Miskolc Mathematical Notes
Vol. 26 (2025), No. 2, pp. 1061–1077

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4696

NEW GENERALIZED CONTRACTION MAPPINGS AND FIXED POINT RESULTS IN FUZZY METRIC SPACES

ABDELLAH TAQBIBT, MOHAMED CHAIB, M'HAMED ELOMARI, AND SAID MELLIANI

Received 17 April, 2023

Abstract. In this paper, we introduce the notions of fuzzy α-$\mathcal{H}$-contractions and fuzzy α-$\mathcal{H}$-Φ-contractions via fuzzy α-admissible mappings, within the framework of complete fuzzy metric spaces $(\mathcal{F}.\mathcal{M}.\mathcal{S})$ as defined by Georg and Veermani. We present several fixed point theorems for this class of contractions and provide practical examples to illustrate our results.

2010 *Mathematics Subject Classification:* 03E72; 55M20; 58C30

Keywords: fixed point, fuzzy α-$\mathcal{H}$-contractive mappings, fuzzy α-$\mathcal{H}$-Φ-contractive mappings, fuzzy α-admissible

1. INTRODUCTION

Fixed point theory is a fundamental area of research in nonlinear functional analysis, with wide-ranging applications in various fields such as physics, mathematics, economics, quantum mechanics, and other domains. It serves as a powerful tool for solving a diverse range of problems and holds significant potential for practical applications. (see e.g. [2, 4, 5, 11, 12, 19, 28]), engineering sciences (see e.g. [17, 22]) and various domains. The most famous theoretical development in the chronology of the fixed point theory, is the famous principle of contraction due to the Polish mathematician Banach, [21] (BCP), this theorem has been developed and generalized with several methods, the reader can see for example [15, 16, 24] and the references therein. Kaleva and Selikkala were the first to define the distance between two objects in a $\mathcal{F}.\mathcal{M}.\mathcal{S}$ in [14], using a non-negative, upper semi-continuous fuzzy number that is both convex and normal. Since then, various researchers have explored the properties of $\mathcal{F}.\mathcal{M}.\mathcal{S}$, including Felbin [6], George et al. [7], Gregory [10], and Hadzic [13], and applied these results to practical problems in the fuzzy domain. In 1988, Grabic [9] introduced a contraction compatible with the structure of $\mathcal{F}.\mathcal{M}.\mathcal{S}$, followed by D. Mihet [18], who generalized it to the ψ-contraction. Meanwhile,

Wardowski introduced the concept of fuzzy $\mathcal{H}$-contractive mapping in [29], proving existence and uniqueness of fixed points in $\mathcal{M}_F$-complete $\mathcal{F}.\mathcal{M}.\mathcal{S}$. Shukla extended this concept to fuzzy $\mathcal{H}$-weak contraction mappings in [27], applying it to some fixed point theorems attributed to Wardowski. Beg et al. defined fuzzy α-$\mathcal{H}$-contraction in [3], demonstrating fixed point results with regard to existence and uniqueness in fuzzy $\mathcal{M}_F$-complete metric spaces. For more information, the reader can refer to [1, 20, 23, 26] and the references therein.

Building on the existing works in this field, we introduce new concepts of fuzzy $\mathcal{H}$-contractive, $\mathcal{H}$-weak contraction, α-$\mathcal{H}$-contraction, and α-$\mathcal{H}$-ψ-contraction mappings within the framework of $\mathcal{F}.\mathcal{M}.\mathcal{S}$. In this work, we establish various results pertaining to the existence and uniqueness of fixed points for these mappings in $\mathcal{F}.\mathcal{M}.\mathcal{S}$. Additionally, we provide practical examples to illustrate these results on existence and uniqueness. The findings presented in this study contribute to the advancement and consolidation of different contraction results that are well-known in the literature of fixed point theory on $\mathcal{F}.\mathcal{M}.\mathcal{S}$.

