mathcal{D})}\right)^{1/\theta}.$$

According to the Poincaré inequality, the norm $\|\cdot\|_{1,\theta}$ on $W_0^{1,\theta}(\mathcal{D})$ is equivalent to the norm $\|\cdot\|_{W_0^{1,\theta}(\mathcal{D})}$ setting by

$$\|\phi\|_{W_0^{1,\theta}(\mathcal{D})} = \|\nabla\phi\|_{L^\theta(\mathcal{D})} \text{ for } \phi \in W_0^{1,\theta}(\mathcal{D}).$$

Remember that the Sobolev space $W_0^{1,\theta}(\mathcal{D})$ is a uniformly convex Banach space and the embedding $W_0^{1,\theta}(\mathcal{D}) \hookrightarrow\hookrightarrow L^\theta(\mathcal{D})$ is compact (see [26]).

Following that, we consider the functional space

$$\mathcal{V} := L^\theta(0,T;W_0^{1,\theta}(\mathcal{D})), \quad (T>0),$$

that is a separable and reflexive Banach space with the norm

$$|\phi|_{\mathcal{V}} = \left(\int_0^T \|\phi\|^\theta_{W_0^{1,\theta}(\mathcal{D})} \mathrm{d}t\right)^{1/\theta}$$

or, by the Poincaré inequality, the equivalent norm in $\mathcal{V}$ given by

$$\|\phi\|_{\mathcal{V}} = \left(\int_0^T \|\nabla\phi\|^{\theta}_{L^{\theta}(\mathcal{D})}\mathrm{d}t\right)^{1/\theta}.$$

2.1. *Topological degree theory for $L+S+C$ operators with $D(L)\subset D(C)$*

In this subsection, we give some results and properties from thedegree theory for $L+S+C$ operators of generalized (S_+) type in real reflexive Banach. In what follows, let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle\cdot,\cdot\rangle$ and given a nonempty subset Ω of X, and $\rightharpoonup$ represents the weak convergence.

Let T from X to 2^{X^*} be a multi-values mapping. We denote by $Gr(T)$ the graph of T, i.e.

$$Gr(T) = \{(u,v)\in X\times X^* \,:\, v\in T(u)\}.$$

Definition 1. The multi-values mapping T is called

(1) monotone, if for each pair of elements $(\eta_1,\theta_1),(\eta_2,\theta_2)$ in $Gr(T)$, we have the inequality
$$\langle\theta_1-\theta_2,\eta_1-\eta_2\rangle\geq 0.$$
(2) maximal monotone, if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from X to 2^{X^*}. An equivalent version of the last clause is that for any $(\eta_0,\theta_0)\in X\times X^*$ for which $\langle\theta_0-\theta,\eta_0-\eta\rangle\geq 0$, for all $(\eta,\theta)\in Gr(T)$, we have $(\eta_0,\theta_0)\in Gr(T)$.

Let Y be another real Banach space.

Definition 2. Let Y be another real Banach space. A mapping $F\colon D(F)\subset X\to Y$ is said to be

(1) Demicontinuous, if for each sequence $(u_n)\subset\Omega$, $u_n\to u$ implies $F(u_n)\rightharpoonup F(u)$.
(2) Of type (S_+), if for any sequence $(u_n)\subset D(F)$ with $u_n\rightharpoonup u$ and $\limsup_{n\to\infty}\langle Fu_n,u_n-u\rangle\leq 0$, we have $u_n\to u$.
(3) Belongs to Γ^{τ}_{σ}, if there exist $\sigma\geq 0$ and $\tau\geq 0$ such that
$$\|Fx\|_Y\leq\tau\|x\|_X+\sigma.$$

Lemma 1 ([6, Lemma 7])**.** *Let G be a nonempty, bounded, and open subset of X. Let $L\colon D(L)\subset X\longrightarrow 2^{X^*}$ be maximal monotone, $L\colon \subset X\longrightarrow 2^{X^*}$ be bounded and of type (S_+) and $C\colon D(C)\subset X\longrightarrow 2^{X^*}$ be compact and $D(L)\subseteq D(C)$ such that C belongs to class Γ^{τ}_{σ}. Assume, further, that $\phi^*\notin(L+S+C)(D(L)\cap\partial G)$. Then there exists $r_0\geq 0$ such that $d(L_r+S+CJ_r,G,\phi^*)$ s well-defined and independent of $r\in(0,r_0]$.*

According to Lemma 1, the associated degree mapping is established as follows:

Definition 3. Let G be a nonempty, bounded, and open subset of X, $L\colon D(L)\subset X\longrightarrow 2^{X^*}$ be maximal monotone, $L\colon \subset X\longrightarrow 2^{X^*}$ be bounded and of type S_+ and $C\colon D(C)\subset X\longrightarrow 2^{X^*}$ be compact and $D(L)\subseteq D(C)$ such that C belongs to class Γ_σ^τ. Assume, further, that $f^*\notin (L+S+C)(D(L)\cap\partial G)$. Then the degree mapping d of $L+S+C$ at $\phi^*\in X^*$ with respect to G is defined by

$$d(L+S+C,G,\phi^*)=\lim_{r\longrightarrow 0^+} d_{S_+}(L_r+S+CJ_r,G,\phi^*),$$

where d_{S_+} is the degree mapping for multivalued bounded operators of type (S_+) from [19].

