Miskolc Mathematical Notes
Vol. 26 (2025), No. 1, pp. 5–15

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4701

# CHARACTERIZATIONS OF TRACE CLASS SEMICLASSICAL FOURIER INTEGRAL OPERATORS

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*Received 25 April, 2023*

*Abstract.* In this work we study a characterizations of trace class and nuclearity of semiclassical Fourier integral operators associated with a particular class of symbols $S^m_{1,0}(\mathbb{R}^{2n})$ on Lebesgue spaces.

2010 *Mathematics Subject Classification:* 35S05; 47G30

*Keywords:* $h$-Fourier integral operators, trace class operators, symbols and phase functions

## 1. INTRODUCTION

A Fourier integral operator or FIO in short is a singular operator defined by

$$I_{a,\phi}u(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)\hat{u}(\xi)\mathrm{d}\xi, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space, i.e. the space of all rapidly decreasing smooth functions. Moreover $\hat{u}$ denote the Fourier transformation of $u$ (we also denote $\hat{u}$ via $\mathcal{F}(u)$). Additionally, $a(x,\xi)$ and $\phi(x,\xi)$ are smooth functions called respectively the amplitude and the phase function.

The theory of FIOs has a lengthy history of study. The regularity of these operators in functional spaces has been studied extensively and there is a huge body of evidence regarding regularity, such as in [8, 13].

According to the theory of FIO developed by Hörmander [13], the phase functions in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus 0)$ homogenous of degree 1 in the frequency variable $\xi$ and with non-vanishing determinant of the mixed Hessian matrix (i.e. $\phi \in \Phi(\mathbb{R}^{2n})$), while the symbols in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

(i.e. $a(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^{2n})$), was initiated in the classical paper of L. Hörmander [13]. Moreover, G. Eskin [9] (in the case $a \in S^0_{1,0}$) and Hörmander [13] (in the case $a \in$

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$S^0_{\rho,1-\rho}$, $\frac{1}{2} < \rho \leq 1$) treated the local $L^2$ boundedness of FIO with non-degenerate phase functions.

Gala-Liu-Ragusa [10], Kheloufi-Ikassoulene [14], Ragusa [18–20] and Seeger-Sogge-Stein [21] have extended local $L^2$-boundedness results to $L^p$ regularity (for $1 \leq p \leq \infty$) since the 1970s, spurred by applications in microlocal analysis and hyperbolic partial differential equations.

In Asada-Fujiwara [7], the initial result was obtained using Fourier integral operators associated to symbols in $S^0_{0,0}$ and inhomogeneous phase functions.

This conclusion was extended to a general class of symbols in $S^m_{\rho,\delta}$ that met certain criteria, as shown in [3–5, 11, 12].

In the semiclassical case an $h$-Fourier integral operator has the following form

$$I_h(a,\phi)u(x) = (2\pi h)^{-n} \iint e^{ih^{-1}[\phi(x,\xi)-\langle y,\xi\rangle]} a(x,\xi)u(y)\mathrm{d}y d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

where $h \in ]0,h_0]$ is a semiclassical parameter.

A fundamental principle of $h$-Fourier integral operator is that linear differential operators known as semi classical Schrodinger operators are used to represent quantum systems including atoms, molecules, solids, and, to a certain extent, nuclei and even stars. For a description of this theory, we point the reader toward the widely used text of Landau and Lifshitz [15]. The Schrodinger operator for a quantum system is the linear partial differential operator

$$H = -\frac{h^2}{2m}\Delta + V,$$

The Laplace operator $\Delta$ is the second-order differential operator that, in Cartesian coordinates on $\mathbb{R}^n$, is given by

$$\Delta = \sum_{k=0}^{n} \frac{\partial^2}{\partial x_i^2}.$$

The constant $m$ is the reduced mass of the system, and the semiclassical parameter $h$ is called Planck's constant. The real-valued function $V$ is called the potential.

In comparison to the research of FIOs, there has been less activity for $h$-Fourier integral operators concerning the study of the corresponding r-nuclearity properties.