2. Preliminaries

During this article, we will be working only the notion of fuzzy metric in the Georg and Veermani sense, that is given as follows:

Definition 1 ([25]). The couple $([0,1],*)$ is said to be a continuous t-norm if $*\colon [0,1]^2 \to [0,1]$ is a binary operation and $([0,1],*)$ is a topological commutative monoid with unit 1 and $\eta \leq \gamma$ and $\beta \leq \lambda$ implies $\eta * \beta \leq \gamma * \lambda \quad (\eta,\beta,\gamma,\lambda \in [0,1])$.

Definition 2 ([7]). Let Σ is an arbitrary set, $*$ is a continuous t-norm and $\mathcal{M}_F$ is a fuzzy set on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

(C1) $\mathcal{M}_F(\sigma,\rho,t) > 0$ for all $\sigma,\rho \in \Sigma$;

(C2) $\mathcal{M}_F(\sigma,\rho,t) = 1$, if and only if $\sigma = \rho$, for every $t > 0$;

(C3) $\mathcal{M}_F(\sigma,\rho,t) = \mathcal{M}_F(\rho,\sigma,t)$;

(C4) $\mathcal{M}_F(\sigma,\delta,t) * \mathcal{M}_F(\rho,\delta,s) \leq \mathcal{M}_F(\sigma,\rho,t+s)$;

(C5) $t \mapsto \mathcal{M}_F(\sigma,\rho,t)$ is continuous from $[0,\infty)$ to $[0,1]$ for every $\sigma,\rho \in \Sigma$ and $s,t > 0$.

Then, the 3-uplet $(\Sigma,\mathcal{M}_F,*)$ is called a $\mathcal{F}.\mathcal{M}.\mathcal{S}$.

Definition 3 ([7]). Let $(\Sigma,\mathcal{M}_F,*)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. The open ball of centre $\sigma \in \Sigma$ and radius ρ, $0 < \rho < 1$, will be denoted by $B(\sigma,\rho,t) = \{\rho \in \Sigma,\ \mathcal{M}_F(\sigma,\rho,t) > 1-\rho\}$. The family $\{B(\sigma,\rho,t)/\sigma \in \Sigma,\ 0 < \rho < 1,\ t > 0\}$ is a basis of neighborhood's for a separate topology on Σ, that can induced by $\mathcal{M}_F$ (fuzzy metric).

In the sequel $(\Sigma, \mathcal{M}_F, *)$ will be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$ in the sense of George and Veeramani (see [7]).

Definition 4 ([7])**.** A sequence $(\sigma_n)_{n\in\mathbb{N}}$ is called $\mathcal{M}_F$-Cauchy in the metric space $(\Sigma, \mathcal{M}_F, *)$, if for every $\varepsilon \in]0,1[$ and $t > 0$, there is $m_0 \in \mathbb{N}$ such that $\mathcal{M}_F(\sigma_n, \sigma_m, t) > 1-\varepsilon$ for all $m, n \geq m_0$.

A $\mathcal{F}.\mathcal{M}.\mathcal{S}$ is complete, when every $\mathcal{M}_F$-Cauchy sequence is convergent.

Theorem 1 ([7])**.** *A sequence $(\sigma_n)_{n\in\mathbb{N}}$ in a $\mathcal{F}.\mathcal{M}.\mathcal{S}$ $(\Sigma, \mathcal{M}_F, *)$ converges to σ if and only if $\lim_{n\to\infty} \mathcal{M}_F(\sigma_n, \sigma, t) = 1$.*

Remark 1.

(1) In the setting of $\mathcal{F}.\mathcal{M}.\mathcal{S}$ $(\Sigma, \mathcal{M}_F, *)$, the limit of any convergent sequence is unique.
(2) As in the classical case, the two notions of convergent sequence and $\mathcal{M}_F$-Cauchy sequence are equivalent.
(3) For all $\sigma, \rho \in \Sigma$, the mapping $\mathcal{M}_F(\sigma, \rho, .)$ is nondecreasing on $(0, \infty)$.
(4) In the space $(\Sigma, \mathcal{M}_F, *)$, the mapping $\mathcal{M}_F$ is continuous on $\Sigma \times \Sigma \times (0, \infty)$.