Theorem 1 ([6, Theorem 9]). *Let G be a nonempty, bounded, and open subset of X, $L\colon D(L)\subset X\longrightarrow 2^{X^*}$ be maximal monotone, $S\colon \subset X\longrightarrow 2^{X^*}$ be bounded and of type (S_+) and $C\colon D(C)\subset X\longrightarrow 2^{X^*}$ be compact and $D(L)\subseteq D(C)$ such that C belongs to class Γ_σ^τ. Then following properties hold:*

(1) *(Normalization) $d(J,G,0)=1$ if $0\in G$ and $d(J,G,0)=0$ if $0\notin \overline{G}$.*
(2) *(Additivity) Let G_1 and G_2 are two disjoint open subsets of G such that $\phi^*\notin (L+S+C)\big((\overline{G}\backslash(G_1\cup G_2))\cap D(L)\big)$ then, we have*

$$d((L+S+C),G,\phi^*)=d((L+S+C),G_1,\phi^*)+d((L+S+C),G_2,\phi^*)\ .$$

(3) *(Existence) If $\phi^*\notin (L+S+C)(D(L)\cap\partial G)$ and $d(L+S+C,G,\phi^*)\neq 0$ then $\phi^*\in (L+S+C)(D(L)\cap G)$.*
(4) *(Translation invariance) Let $\phi^*\notin (L+S+C)(D(L)\cap\partial G)$. Then we have*

$$d(L+S+C-\phi^*,G,0)=d(L+S+C,G,\phi^*).$$

(5) *Let $H(t,x)=Lx+t(S_1x+Cx)+(1-t)S_2x$, $(t,x)\in[0,1]$, where $S_i\colon X\longrightarrow 2^{X^*}$ $(i=1,2)$ is bounded and of type (S_+) and $0\notin H(t,D(L)\cap\partial G)$ for all $t\in[0,t]$. Then $d(H(t,\cdot),G,0)$ is indepondent of $t\in[0,1]$.*

Theorem 2 ([6, Theorem 12]). *Let $L\colon D(L)\subset X\longrightarrow 2^{X^*}$ be maximal monotone with $0\in L(0)$, $S\colon \subset X\longrightarrow 2^{X^*}$ be bounded pseudomonotone and $C\colon D(C)\subset X\longrightarrow 2^{X^*}$ be compact and $D(L)\subseteq D(C)$ such that C belongs to class Γ_σ^τ. Let $\phi^*\in X^*$. Assume further, that there exists $R>0$ such that*

$$\langle Lx+Sx+Cx-\phi^*,x\rangle>0$$

for all $x\in D(L)\cap\partial B_{R(0)}$. Then $\phi^*\in\overline{(L+S+C)(D(L)\cap\partial B_R(0))}$. Furthermore $R\overline{(L+S+C)}=X^*$ provided that $L+S+C$ is coercive.

3. Basic assumptions and main result

Throughout this paper, we use the following assumptions: $b(x,t,\xi)\colon Q\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ is a Carathéodory, and a continuous derivative with respect to ξ of the continuous mapping $\mathcal{B}(x,t,\xi)\colon Q\times\mathbb{R}^N\longrightarrow\mathbb{R}$. for all $(x,t)\in Q$ and all $\xi,\xi'\in\mathbb{R}^N$, with $(\xi\neq\xi')$.

(h_1) $\mathcal{B}(x,t,0)=0$ and $b(x,t,\xi)=\nabla_\xi\mathcal{B}(x,t,\xi)$,

(h_2) $\alpha|\xi|^{\theta} \leq b(x,t,\xi)\cdot\xi \leq \theta\mathcal{B}(x,t,\xi)$,
(h_3) $|b(x,t,\xi)| \leq k(x,t)+\beta\,|\xi|^{\theta-1}$,
(h_4) $[b(x,t,\xi)-b(x,t,\xi')]\cdot(\xi-\xi') > 0$,

where α, β are some real positive number and $k(x,t)$ is a positive function in $L^{\theta'}(Q)$.