Many results of boundedness and compactness (on $L^p$, on Bezov spaces and on Bessel potential spaces) are obtained for semiclassical Fourier integral operators with symbols introduced by Hörmander and non degenerate and homogeneous phase functions, see [1, 2].

The lack of $r$-nuclearity results for $h$-FIOs in semiclassical analysis literature prompted this study.

The aim of this work is to extend results obtained in [6] for a general class of semiclassical Fourier integral operators. Our investigation is to study characterizations of nuclear $h$-FIOs on Lebesgue spaces with the same conditions on the phase functions and on the symbols are kept.

Now let us go over this article's structure. We introduce the pertinent preliminaries and notations that will be used throughout the work in the second section.

Following that, we'll go over some fundamental theorems from the theory of $r$-nuclear operators and trace class operators (for $r = 1$), which will serve as the starting point for our research. Finally, the last section is devoted to prove our main result.

## 2. NOTATIONS AND PRELIMINARIES

We assume $n \in \mathbb{N}$ throughout the whole paper unless otherwise noted. In particular $n \neq 0$. For all $x, \xi \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ we define

$$\langle x, \xi \rangle = \sum_{j=0}^{n} x_j \xi_j \quad \text{and} \quad \widehat{d_h}\xi = (2\pi h)^{-n} \mathrm{d}\xi.$$

Moreover, let us recall weight and characteristic functions defined by

$$\langle \xi \rangle = \left(1 + |\xi|^2\right)^{1/2} \quad \text{and} \quad 1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{else.} \end{cases}$$

Furthermore, we define the semiclassical Fourier transform $\mathcal{F}_h$ and its inverse $\mathcal{F}_h^{-1}$ by

$$\mathcal{F}_h f(\xi) = \int_{\mathbb{R}^n} e^{-ih^{-1}\langle x, \xi \rangle} f(x) \mathrm{d}x \quad \text{and} \quad \mathcal{F}_h^{-1} f(x) = \int_{\mathbb{R}^n} e^{ih^{-1}\langle x, \xi \rangle} f(\xi) \, \widehat{d_h}\xi,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$.

Additionally we scale partial derivatives with respect to a variable $x \in \mathbb{R}^n$ with the factor $-i$ are denoted by

$$D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha = (-i)^{|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index and $|\alpha| = \sum_{j=1}^{n} \alpha_j$ is the length of $\alpha$.

We note that the dual space of a topological vector space $E$ is denoted by $E'$.

Finally, let us denote by $B(x, r)$ the Euclidean ball centered at $x$ with radius $r > 0$, and by $m\left(B(x, r)\right)$ the Lebesgue measure of $B(x, r)$.

To present our main results, we will need the notion of trace class operators.

**Definition 1.** Let $X$ and $Y$ be complex Banach spaces and let $T \colon D(T) \subset X \to Y$ (where $D(T)$ is the domain of $T$) be a linear operator. Suppose that we can find sequences $\left\{x'_k\right\}_{k \geq 0}$ in the dual space $X'$ and $\{y_k\}_{k \geq 0}$ in $Y$ such that

$$\sum_{k=0}^{\infty} \|x'_k\|_{X'}^r \|y_k\|_Y^r < \infty,$$

and

$$Tx = \sum_{k=0}^{\infty} x'_k(x) y_k, \quad \forall x \in X.$$

Then we call $T \colon D(T) \subset X \to Y$ a $r$-nuclear operator.

The class of $r$-nuclear operators is usually endowed with the natural semi-norm

$$q_r(T) = \inf\left\{ \left( \sum_{k=0}^{\infty} \|x_k'\|_{X'}^r \|y_k\|_Y^r \right)^{1/r} ; T = \sum_{k=0}^{\infty} x_k' \otimes y_k \right\}.$$

*Remark* 1. When $r = 1$, $q_1$ is a norm and we obtain the ideal of trace class operators.