Definition 5 ([9])**.** Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A self-mapping $Q\colon \Sigma \to \Sigma$ is said to be fuzzy contractive if there is $\lambda > 1$ such that

$$\mathcal{M}_F(Q\sigma, Q\rho, t) \geq \mathcal{M}_F(\sigma, \rho, \lambda t) \text{ for all } \sigma, \rho \in \Sigma \text{ and } t > 0.$$

Definition 6 ([8])**.** Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A self-mapping $Q\colon \Sigma \to \Sigma$ is said to be fuzzy α-admissible if there is a function $\alpha\colon \Sigma \times \Sigma \times (0, \infty) \to (0, \infty)$ such that

$$\alpha(\sigma, \rho, t) \geq 1, \text{ implies that } \alpha(Q\sigma, Q\rho, t) \geq 1 \text{ for all } \sigma, \rho \in \Sigma \text{ and } t > 0.$$

3. Main Results

In the rest of this paper, we extend certain results concerning the notion of contraction, in $\mathcal{F}.\mathcal{M}.\mathcal{S}$.

Definition 7 ([9])**.** Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A self-mapping $Q\colon \Sigma \to \Sigma$ is said to be fuzzy contractive, if there is $\lambda > 1$ such that

$$\mathcal{M}_F(Q\sigma, Q\rho, t) \geq \mathcal{M}_F(\sigma, \rho, \lambda t) \text{ for all } \sigma, \rho \in \Sigma \text{ and } t > 0.$$

We are going to introduce our first contraction named fuzzy $\mathcal{H}$-contraction, let $\mathcal{H} = \{\eta\colon (0,1] \to [0, \infty) / (\mathcal{H}_1) - (\mathcal{H}_3) \text{ hold }\}$, where

$(\mathcal{H}_1)$ η is continuous bijective;
$(\mathcal{H}_2)$ η is strictly decreasing;
$(\mathcal{H}_3)$ $\eta(s) = 0$ if and only if $s = 1$.

Definition 8. Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$, a mapping $Q\colon \Sigma \to \Sigma$ is called fuzzy $\mathcal{H}$-contractive if there are $\eta \in \mathcal{H}$ and $\lambda > 1$ such that

$$\eta(\mathcal{M}_F(Q\sigma, Q\rho, t)) \leq \eta(\mathcal{M}_F(\sigma, \rho, \lambda t)), \text{ for all } \sigma, \rho \in \Sigma, t > 0. \tag{3.1}$$

Proposition 1. *Every fuzzy $\mathcal{H}$-contraction mapping $Q\colon \Sigma \to \Sigma$ is continuous.*

Proof. Let $(\sigma_n)_{n\in\mathbb{N}} \subset \Sigma$, with

$$\lim_{n\to+\infty} \mathcal{M}_F(\sigma_n,\sigma,t) = 1 \text{ and } \mathcal{M}_F(Q\sigma_n, Q\sigma,t) < 1 \text{ for all } n \in \mathbb{N},$$

for some $\sigma \in \Sigma$. According to the strictly decreasing of η and (3.1), we can obtain

$$\eta(\mathcal{M}_F(Q\sigma_n, Q\sigma,t)) \leq \eta(\mathcal{M}_F(\sigma_n,\sigma,\lambda t)) < \eta(\mathcal{M}_F(\sigma_n,\sigma,t)),$$

which implies that

$$\mathcal{M}_F(Q\sigma_n, Q\sigma,t) > \mathcal{M}_F(\sigma_n,\sigma,t).$$

Hence,

$$\lim_{n\to+\infty} \mathcal{M}_F(Q\sigma_n, Q\sigma,t) = 1.$$

Therefore, Q is continuous. □

The theorem which will come is our first contribution in this framework.