(h_5) $g\colon Q\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is Carathéodory function and there exist $\rho > 0$ and $\gamma \in L^{\theta'}(Q)$ such that

$$|g(x,t,\xi,\eta)| \leq \rho(|\xi|^{\theta-1}+|\eta|^{\theta-1})+\gamma(x,t),$$
$$g(x,t,\xi,\eta)\cdot\xi \geq |\xi|^{\theta},$$

for all $(x,t)\in Q$, $\xi\in\mathbb{R}$ and $\eta\in\mathbb{R}^N$ with $\eta\neq\xi$.

In order to obtain the existence of weak solutions, the authors always assume that the Kirchhoff function $\mathcal{M}\colon\mathbb{R}^+\to\mathbb{R}^+$ is continuous and non-decreasing function, and satisfies the following conditions:

(M_0) there exist two positive constant m_0 and m_1, such that,

$$m_0 \leq \mathcal{M}(t) \leq m_1,$$

for all $t\in[0,+\infty[$.

Let G be a nonempty, bounded, a

Lemma 2 ([3, Lemma 2.10]). *Assume that* (h_2)*-*(h_4) *hold, let* $(u_n)_n$ *be a sequence in* $L^{\theta}\left(0,T,W_0^{1,\theta}(\Omega)\right)$ *such that* $u_n \rightharpoonup u$ *weakly in* $L^{\theta}\left(0,T,W_0^{1,\theta}(\Omega)\right)$ *and*

$$\int_Q [b(x,t,\nabla u_n)-b(x,t,\nabla u)]\,\nabla(u_n-u)\mathrm{d}x \longrightarrow 0.$$

Then $u_n \longrightarrow u$ *strongly in* $L^{\theta}\left(0,T,W_0^{1,\theta}(\Omega)\right)$.

Let us consider the following functional

$$\mathcal{T}(u)=\int_0^T \widehat{\mathcal{M}}\left(\int_\Omega\left(\mathcal{B}(x,\nabla u)+\frac{1}{\theta}|\nabla u|^{\theta}\right)\mathrm{d}x\right)\mathrm{d}t \quad \text{for all } u\in\mathcal{V},$$

where $\widehat{\mathcal{M}}\colon[0,+\infty[\longrightarrow[0,+\infty[$ be the primitive of the function $\mathcal{M}$, defned by

$$\widehat{\mathcal{M}}(n)=\int_0^n \mathcal{M}(\xi)d\xi.$$

It is well known that $\mathcal{T}$ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u\in\mathcal{V}$ is the functional $\mathtt{S}(u)\in\mathcal{V}^*$ setting by

$$\begin{aligned}\langle \mathtt{S}u\,,\varphi\rangle = \int_0^T &\left\{\mathcal{M}\left(\int_\Omega(\mathcal{B}(x,t,\nabla u)+\frac{1}{\theta}|\nabla u|^{\theta})\mathrm{d}x\right)\right.\\ &\left.\times\left[\int_\Omega b(x,t,\nabla u)\nabla\varphi\mathrm{d}x+\int_\Omega|\nabla u|^{\theta-2}\nabla u\nabla\varphi\mathrm{d}x\right]\right\}\mathrm{d}t\end{aligned}$$

for all $\varphi \in \mathcal{V}$.

Lemma 3. *Suppose that the assumption (h_2)-(h_4) and (M_0) hold, then*

- $\mathtt{S}$ *is continuous and bounded mapping.*
- *the mapping* $\mathtt{S}$ *is of class* (S_+).

Proof.

- Given that $\mathcal{T}$ is continuously Gâteaux differentiable and whose Gâteaux derivatives at point $u \in \mathcal{V}$, is $\mathtt{S}u \in \mathcal{V}^*$ with

$$\langle \mathcal{T}'u, \varphi\rangle = \langle \mathtt{S}u, \varphi\rangle, \quad \forall \varphi \in \mathcal{V}.$$

Therefore $\mathtt{S}$ is the Fréchet derivative of $\mathcal{T}$. So we can conclude that the operator $\mathtt{S}$ is continuous.

Now, we will prove that $\mathtt{S}$ is bounded.