**Proposition 1.** *For the case of Hilbert spaces $H$, the set of $r$-nuclear operators is identical to with the Schatten-von Neumann class of order $r$.*

*Proof.* See Pietsch [16, 17]. ☐

The following theorem will be required to investigate the trace class properties of the $h$-Fourier integral operator.

**Theorem 1.** *Let $(X_1, \mu_1)$ and $(X_2, \mu_2)$ be two $\sigma$-finite measure spaces. Let us consider $p_1, p_2 \in [1, \infty[$,, and let $p_1', p_2'$ be conjugates of $p_1, p_2$ respectively $\Big($i.e. $\frac{1}{p_i} + \frac{1}{p_i'}$ $= 1\ \forall i \in \{1, 2\}$). An operator $T : L^{p_1}(X_1, \mu_1) \to L^{p_2}(X_2, \mu_2)$ is trace class if, and only if, there exist sequences $\{u_k\}_{k\geq 0}$ in $L^{p_2}(X_2, \mu_2)$, and $\{v_k\}_{k\geq 0}$ in $L^{p_1'}(X_1, \mu_1)$, such that*

$$\sum_{k=0}^{\infty} \|u_k\|_{L^{p_2}} \|v_k\|_{L^{p_1'}} < \infty,$$

*and*

$$Tf(x) = \int_{X_1} \sum_{k=0}^{\infty} u_k(x) v_k(y) f(y) \mathrm{d}y, \quad \forall f \in L^{p_1}(X_1, \mu_1).$$

*Proof.* Let be $T$ a trace class operator from $L^{p_1}$ to $L^{p_2}$. Then there are sequences $\{u_k\}_{k\geq 0}$ in $L^{p_2}$ and $\{v_k\}_{k\geq 0}$ in $L^{p_1'}$ so that

$$\sum_{k=0}^{\infty} \|u_k\|_{L^{p_2}} \|v_k\|_{L^{p_1'}} < \infty$$

and

$$Tf = \sum_{k=0}^{\infty} \langle f, v_k \rangle u_k.$$

Now

$$Tf = \sum_{k=0}^{\infty} \langle f, v_k \rangle u_k = \sum_{k=0}^{\infty} \left( \int_{X_1} f(y) v_k(y) \mathrm{d}y \right) u_k,$$

where the sum converges in $L^{p_2}$ norm. There exist two subsequences $\{\tilde{u}_k\}_{k\geq 0}$ and $\{\tilde{v}_k\}_{k\geq 0}$ of $\{u_k\}_{k\geq 0}$ and $\{v_k\}_{k\geq 0}$ respectively, so for all $x \in X_2$

$$Tf(x) = \sum_{k=0}^{\infty} \langle f, \tilde{v}_k \rangle \tilde{u}_k(x) = \sum_{k=0}^{\infty} \left( \int_{X_1} f(y) \tilde{v}_k(y) \mathrm{d}y \right) \tilde{u}_k(x).$$

Since $\{\tilde{u}_k\}_{k\geq 0}$ and $\{\tilde{v}_k\}_{k\geq 0}$ satisfie

$$\sum_{k=0}^{\infty} \|\tilde{u}_k\|_{L^{p_2}} \|\tilde{v}_k\|_{L^{p'_1}} < \infty,$$

then

$$\begin{aligned} Tf(x) &= \sum_{k=0}^{\infty} \langle f, \tilde{v}_k \rangle \tilde{u}_k(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \left( \int_{X_1} f(y)\tilde{v}_k(y)\mathrm{d}y \right) \tilde{u}_k(x) \\ &= \lim_{n\to\infty} \int_{X_1} \left( \sum_{k=0}^{n} f(y)\tilde{u}_k(x)\tilde{v}_k(y) \right) \mathrm{d}y = \int_{X_1} \left( \sum_{k=0}^{\infty} \tilde{u}_k(x)\tilde{v}_k(y) \right) f(y)\mathrm{d}y, \end{aligned}$$

for all $x \in X_2$. The proof of the *"if"* part is complete. Thus, let us consider the *"only if"* part.