Theorem 2. *Let $(\Sigma, \mathcal{M}_F, *)$ be an $\mathcal{M}_F$-complete $\mathcal{F}.\mathcal{M}.\mathcal{S}$. Assume that $Q\colon \Sigma \to \Sigma$ is a fuzzy $\mathcal{H}$-contraction mapping. Then, Q possesses a fixed point, and moreover it is unique.*

Proof. Let $\sigma_0 \in \Sigma$, $\sigma_n = Q^n\sigma_0$. By using the contraction given in condition (3.1), we can obtain

$$0 < \eta(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t)) \leq \eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t))$$

and by induction, we have

$$\eta(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t)) \leq \eta(\mathcal{M}_F(\sigma_0,\sigma_1,\lambda^n t)).$$

Since $\lim_{n\to+\infty} \mathcal{M}_F(\sigma_0,\sigma_1,t) = 1$ and η is continuous, then

$$\lim_{n\to+\infty} \mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t) = 0,$$
$$\lim_{n\to+\infty} \eta(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t)) = 0$$

and thus

$$\lim_{n\to+\infty} \mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t) = 1. \tag{3.2}$$

Now, let us prove that $(\sigma_n)_{n\in\mathbb{N}}$ is an $\mathcal{M}_F$-Cauchy sequence. Assume in the contrast that $(\sigma_n)_{n\in\mathbb{N}}$ is not $\mathcal{M}_F$-Cauchy, which implies that there are $t_0 > 0$, $\varepsilon \in (0,1)$ and two subsequences (σ_{m_k}), (σ_{n_k}) of $(\sigma_n)_{n\in\mathbb{N}}$ such that $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ and

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) \leq 1-\varepsilon, \tag{3.3}$$

since the mapping $\mathcal{M}_F(\sigma,\rho,.)$ is nondecreasing on $(0,\infty)$ for all $\sigma,\rho \in \Sigma$, we obtain

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},\frac{t_0}{2}) \leq 1-\varepsilon. \tag{3.4}$$

The smallest integer exceeding m_k and satisfying inequality (3.4) will be noted n_k, so thus, we obtain

$$\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2})>1-\varepsilon. \tag{3.5}$$

By the contractive inequality given in (3.1) we take $\sigma=\sigma_{m_k-1}$, $\rho=\rho_{n_k-1}$ and $t=t_0$, it follows

$$\eta(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0))\leq\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0))<\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)).$$

Thanks to the strictly decreasing of η, we have

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)>\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0). \tag{3.6}$$

We use (3.3) and (3.5) in (3.6) and using (C4), we can write

$$\begin{aligned}1-\varepsilon&\geq\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)\\&>\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)\\&>\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2})*\mathcal{M}_F(\sigma_{n_k},\sigma_{n_k-1},\frac{t_0}{2})\\&>(1-\varepsilon)*\mathcal{M}_F(\sigma_{n_k},\sigma_{n_k-1},\frac{t_0}{2}).\end{aligned}$$

By passing to the limit as k tends to $+\infty$ and use (3.2), we can deduce

$$\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)=1-\varepsilon$$

and

$$\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)=1-\varepsilon.$$

On the other side, we have

$$\begin{aligned}\eta(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0))&\leq\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0))\\&\leq\eta(\mathcal{M}_F(\sigma_{m_k-2},\sigma_{n_k-2},\lambda^2 t_0))\\&\vdots\\&\leq\eta(\mathcal{M}_F(\sigma_0,\sigma_{n_1},\lambda^{n_k}t_0)).\end{aligned}$$