$$\begin{aligned}
|\ \langle \mathtt{S}u, \varphi\rangle\ | = &\left| \int_0^T \left\{ \mathcal{M}\left(\int_\Omega \mathcal{B}(x,t,\nabla u) + \frac{1}{\theta}|\nabla u|^\theta \right) dx \right) \right.\\
&\left. \times \left(\int_\Omega b(x,t,\nabla u)\nabla\varphi \mathrm{d}x + \int_\Omega |\nabla u|^{\theta-2}\nabla u \nabla \varphi \mathrm{d}x \right\} \mathrm{d}t \right|\\
&\le \int_0^T m_1 \times \left(\int_\Omega |b(x,t,\nabla u)| \cdot |\nabla\varphi| \mathrm{d}x + \int_\Omega |\nabla u|^{\theta-1} \cdot |\nabla\varphi| \mathrm{d}x \right) \mathrm{d}t\\
&\le m_1 \left(\int_0^T \int_\Omega |b(x,t,\nabla u)| \cdot |\nabla\varphi| \mathrm{d}x\, dt + \int_0^T \left(\int_\Omega |\nabla u|^{\theta-1} \cdot |\nabla\varphi| \mathrm{d}x \right) \mathrm{d}t \right)\\
&\le 2m_1 \int_0^T \left(\|b(x,t,\nabla u)\|_{L^{\theta'}(\Omega)} \cdot \|\nabla\varphi\|_{L^\theta(\Omega)} \right) dt\\
&\quad + \int_0^T \|\nabla u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}} \cdot \|\nabla\varphi\|_{L^\theta(\Omega)} \mathrm{d}t\\
&\le Cst \int_0^T \|b(x,t,\nabla u)\|_{L^{\theta'}(\Omega)} \|\varphi\|_{1,\theta} dt + Cst \int_0^T \|u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}} \|\varphi\|_{1,\theta}\, dt.
\end{aligned}$$

From the growth condition (h_2), we can easily show that $\|b(x,t,\nabla u)\|_{L^{\theta'}(\Omega)}$ is bounded for all $u \in W_0^{1,\theta}(\Omega)$. Then

$$|\langle \mathtt{S}u, \varphi\rangle| \le C_1 \int_0^T \|\varphi\|_{1,\theta} + C_2 \int_0^T \|\varphi\|_{1,\theta}.$$

By the continuous embedding $\mathcal{V} \hookrightarrow L^1\left(0,T,W^{1,\theta}(\Omega)\right)$, we concludes that

$$|\langle \mathtt{S}u, \varphi\rangle| \le \text{Const}\ \|\varphi\|_{\mathcal{V}}.$$

Which means that the operator $\mathtt{S}$ is bounded.

- Next, we verify that the operator $\mathtt{S}$ is of type (S_+).

Assume that $(u_n)_n \subset \mathcal{V}$ and

$$\begin{cases} u_n \rightharpoonup u \quad \text{in } \mathcal{V}; \\ \limsup\limits_{n\to\infty} \langle \mathrm{S}u_n, u_n - u\rangle \leq 0. \end{cases}$$

We will show that $u_n \to u$ in $\mathcal{V}$.

On the one hand, since $u_n \rightharpoonup u$ in $\mathcal{V}$, so $(u_n)_n$ is a bounded sequence in $\mathcal{V}$ and since $\mathcal{V}$ embeds compactly in $L^\theta(Q)$, then there exist a subsequence still denoted by $(u_n)_n$ such that $u_n \to u$ in $L^\theta(Q)$.

On the other hand, we have

$$\limsup_{n\to\infty} \langle \mathrm{S}u_n, u_n - u\rangle = \limsup_{n\to\infty} \langle \mathrm{S}u_n - \mathrm{S}u,\ u_n - u\rangle = \lim_{n\to\infty} \langle \mathrm{S}u_n - \mathrm{S}u,\ u_n - u\rangle \leq 0.$$

Then,

$$\begin{aligned} &\lim_{n\to\infty} \langle \mathrm{S}u_n - \mathrm{S}u,\ u_n - u\rangle \\ &= \lim_{n\to\infty} \Bigg(\int_0^T \mathcal{M}\left(\mathcal{E}(u_n)\right) \bigg[\int_\Omega b(x,t,\nabla u_n)\nabla(u_n - u)\mathrm{d}x \\ &\qquad\qquad + \int_\Omega |\nabla u_n|^{\theta-2}\nabla u_n \nabla(u_n - u)\mathrm{d}x \bigg] \mathrm{d}t \\ &\qquad - \int_0^T \mathcal{M}\left(\mathcal{E}(u)\right) \bigg[\int_\Omega b(x,t,\nabla u)\nabla(u_n - u)\mathrm{d}x \\ &\qquad\qquad + \int_\Omega |\nabla u|^{\theta-2}\nabla u \nabla(u_n - u)\mathrm{d}x \bigg] \mathrm{d}t \Bigg) \leq 0. \end{aligned}$$

Or by (A_1) we have for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$

$$\mathcal{B}(x,\xi) = \int_0^1 \frac{d}{ds}\mathcal{B}(x,s\xi)ds = \int_0^1 b(x,t,s\xi)\xi ds,$$

by combining (h_3), Fubini's theorem and Young's inequality we have

$$\begin{aligned} \int_\Omega \mathcal{B}(x,t,\nabla u)\mathrm{d}x &= \int_\Omega \int_0^1 b(x,t,s\nabla u_n)\nabla u ds\mathrm{d}x \\ &\leq \int_0^1 \left[C_{\theta'} \int_\Omega |b(x,t,s\nabla u)|^{\theta'} \mathrm{d}x + C_\theta \int_\Omega |\nabla u|^\theta dx \right] \mathrm{d}s \\ &\leq C_1 + C' \int_0^1 \int_\Omega |s\nabla u|^\theta \mathrm{d}x\mathrm{d}s + C_\theta \|u\|^\theta_{L^\theta(\Omega)} \\ &\leq C_1 + C_2 \int_\Omega |\nabla u|^\theta \mathrm{d}x + C_\theta \|u\|^\theta_{1,\theta} \\ &\leq C\|u\|^\theta_{1,\theta}. \end{aligned} \tag{3.1}$$

by (3.1), Then $\int_\Omega (\mathcal{B}(x,t,\nabla u_n)\mathrm{d}x$ is bounded.