We assume that there exists sequences $\{u_k\}_{k\geq 0}$ in $L^{p_2}$ and $\{v_k\}_{k\geq 0}$ in $L^{p'_1}$ such that

$$\sum_{k=0}^{\infty} \|u_k\|_{L^{p_2}} \|v_k\|_{L^{p'_1}} < \infty.$$

So for all $f \in L^{p_1}$, we have

$$\begin{aligned} \int_{X_1} \left( \sum_{k=0}^{\infty} u_k(x)v_k(y) \right) f(y)\mathrm{d}y &= \lim_{n\to\infty} \int_{X_1} \left( \sum_{k=0}^{n} f(y)u_k(x)v_k(y) \right) \mathrm{d}y \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \left( \int_{X_1} f(y)v_k(y)\mathrm{d}y \right) u_k(x) \\ &= \sum_{k=0}^{\infty} \left( \int_{X_1} f(y)v_k(y)\mathrm{d}y \right) u_k(x) \\ &= \sum_{k=0}^{\infty} \langle f, v_k \rangle u_k(x) = Tf(x). \end{aligned}$$

We put

$$g_n = \sum_{k=0}^{n} \langle f, \tilde{v}_k \rangle \tilde{u}_k,$$

then $\{g_n\}_{n\geq 0}$ is a sequence in $L^{p_2}$ and

$$|g_n(x)| \leq h_n(x) \leq h(x),$$

where for all $x \in X_2$

$$h_n(x) = \|f\|_{L^{p_1}} \sum_{k=0}^{n} \|v_k\|_{L^{p'_1}} |u_k(x)|, \quad \forall n \in \mathbb{N}$$

and

$$h(x) = \lim_{n\to\infty} h_n(x).$$

Since we have

$$\|h_n\|_{L^{p_2}} \leq \|h\|_{L^{p_2}} \leq \|f\|_{L^{p_1}} \sum_{k=0}^{\infty} \|v_k\|_{L^{p_1'}} \|u_k\|_{L^{p_2}} < \infty,$$

then $h \in L^{p_2}$ is a limit of $\{h_n\}_{n\geq 0}$ of $L^{p_2}$-functions.

So by Lebesgue dominated convergence theorem we get that $\{g_n\}_{n\geq 0}$ converge to $Tf$ in $L^{p_2}$. ☐

**Corollary 1.** *When $p_1 = p_2$ and $\mu_1 = \mu_2 = \mu$, the trace of $T$ is given by*

$$Tr(T) = \int_{X_1} \sum_{k=0}^{\infty} u_k(x) v_k(x) \mathrm{d}\mu.$$

## 3. CHARACTERIZATION OF TRACE CLASS $h$-FOURIER INTEGRAL OPERATORS

In this section we prove our main result concerning the trace class property of the $I_h(a,\phi)$ defined as in (1.1). First let us observe that $I_h(a,\phi)$ is a integral operator of distribution kernel

$$k_h(x,\xi) = \int_{\mathbb{R}^n} e^{ih^{-1}[\phi(x,\xi)-\langle y,\xi\rangle]} a(x,\xi)\, \widehat{d_h\xi}, \quad \text{for all } x,\xi \in \mathbb{R}^n.$$

**Theorem 2.** *Let $p_1 \in [2,\infty[$ and $p_2 \in [1,\infty[$ and let $I_h(a,\phi)$ defined by*

$$I_h(a,\phi)f(x) = \iint e^{ih^{-1}[\phi(x,\xi)-\langle y,\xi\rangle]} a(x,\xi) f(y) \mathrm{d}y\, \widehat{d_h\xi},$$

*where $a \in S^m_{1,0}(\mathbb{R}^{2n})$ and $\phi \in \Phi(\mathbb{R}^{2n})$ Then $I_h(a,\phi) \colon L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is a trace class operator if and only if the symbol $a$ admits a decomposition of the form*

$$a(x,\xi) = e^{-ih^{-1}\phi(x,\xi)} \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi) \tag{3.1}$$