Since η is continuous, we get

$$\eta(\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0))\leq\eta(\lim_{k\to\infty}\mathcal{M}_F(\sigma_0,\sigma_1,\lambda^{n_k}t_0)),$$

this implies

$$\eta(1)\geq\eta(1-\varepsilon),$$

that constitutes is a contradiction, and thus (σ_n) is an $\mathcal{M}_F$-Cauchy sequence in Σ. The completeness of $(\Sigma,\mathcal{M}_F,*)$, implies that there is $\sigma\in\Sigma$ satisfying

$$\lim_{n\to\infty}\mathcal{M}_F(\sigma_n,\sigma,t)=1,t>0.$$

From Proposition 1, we have $\lim_{n\to\infty} \mathcal{M}_F(Q\sigma_n, Q\sigma, t) = 1$. According to the uniqueness of the limit, we obtain $Q\sigma = \sigma$. Now let us prove the uniqueness. Let σ_1 and σ_2 $(\sigma_1 \neq \sigma_2)$ such that $Q\sigma_1 = \sigma_1$ and $Q\sigma_2 = \sigma_2$, then we have

$$\eta(\mathcal{M}_F(Q\sigma_1, Q\sigma_2, t_0)) \leq \eta(\mathcal{M}_F(\sigma_1, \sigma_2, \lambda t_0)) < \eta(\mathcal{M}_F(\sigma_1, \sigma_2, t_0)),$$

which is a contradiction. □

Example 1. Consider the following fuzzy set

$$\mathcal{M}_F : \mathbb{R}^2 \times \mathbb{R}^+ \to [0,1]; \quad (\sigma, \rho, t) \mapsto \mathcal{M}_F(\sigma, \rho, t) = \exp\{-\frac{|\sigma - \rho|}{t}\}.$$

It is clear that $(\mathbb{R}, \mathcal{M}_F, *)$ is an $\mathcal{M}_F$-complete $\mathcal{F}.\mathcal{M}.\mathcal{S}$ with the product t-norm $*$ $(b * d = bd)$, $b, d \in [0,1]$. We define a self mapping Q by

$$Q_F : \mathbb{R} \to \mathbb{R}; \quad \sigma \mapsto Q\sigma = \frac{\sigma}{5},$$

and a function η in $\mathcal{H}$ given by

$$\eta : (0,1] \to \mathbb{R}^+; \quad \beta \mapsto \eta(\beta) = -\ln(\beta).$$

For each $t > 0$, $\sigma, \rho \in \mathbb{R}$, we have

$$\begin{aligned}
\eta\big(\mathcal{M}_F(Q\sigma, Q\rho, t)\big) &= \eta\big(\exp\{-\frac{|Q\sigma - Q\rho|}{t}\}\big) \\
&\leq \ln\big(\exp\{\frac{|\sigma - \rho|}{5t}\}\big) \\
&\leq -\ln\big(\exp\{-\frac{|\sigma - \rho|}{2t}\}\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma, \rho, \lambda t),
\end{aligned}$$

where $\lambda = 2 > 1$. Then, Q is a fuzzy $\mathcal{H}$-contractive mapping. So, from Theorem 2, Q admits 0 as a fixed point.

Definition 9. A mapping $Q : \Sigma \to \Sigma$ over a fuzzy metric space $(\Sigma, \mathcal{M}_F, *)$, is called fuzzy $\mathcal{H}$-weak contractive if there are $\eta \in \mathcal{H}$ and $\lambda > 1$ such that

$$\eta(\mathcal{M}_F(Q\sigma, Q\rho, t)) \leq \mathcal{N}(\sigma, \rho, \lambda t), \tag{3.7}$$

where $\mathcal{N}(\sigma, \rho, \lambda t) = \max\big\{\eta(\mathcal{M}_F(\sigma, \rho, t)), \eta(\mathcal{M}_F(\sigma, Q\sigma, t)), \eta(\mathcal{M}_F(\rho, Q\rho, t))\big\}$.