As $\mathcal{M}$ is continuous, up to a subsequence there is $k \geq 0$ by

$$\mathcal{M}\left(\mathcal{E}(u_n)\right) \longrightarrow \mathcal{M}(k) \geq m_0 \quad \text{as } n \to \infty.$$

In addition by appliying the assumption (M_0), we get

$$\begin{aligned}\lim_{n\to\infty} m_0 \Bigg(\int_0^T \Bigg[\int_\Omega & b(x,t,\nabla u_n)\nabla(u_n - u)\mathrm{d}x \\ & + \int_\Omega |\nabla u_n|^{\theta-2}\nabla u_n \nabla(u_n - u)\,dx \Bigg] \mathrm{d}t \\ & - \int_0^T \Bigg[\int_\Omega b(x,t,\nabla u)\nabla(u_n - u)\,dx \\ & + \int_\Omega |\nabla u|^{\theta-2}\nabla u \nabla(u_n - u)\mathrm{d}x \Bigg] \mathrm{d}t \Bigg) \leq 0.\end{aligned}$$

Using the compact embedding $\mathcal{V} \hookrightarrow\hookrightarrow L^\theta(Q)$, we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{\theta-2}\nabla u_n(\nabla u_n - \nabla u)\mathrm{d}x = 0, \qquad \lim_{n\to\infty} \int_\Omega |\nabla u|^{\theta-2}\nabla u(\nabla u_n - \nabla u)\mathrm{d}x = 0.$$

Since $m_0 \geq 0$ then, we have

$$\lim_{n\to\infty} \left(\int_0^T \int_\Omega b(x,t,\nabla u_n)\nabla(u_n - u)\mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega b(x,t,\nabla u)\nabla(u_n - u)\,dx\mathrm{d}t \right) \leq 0,$$

which means

$$\lim_{n\to\infty} \int_0^T \int_\Omega \left[b(x,t,\nabla u_n) - b(x,t,\nabla u)\right](\nabla u_n - \nabla u)\mathrm{d}x\mathrm{d}t \leq 0, \tag{3.2}$$

By combining (3.2) and (h_4) we deduce that

$$\lim_{n\to\infty} \int_0^T \int_\Omega \left[b(x,t,\nabla u_n) - b(x,t,\nabla u)\right](\nabla u_n - \nabla u)\mathrm{d}x\mathrm{d}t = 0,$$

In light of Lemma 2, we obtain

$$u_n \longrightarrow u \quad \text{in } \mathcal{V},$$

which implies that $\mathtt{S}$ is of type (S_+). □

Lemma 4. *Suppose that the assumption (h_5) hold, then the mapping $\mathtt{C}\colon D(\mathtt{C}) \subset \mathcal{V} \longrightarrow \mathcal{V}^*$ defined by*

$$\langle \mathtt{C}u, \varphi\rangle = \int_Q g(x,t,u,\nabla u)\varphi\mathrm{d}x\mathrm{d}t, \text{ and } D(\mathtt{C}) = \{v \in \mathcal{V} : \ v' \in \mathcal{V}^*\},$$

is compact and belong to the class Γ_σ^τ.

Proof. We first show that $\mathtt{C}$ is compact.

It is know that $D(\mathtt{C})$ is compactly embedded in $L^\theta(Q)$, because $W^{1,\theta}(\Omega)$ is compactly embedded in $L^\theta(\Omega)$ (see Proposition 1.3 in [21]), as a result $\mathtt{C}$ is completly continuous, that is $\mathtt{C}$ is compact operator.