*where $\{u_{h,k}\}_{k\geq 0}$ and $\{v_{h,k}\}_{k\geq 0}$ are sequences of functions satisfying*

$$\sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} \|v_{h,k}\|_{L^{p_1'}} < \infty. \tag{3.2}$$

*Proof.* Let us assume that $I_h(a,\phi) \colon L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is a trace class operator. Then by using the theorem 1, there exist sequences $\{u_{h,k}\}_{k\geq 0}$ in $L^{p_2}$ and $\{v_{h,k}\}_{k\geq 0}$ in $L^{p_1'}$ satisfying

$$I_h(a,\phi)\varphi(x) = \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) \varphi(y) \mathrm{d}y, \quad \forall \varphi \in L^{p_1}$$

with

$$\sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} \|v_{h,k}\|_{L^{p_1'}} < \infty.$$

Let us choose $\xi_0 \in \mathbb{R}^n$ and $r > 0$, then we define a useful function by

$$\lambda_{\xi_0,r}(x) = \frac{1_{B(\xi_0,r)}(x)}{m(B(\xi_0,r))}$$

By using the Hausdorff-Young inequality we have

$$\|\mathcal{F}_h^{-1}(\lambda_{\xi_0,r})\|_{L^{p_1}} \leq \|\lambda_{\xi_0,r}\|_{L^{p_1'}} = 1.$$

So, for every $\xi_0 \in \mathbb{R}^n$ and $r > 0$, the function $\mathcal{F}_h^{-1}(\lambda_{\xi_0,r}) \in L^{p_1}(\mathbb{R}^n)$, and we get,

$$I_h(a,\phi)(\mathcal{F}_h^{-1}(\lambda_{\xi_0,r}))(x) = \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} u_{h,k}(x)v_{h,k}(y)\right) \mathcal{F}_h^{-1}(\lambda_{\xi_0,r})(y)\mathrm{d}y,$$

$$\lim_{r\to 0^+} \mathcal{F}_h^{-1}(\lambda_{\xi_0,r})(x) = \lim_{r\to 0^+} \frac{1}{m(B(\xi_0,r))} \int_{B(\xi_0,r)} e^{ih^{-1}\langle x,\xi\rangle}\mathrm{d}\xi, = e^{ih^{-1}\langle x,\xi_0\rangle},$$

The dominated convergence theorem now gives us

$$\lim_{r\to 0^+} I_h(a,\phi)\mathcal{F}_h^{-1}(\lambda_{\xi_0,r})(x) = I_h(a,\phi)\left(\lim_{r\to 0^+} \mathcal{F}_h^{-1}(\lambda_{\xi_0,r})(x)\right)$$
$$= \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} u_{h,k}(x)v_{h,k}(y)\right) e^{ih^{-1}\langle y,\xi_0\rangle}\mathrm{d}y, = \sum_{k=0}^{\infty} u_{h,k}(x)\left(\mathcal{F}_h^{-1}v_{h,k}\right)(\xi_0).$$

In fact, for all $x \in \mathbb{R}^n$,

$$\left|\left(\sum_{k=0}^{\infty} u_{h,k}(x)v_{h,k}(y)\right) \mathcal{F}_h^{-1}(\lambda_{\xi_0,r})(y)\right|$$
$$= \left|\left(\sum_{k=0}^{\infty} u_{h,k}(x)v_{h,k}(y)\right) \frac{1}{m(B(\xi_0,r))} \int_{B(\xi_0,r)} e^{ih^{-1}\langle x,\xi\rangle}\mathrm{d}\xi\right|$$
$$\leq \left|\sum_{k=0}^{\infty} u_{h,k}(x)v_{h,k}(y)\right| \leq |k_h(x,y)|.$$