Theorem 3. *Let $(\Sigma, \mathcal{M}_F, *)$ be an $\mathcal{M}_F$-complete $\mathcal{F}.\mathcal{M}.\mathcal{S}$. Assume that $Q : \Sigma \to \Sigma$ is a fuzzy $\mathcal{H}$-weak contractive mapping, then Q admits a unique fixed point.*

Proof. Let us consider σ_0 in Σ, define a sequence $(\sigma_n)_{n\in\mathbb{N}}$ by

$$\sigma_n = Q\sigma_{n-1}.$$

We have

$$\eta(\mathcal{M}_F(\sigma_n, \sigma_{n+1}, t)) = \eta(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, t))$$

$$\begin{aligned}
&\leq \mathcal{N}(\sigma_{n-1},\sigma_n,\lambda t)\\
&\leq \max\left\{\eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)),\eta(\mathcal{M}_F(\sigma_{n-1},Q\sigma_{n-1},\lambda t)),\eta(\mathcal{M}_F(\sigma_n,Q\sigma_n,t))\right\}\\
&= \max\left\{\eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)),\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t))\right\}.
\end{aligned}$$

If for example $\eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)) \leq \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},\lambda t))$, then

$$\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) \leq \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},\lambda t)).$$

Since $\lambda t > t$ and η is strictly decreasing, what is absurd. Hence,

$$\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) \leq \eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)).$$

Then, we find that

$$\begin{aligned}
0 &< \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t))\\
&\leq \mathcal{N}(\sigma_{n-1},\sigma_n,\lambda t)\\
&\leq \eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t))\\
&\leq \mathcal{N}(\mathcal{M}_F(\sigma_{n-2},\sigma_{n-1},\lambda^2 t)),
\end{aligned}$$

and by induction, we obtain

$$0 < \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) \leq \eta(\mathcal{M}_F(\sigma_0,\sigma_1,\lambda^n t)).$$

So,

$$\lim_{n\to+\infty} \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) = 0.$$

Which means that

$$\lim_{n\to+\infty} (\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) = 1. \tag{3.8}$$

Now, we prove by contradiction that $(\sigma_n)_{n\in\mathbb{N}}$ is $\mathcal{M}_F$-Cauchy. Assume that $(\sigma_n)_{n\in\mathbb{N}}$ is not $\mathcal{M}_F$-Cauchy, so there are $\varepsilon \in (0,1)$, $t_0 > 0$ and two subsequences $(\sigma_{n_k})_{k\in\mathbb{N}}$, $(\sigma_{m_k})_{k\in\mathbb{N}}$ of $(\sigma_n)_{n\in\mathbb{N}}$, such that $m_k \geq n_k$ for all $k \in \mathbb{N}$ and

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) \leq 1-\varepsilon. \tag{3.9}$$

Then, we have

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},\frac{t_0}{2}) \leq \mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) \leq 1-\varepsilon. \tag{3.10}$$

The smallest integer exceeding m_k and satisfying inequality (3.4) will be denoted by n_k. So, we can have

$$\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2}) > 1-t.$$

From inequality (3.7), we get

$$\eta(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)) = \eta(\mathcal{M}_F(Q\sigma_{m_k-1},Q\sigma_{n_k-1},t_0) \leq \mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0), \tag{3.11}$$

where

$\mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0)$

$$= \max\{\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0)),\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{m_k},\lambda t_0)),\eta(\mathcal{M}_F(\sigma_{n_k-1},\sigma_{n_k},\lambda t_0))\}.$$

If $\mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0) = \eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{m_k},\lambda t_0))$, then (3.11) becomes

$$\eta(\mathcal{M}_F(\sigma_{m_k},\sigma_{m_k},t_0)) \leq \eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{m_k},\lambda t_0)). \tag{3.12}$$

Passing to the limit as k tends to $+\infty$ in (3.12) and by using (3.8), Remark 1 and taking into consideration the fact that η is continuous, we obtain

$$\eta\Big(\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)\Big) = 0.$$

Therefore, $\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) = 1$, which contradicts the formula (3.9). In the same way, if we consider $\mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},t_0) = \eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},t_0))$, then we come to an absurdity again. We must have

$$\mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},t_0) = \eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0))$$

and thus (3.11) gives

$$\eta\Big(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)\Big) \leq \eta\Big(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},\lambda t_0)\Big) < \eta\Big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)\Big). \tag{3.13}$$

Thanks to the monotony of η

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) > \mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0). \tag{3.14}$$