Next, we prove that $\mathtt{C}$ belongs to Γ_σ^τ. By apliying condition (h_4), we get

$$
\begin{aligned}
|\langle \mathtt{C}u,\varphi\rangle| &= \left|\int_0^T \left(\int_\Omega g(x,t,u,\nabla u)\varphi\, dx\right) \mathrm{d}t\right| \\
&= \int_0^T \left(\int_\Omega |g(x,t,u,\nabla u)|\,|\varphi| \mathrm{d}x\right) \mathrm{d}t \\
&\leq \int_0^T \left[\rho \int_\Omega \left(|u|^{\theta-1} + |\nabla u|^{\theta-1}\right)|\varphi|\, dx\right] \mathrm{d}t + \int_Q |\gamma(x,t)||\varphi| \mathrm{d}x\mathrm{d}t \\
&\leq \rho \int_0^T \|u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}} \|\varphi\|_{L^\theta(\Omega)} \mathrm{d}t + \rho \int_0^T \|\nabla u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}} \|\varphi\|_{L^\theta(\Omega)} \mathrm{d}t \\
&\quad + \int_0^T \|\gamma(x,t)\|_{L^{\theta'}(\Omega)} \|\varphi\|_{L^\theta(\Omega)} \mathrm{d}t \\
&\leq 2\rho \int_0^T \|\nabla u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}} \|\varphi\|_{L^\theta(\Omega)} + \int_0^T \|\gamma(x,t)\|_{L^{\theta'}(\Omega)} \|\varphi\|_{L^\theta(\Omega)} \mathrm{d}t \\
&\leq 2\rho \int_0^T \|\nabla u\|_{1,\theta}^{\theta-1} \|\varphi\|_{1,\theta} + \int_0^T \|\gamma(x,t)\|_{L^{\theta'}(\Omega)} \|\varphi\|_{1,\theta} \mathrm{d}t \\
&\leq 2\rho \|u\|_{\mathcal{V}}^{\theta-1} \|\varphi\|_{\mathcal{V}} + \|\gamma\|_{L^{\theta'}(Q)} \|\varphi\|_{\mathcal{V}} \\
&\leq Cst \|u\|_{\mathcal{V}} \|\varphi\|_{\mathcal{V}} + \|\gamma\|_{L'^{\theta}(Q)} \|\varphi\|_{\mathcal{V}} \quad \text{for all } u,\varphi \in \mathcal{V}.
\end{aligned}
$$

Consequently, taking supremum overall $\varphi \in \mathcal{V}$ with $\|\varphi\|_{\mathcal{V}} \leq 1$, we conclude that

$$
\|\mathtt{C}u\|_{\mathcal{V}^*} \leq \tau \|u\|_{\mathcal{V}} + \sigma \quad \forall\, u \in \mathcal{V}.
$$

Where $\tau = Cst \geq 0$ and $\sigma = \|\gamma\|_{L^{\theta'}(Q)} \geq 0$, that is $\mathtt{C}$ belong to the class Γ_σ^τ. □

Now, let us consider the following operator $\mathtt{L}$ defined from the subset $D(\mathtt{L})$ of $\mathcal{V}$ into its dual $\mathcal{V}^*$, such that

$$
D(\mathtt{L}) = \left\{\varphi \in \mathcal{V} : \ \varphi' \in \mathcal{V}^*,\ \varphi(0) = 0\right\}, \text{ that is } D(\mathtt{L}) \subset D(\mathtt{C}),
$$

by

$$
\langle \mathtt{L}u,\varphi\rangle = -\int_Q u\varphi_t \mathrm{d}x\mathrm{d}t, \quad \text{for all } u \in D(\mathtt{L}),\ \varphi \in \mathcal{V}.
$$

Consequently, the operator $\mathtt{L}$ is generated by $\partial/\partial t$ by means of the relation

$$
\langle \mathtt{L}u,\varphi\rangle = \int_0^T \langle u'(t),\varphi(t)\rangle \mathrm{d}t, \ \text{for all } u \in D(\mathtt{L}),\ \varphi \in \mathcal{V}.
$$

Lemma 5 ([26, Theorem 32.L])**.** $\mathtt{L}$ *is linear, densely defined and maximal monotone map.*

Our main result is the following existence theorem:

Theorem 3. *Let $f \in \mathcal{V}^*$, $u_0 \in L^2(\Omega)$ and assume that the hypothesis $(h_1)-(h_5)$ and (M_0) hold. Then, there exists at least one weak solution $u \in D(\mathtt{L})$ of problem $(\mathcal{P})$ in the following sense*

$$-\int_Q u\varphi_t \mathrm{d}x\mathrm{d}t + \int_0^T \left\{ \mathcal{M}(\mathcal{E}(u)) \left[\int_\Omega b(x,t,\nabla u)\nabla\varphi|\, dx + \int_\Omega |\nabla u|^{\theta-2}\nabla u \nabla\varphi \mathrm{d}x \right] \right\} \mathrm{d}t$$
$$+ \int_Q g(x,t,u,\nabla u)\varphi\, dx\mathrm{d}t = \int_0^T \langle f,\varphi\rangle\, dt,$$

for all $\varphi \in \mathcal{V}$.