Now, by the dominated convergence theorem, we have

$$\lim_{r\to 0^+} I_h(a,\phi)\left(\mathcal{F}_h^{-1}(\lambda_{\xi_0,r})\right)(x) = \lim_{r\to 0^+} \int_{\mathbb{R}^n} k_h(x,y)\mathcal{F}_h^{-1}(\lambda_{\xi_0,r})(y)\mathrm{d}y$$
$$= \int_{\mathbb{R}^n} k_h(x,y) \lim_{r\to 0^+} \mathcal{F}_h^{-1}(\lambda_{\xi_0,r})(y)\mathrm{d}y = \int_{\mathbb{R}^n} e^{ih^{-1}\langle y,\xi_0\rangle}\mathrm{d}y$$
$$= \lim_{l\to\infty} \int_{|y|\leq l} e^{ih^{-1}\langle y,\xi_0\rangle}\mathrm{d}y$$
$$= \lim_{l\to\infty} \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} u_{h,k}(x)v_{h,k}(y)\right) e^{ih^{-1}\langle y,\xi_0\rangle} 1_{B(0,l)}(y)\mathrm{d}y$$

$$
\begin{aligned}
&= \lim_{l\to\infty} \int_{\mathbb{R}^n} \left( \lim_{m\to\infty} \sum_{k=0}^{m} u_{h,k}(x) v_{h,k}(y) \right) e^{ih^{-1}\langle y,\xi_0\rangle} 1_{B(0,l)}(y) \mathrm{d}y \\
&= \lim_{l\to\infty} \lim_{m\to\infty} \sum_{k=0}^{m} u_{h,k}(x) \int_{\mathbb{R}^n} v_{h,k}(y) e^{ih^{-1}\langle y,\xi_0\rangle} 1_{B(0,l)}(y) \mathrm{d}y \\
&= \lim_{l\to\infty} \lim_{m\to\infty} \sum_{k=0}^{m} u_{h,k}(x) \int_{|y|\le l} v_{h,k}(y) e^{ih^{-1}\langle y,\xi_0\rangle} \mathrm{d}y \\
&= \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi_0).
\end{aligned}
$$

On the other hand, if we compute $I_h(a,\phi)\left(\mathcal{F}_h^{-1}(\lambda_{\xi_0,r})\right)$, we have

$$
\lim_{r\to 0^+} I_h(a,\phi)\left(\mathcal{F}_h^{-1}(\lambda_{\xi_0,r})\right)(x) = e^{ih^{-1}\phi(x,\xi_0)} a(x,\xi_0).
$$

Consequently, we deduce the identity

$$
e^{ih^{-1}\phi(x,\xi_0)} a(x,\xi_0) = \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi_0),
$$

which in turn is equivalent to

$$
a(x,\xi_0) = e^{-ih^{-1}\phi(x,\xi_0)} \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi_0).
$$

We've now established the first half of the theorem. Now, if we suppose that the symbol $a$ of $I_h(a,\phi)$ fulfills the decomposition formula (3.1) for fixed sequences $u_{h,k}$ in $L^{p_2}$ and $v_{h,k}$ in $L^{p_1'}$ fulfilling (3.2), then from (1.1) we can write (in the sense of distributions)

$$
\begin{aligned}
I_h(a,\phi)f(x) &= \int_{\mathbb{R}^n} e^{ih^{-1}\phi(x,\xi)} a(x,\xi) \mathcal{F}_h f(\xi) \widehat{d_h\xi} \\
&= \int_{\mathbb{R}^n} \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi) \mathcal{F}_h f(\xi) \widehat{d_h\xi} \\
&= \int_{\mathbb{R}^n} \sum_{k=0}^{\infty} u_{h,k}(x) \int_{\mathbb{R}^n} e^{ih^{-1}\langle y,\xi_0\rangle} v_{h,k}(y) \mathrm{d}y \, \mathcal{F}_h f(\xi) \widehat{d_h\xi} \\
&= \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) \left( \int_{\mathbb{R}^n} e^{ih^{-1}\langle y,\xi_0\rangle} v_{h,k}(y) \mathcal{F}_h f(\xi) \widehat{d_h\xi} \right) \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) f(y) \mathrm{d}y.
\end{aligned}
$$

So, by Theorem 1 we finish the proof. □

Furthermore, as a result of Theorem 2, we gain the following result.