By using (3.14) and (3.9)

$$\begin{aligned}
1-\varepsilon &\geq \mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)\\
&> \mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)\\
&\geq \mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2}) * \mathcal{M}_F(\sigma_{n_k},\sigma_{n_k-1},\frac{t_0}{2})\\
&> (1-\varepsilon) * \mathcal{M}_F(\sigma_{n_k},\sigma_{n_k-1},\frac{t_0}{2}),
\end{aligned}$$

which yields by letting $k\to\infty$ and from (3.8) and (C3) that

$$\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0) = 1-\varepsilon = \lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0).$$

Now, using (3.13), (3.8), the continuity of η and taking $k\to\infty$, we get

$$\eta(1) > \eta(1-\varepsilon),$$

which gives an absurdity, and thus $(\sigma_n)_{n\in\mathbb{N}}$ is $\mathcal{M}_F$-Cauchy in Σ. The completeness of $(\Sigma,\mathcal{M}_F,*)$ implies that there is $\sigma\in\Sigma$ such that

$$\lim_{k\to\infty}\mathcal{M}_F(\sigma_n,\sigma,t) = 1, \quad \text{for all } t>0.$$

Next, we prove that σ is a fixed point of Q. Assume that there is $t_1>0$, such that $\mathcal{M}_F(\sigma,Q\sigma,t_1) < 1$, then $\eta(\mathcal{M}_F(\sigma,Q\sigma,t_1)) > 0$. Also, as Q is $\mathcal{H}$-weak contraction mapping in fuzzy setting, we have

$$\eta((\mathcal{M}_F(\sigma_{n+1},Q\sigma,t_1)) = \eta(\mathcal{M}_F(Q\sigma_n,Q\sigma,t_1)) \leq \mathcal{N}(\sigma_n,\sigma,\lambda t_1),$$

with $\mathcal{N}(\sigma_n,\sigma,t_1)=\max\{\eta(\mathcal{M}_F(\sigma_n,\sigma,t_1)),\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t_1)),\eta(\mathcal{M}_F(\sigma,Q\sigma,t_1))\}$. According to the continuity of η, we have

$$\lim_{n\to\infty}\eta(\mathcal{M}_F(\sigma_n,\sigma,t_1))=\lim_{n\to\infty}\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t_1))=0$$

for every $t>0$, and thus

$$\lim_{n\to\infty}\mathcal{N}(\sigma_n,x,t_1)=\max\{0,0,\mathcal{M}_F(x,Q\sigma,t_1)\}=\eta(\mathcal{M}_F(x,Q\sigma,t_1)).$$

If we tend n to infinity, we get

$$\eta(\mathcal{M}_F(\sigma,Q\sigma,t_1))\le\eta(\mathcal{M}_F(\sigma,Q\sigma,\lambda t_1))<\eta(\mathcal{M}_F(\sigma,Q\sigma,t_1)),\quad \text{as } \lambda>1,$$

which is a contradiction. Hence, we have $\mathcal{M}_F(\sigma,Q\sigma,t)=1$, $t>0$, then σ is a fixed point of Q. To show the uniqueness of σ, let σ_1,σ_2 be two different fixed points of Q. So, using (3.7), we get

$$\begin{aligned}\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t))&=\eta(\mathcal{M}_F(Q\sigma_1,Q\sigma_2,t))\\&\le\max\left\{\eta(\mathcal{M}_F(\sigma_1,\sigma_2,\lambda t)),\eta(\mathcal{M}_F(\sigma_1,Q\sigma_1,\lambda t)),\eta(\mathcal{M}_F(\sigma_2,Q\sigma_2,\lambda t))\right\}\\&=\eta(\mathcal{M}_F(\sigma_1,\sigma_2,\lambda t)).\end{aligned}\tag{3.15}$$