Proof. On the one hand, from the Lemma 5, the operator

$$\mathtt{L}\colon D(\mathtt{L}) \subset \mathcal{V} \longrightarrow \mathcal{V},$$
$$\langle \mathtt{L}u, \varphi\rangle_{\mathcal{V}} = \int_0^T \langle u'(t), \varphi(t)\rangle\, dt, \ \text{ for all } u \in D\mathtt{L}),\ \varphi \in \mathcal{V}.$$

is a densely defined maximal monotone operator.

By the monotonicity of L we have

$$\langle \mathtt{L}u, u\rangle \geq 0 \ \text{ for all } u \in D(\mathtt{L}),$$

then we obtain

$$\begin{aligned}\langle \mathtt{L}u + \mathtt{S}u + \mathtt{C}u, u\rangle &\geq \langle \mathtt{S}u + \mathtt{C}u, u\rangle \\ &= \int_0^T \left\{ \mathcal{M}\left(\int_\Omega A(x,t,\nabla u) + \frac{1}{p}|\nabla u|^p \mathrm{d}x \right) \int_\Omega a(x,t,\nabla u)\nabla u \mathrm{d}x \right. \\ &\qquad \left. + \int_\Omega |\nabla u|^p \mathrm{d}x \right\} \mathrm{d}t \\ &\quad + \int_Q g(x,t,u,\nabla u)u \mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.3}$$

by using th assumptions (h_2), (M_0) and (h_4), we get

$$\begin{aligned}\langle \mathtt{L}u + \mathtt{S}u + \mathtt{C}u, u\rangle &\geq \int_0^T m_0 \int_\Omega b(x,t,\nabla u)\nabla u \mathrm{d}x\mathrm{d}t + \int_0^T m_0 \int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}t \\ &\quad + \int_Q g(x,t,u,\nabla u)u \mathrm{d}x\mathrm{d}t \\ &\geq m_0 \int_Q b(x,t,\nabla u)\nabla u \mathrm{d}x\mathrm{d}t + m_0 \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + \int_Q |u|^\theta \mathrm{d}x\mathrm{d}t \\ &\geq m_0\alpha \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + m_0 \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + \int_Q |u|^\theta \mathrm{d}x\mathrm{d}t \\ &\geq C_{min} \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + C_{min} \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + \int_Q |u|^\theta \mathrm{d}x\mathrm{d}t \\ &\geq C_{min} \int_0^T \left(\int_\Omega \left(|u|^\theta + |\nabla u|^\theta \right) dx \right) \mathrm{d}t\end{aligned}$$

$$\geq C_{min}\|u\|_{\mathcal{V}}^{\theta},$$

for all $u \in \mathcal{V}$.

Since the right side of the above inequality (3.3) tends to ∞ as $\|u\|_{\mathcal{V}} \to \infty$, then for each $f \in \mathcal{V}^*$ there exists $R = R(f)$ such that

$$\langle \mathtt{L}u + \mathtt{S}u + \mathtt{C}u - f, u \rangle > 0,$$

for all $u \in \mathtt{B}_R(0) \cap D(\mathtt{L})$.

By appliying Lemma 2, we infer that the equation

$$\mathtt{L}u + \mathtt{S}u + \mathtt{C}u = f,$$

is solvable in $D(\mathtt{L})$. That is the problem $(\mathcal{P})$ admits at least one-weak solution.

Which implies that the problem $(\mathcal{P})$ admits at least one weak solution. This ends the proof. □

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REFERENCES

[1] D. R. Adhikari, "Topological degree for quasibounded multivalued $(\widetilde{S}_+)$- perturbations of maximal monotone operators," *Applicable Analysis*, vol. 99, no. 13, pp. 2339–2360, 2020, doi: 10.1080/00036811.2018.1562058.

[2] R. P. Agarwal, A. M. Alghamdi, S. Gala, and M. A. Ragusa, "On the regularity criterion on one velocity component for the micropolar fluid equations," *Mathematical Modelling and Analysis*, vol. 28, no. 2, pp. 271–284, 2023, doi: 10.3846/mma.2023.15261.

[3] Y. Akdim, C. Allalou, and N. El Gorch, "Solvability of degenerated $p(x)$-parabolic equations with three unbounded nonlinearities," *Electronic Journal of Mathematical Analysis and Applications*, vol. 4, no. 1, pp. 42–62, 2016.

[4] C. Allalou, K. Hilal, and S. Yacini, "Existence of weak solution for p-Kirchoff type problem via topological degree," *Journal of Elliptic and Parabolic Equations*, pp. 1–14, 2023, doi: 10.1007/s41808-023-00220-0.

[5] M. A. Alyami, "Existence of solution for a singular fractional boundary value problem of Kirchhoff type," *Filomat*, vol. 36, no. 17, pp. 5803–5812, 2022, doi: 10.2298/FIL2217803A.

[6] T. M. Asfaw, "A degree theory for compact perturbations of monotone type operators and application to nonlinear parabolic problem," in *Abstract and Applied Analysis*, vol. 2017, doi: 10.1155/2017/7236103.