**Corollary 2.** *Let $I_h(a,\phi)$ be a h-Fourier integral operator. So, if $I_h(a,\phi)\colon L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is a trace class operator, then $a \in L_x^{p_2}L_\xi^{p_1}(\mathbb{R}^{2n}) \cap L_\xi^{p_1}L_x^{p_2}(\mathbb{R}^{2n})$; this means that*

$$\|a\|_{L_\xi^{p_1}L_x^{p_2}} = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|a(x,\xi)|^{p_1}\mathrm{d}\xi\right)^{\frac{p_2}{p_1}}\mathrm{d}x\right)^{\frac{1}{p_2}} < \infty,$$

*and*

$$\|a\|_{L_x^{p_2}L_\xi^{p_1}} = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|a(x,\xi)|^{p_2}\mathrm{d}x\right)^{\frac{p_1}{p_2}}\mathrm{d}\xi\right)^{\frac{1}{p_1}} < \infty.$$

*Proof.* If $I_h(a,\phi)\colon L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is a trace class operator, then Theorem 2 guarantees the decomposition

$$a(x,\xi) = e^{-ih^{-1}\phi(x,\xi)}\sum_{k=0}^{\infty}u_{h,k}(x)\mathcal{F}_h^{-1}v_{h,k}(\xi),$$

where $u_{h,k}(x)$ and $v_{h,k}(\xi)$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty}\|u_{h,k}\|_{L^{p_2}}\|v_{h,k}\|_{L^{p_1'}} < \infty.$$

So, if we take the $L_x^{p_2}$-norm, we have,

$$\begin{aligned}\|a(.,\xi)\|_{L_x^{p_2}} &= \left\|e^{-ih^{-1}\phi(.,\xi)}\sum_{k=0}^{\infty}u_{h,k}(.)\mathcal{F}_h^{-1}v_{h,k}(\xi)\right\|_{L_x^{p_2}}\\ &= \left\|\sum_{k=0}^{\infty}u_{h,k}(x)\mathcal{F}_h^{-1}v_{h,k}(\xi)\right\|_{L_x^{p_2}} \le \sum_{k=0}^{\infty}\|u_{h,k}\|_{L^{p_2}}|\mathcal{F}_h^{-1}v_{h,k}(\xi)|.\end{aligned}$$

Using the Hausdorff-Young inequality, we can now deduce that $\|\mathcal{F}_h^{-1}v_{h,k}\|_{L^{p_1}} \le \|v_{h,k}\|_{L^{p_1'}}$. Consequently,

$$\begin{aligned}\|a(.,.)\|_{L_x^{p_2}L_\xi^{p_1}} &= \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|a(x,\xi)|^{p_2}\mathrm{d}x\right)^{\frac{p_1}{p_2}}\mathrm{d}\xi\right)^{\frac{1}{p_1}} \le \left\|\sum_{k=0}^{\infty}\|u_{h,k}\|_{L^{p_2}}|\mathcal{F}_h^{-1}v_{h,k}(\xi)|\right\|_{L_\xi^{p_1}}\\ &\le \sum_{k=0}^{\infty}\|u_{h,k}\|_{L^{p_2}}\|\mathcal{F}_h^{-1}v_{h,k}\|_{L^{p_1}} \le \sum_{k=0}^{\infty}\|u_{h,k}\|_{L^{p_2}}\|v_{h,k}\|_{L^{p_1'}} < \infty.\end{aligned}$$

In a similar vein, we can show that

$$\|a(.,.)\|_{L_\xi^{p_1}L_x^{p_2}} \le \sum_{k=0}^{\infty}\|u_{h,k}\|_{L^{p_2}}\|v_{h,k}\|_{L^{p_1'}} < \infty.$$

As a result, we've completed the proof. $\square$

## ACKNOWLEDGMENTS

The authors would like to express their deep thanks to the referees for their very careful reading and useful comments which do improve the presentation of this article.

The research of authors is supported by the general direction of scientific research and technological development DGRSDT, Algeria.

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