[7] T. M. Asfaw, "A topological degree theory for constrained problems with compact perturbations and application to nonlinear parabolic problem," *Partial Differential Equations in Applied Mathematics*, vol. 3, p. 100019, 2021, doi: 10.1016/j.padiff.2020.100019.

[8] J. Berkovits and V. Mustonen, "On the topological degree for mappings of monotone type," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 10, no. 12, pp. 1373–1383, 1986, doi: 10.1016/0362-546X(86)90108-2.

[9] M. El Ouaarabi, C. Allalou, and S. Melliani, "Existence of weak solutions to a class of nonlinear degenerate parabolic equations in weighted Sobolev space," *Electronic Journal of Mathematical Analysis and Applications*, vol. 11, no. 1, pp. 45–58, 2023.

[10] M. El Ouaarabi, C. Allalou, and S. Melliani, "Existence result for a Neumann boundary value problem governed by a class of $p(x)$-Laplacian-like equation," *Asymptotic Analysis*, vol. 132, no. 1-2, pp. 245–259, 2023, doi: 10.3233/ASY-221791.

[11] A. Fiscella and E. Valdinoci, "A critical Kirchhoff type problem involving a nonlocal operator," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 94, pp. 156–170, 2014, doi: 10.1016/j.na.2013.08.011.

[12] Y. Fu and M. Xiang, "Existence of solutions for parabolic equations of Kirchhoff type involving variable exponent," *Applicable Analysis*, vol. 95, no. 3, pp. 524–544, 2016, doi: 10.1080/00036811.2015.1022153.

[13] M. Ghergu and V. D. Rădulescu, *Nonlinear PDEs. Mathematical models in biology, chemistry and population genetics*, ser. Springer Monogr. Math. Berlin: Springer, 2012. doi: 10.1007/978-3-642-22664-9.

[14] G. Kirchhoff, "Mechanik, Teubner, Leipzig," *Kirchhoff G*, 1883.

[15] J. Leray and J. Schauder, "Topologie et équations fonctionnelles," in *Annales scientifiques de l'École normale supérieure*, vol. 51, 1934, pp. 45–78.

[16] J. Limaco Ferrel and L. A. Medeiros, "Kirchhoff-Carrier elastic strings in noncylindrical domains." *Portugaliae Mathematica*, vol. 56, no. 4, pp. 465–500, 1999.

[17] B. Lovat, "Etudes de quelques problemes paraboliques non locaux," Ph.D. dissertation, Université Paul Verlaine-Metz, 1995.

[18] A. Matallah, H. Benchira, and M. El Mokhtar, "Existence of solutions for p-Kirchhoff problem of Brézis-Nirenberg type with singular terms," *Journal of Function Spaces*, vol. 2022, 2022, doi: 10.1155/2022/7474777.

[19] D. O'Regan, Y. J. Cho, and Y.-Q. Chen, *Topological degree theory and applications.*, ser. Ser. Math. Anal. Appl. Boca Raton, FL: Chapman & Hall/CRC, 2006, vol. 10.

[20] P. Pucci and S. Saldi, "Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators," *Revista Matematica Iberoamericana*, vol. 32, no. 1, pp. 1–22, 2016, doi: 10.4171/RMI/879.

[21] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, ser. Math. Surv. Monogr. Providence, RI: American Mathematical Soiety, 1997, vol. 49. [Online]. Available: www.ams.org/online_bks/surv49/

[22] A. Tudorascu and M. Wunsch, "On a nonlinear, nonlocal parabolic problem with conservation of mass, mean and variance," *Communications in Partial Differential Equations*, vol. 36, no. 8, pp. 1426–1454, 2011, doi: 10.1080/03605302.2011.563402.

[23] M. Xiang, B. Zhang, and M. Ferrara, "Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 424, no. 2, pp. 1021–1041, 2015, doi: 10.1016/j.jmaa.2014.11.055.

[24] S. Yacini, C. Allalou, and K. Hilal, "Weak solution to $p(x)$-Kirchoff type problems under no-flux boundary condition by topological degree," *Boletim da Sociedade Paranaense de Matemática*, vol. 41, pp. 1–12, 2023, doi: 10.5269/bspm.63341.

[25] S. Zediri, R. Guefaifia, and S. Boulaaras, "Existence of positive solutions of a new class of nonlocal p- Kirchhoff parabolic systems via sub-super-solutions concept," *Journal of Applied Analysis*, vol. 26, no. 1, pp. 49–58, 2020, doi: 10.1515/jaa-2020-2002.

[26] E. Zeidler, *Nonlinear functional analysis and its applications. IV: Applications to mathematical physics. Transl. from the German by Juergen Quandt*. New York etc.: Springer-Verlag, 1988.

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