If $\mathcal{M}_F(\sigma_1,\sigma_2,t)<1$, then $\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t))>0$ and hence (3.15) becomes

$$\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t))\le\eta(\mathcal{M}_F(\sigma_1,\sigma_2,\lambda t))<\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t)),$$

that is a contradiction. Therefore, $\mathcal{M}_F(\sigma_1,\sigma_2,t)=1$, it yields $\sigma_1=\sigma_2$, which is the desired result. □

Definition 10. Let $(\Sigma,\mathcal{M}_F,*)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A mapping $Q\colon\Sigma\to\Sigma$ is called fuzzy α-$\mathcal{H}$-contractive if there exist $\eta\in\mathcal{H}$ and $\lambda>1$ such that

$$\eta\big(\mathcal{M}_F(Q\sigma,Q\rho,\frac{t}{\alpha(\sigma,\rho,t)})\big)\le\eta(\mathcal{M}_F(\sigma,\rho,\lambda t)),\ \text{for every } \rho,\sigma\in\Sigma,t>0.\tag{3.16}$$

Theorem 4. *Let $Q\colon\Sigma\to\Sigma$ is a mapping as described in Definition 10. We further assume that it satisfies conditions (i)-(iv), where*

(i) *there is $\sigma_0\in\Sigma$ with $\alpha(\sigma_0,Q\sigma_0,t)\ge1$, $t\ge0$;*
(ii) *Q is fuzzy α-admissible;*
(iii) *for each subsequence $(\sigma_{n_k})_{k\in\mathbb{N}}\subset(\sigma_n)_{n\in\mathbb{N}}=(Q^n\sigma_0)_{n\in\mathbb{N}}$, we have*

$$\alpha(\sigma_{n_k},\sigma_{n_l},t)\ge1,\quad \text{where } l,k\in\mathbb{N},\ l<k \text{ and } t>0;$$

(iv) *if $(\sigma_n)_{n\in\mathbb{N}}\subset\Sigma$ with $\alpha(\sigma_n,\sigma_{n+1},t)\ge1$, and $\lim_{n\to\infty}\sigma_n=\sigma$, then*

$$\alpha(\sigma_n,\sigma_{n+1},t)\ge1,\ \text{for all } t>0,n\in\mathbb{N}.$$

Then, Q possesses a unique fixed point.

Proof. From (i), there is $\sigma_0 \in \Sigma$ with $\alpha(\sigma_0, Q\sigma_0, t) \geq 1,\ t \geq 0$. We define a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in Σ by $\sigma_n = Q\sigma_{n-1}$, for all $n \in \mathbb{N}$. Since Q is fuzzy α-admissible, then we have

$$\alpha(\sigma_0, \sigma_1, t) = \alpha(\sigma_0, Q\sigma_0, t) \geq 1,$$

implying

$$\alpha(Q\sigma_0, Q\sigma_1, t) = \alpha(\sigma_1, \sigma_2, t) \geq 1,\ t > 0.$$

By induction, we get

$$\alpha(\sigma_n, \sigma_{n+1}, t) \geq 1, \quad t > 0,\ n \in \mathbb{N}. \tag{3.17}$$

Since Q is a fuzzy α-$\mathcal{H}$-contractive mapping and by using (3.17), we get

$$\begin{aligned}
0 &\leq \eta\big(\mathcal{M}_F(\sigma_n, \sigma_{n+1}, t)\big) \\
&= \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, t)\big) \\
&\leq \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, \frac{t}{\alpha(\sigma_{n-1}, \sigma_n, t)})\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{n-1}, \sigma_n, \lambda t)\big) \\
&= \eta\big(\mathcal{M}_F(Q\sigma_{n-2}, Q\sigma_{n-1}, \lambda t)\big) \\
&\leq \eta\big(\mathcal{M}_F(Q\sigma_{n-2}, Q\sigma_{n-1}, \lambda\frac{t}{\alpha(\sigma_{n-2}, \sigma_{n-1}, t)})\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{n-2}, \sigma_{n-1}, \lambda^2 t)\big) \\
&\ \ \vdots \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{n-2}, \sigma_{n-1}, \lambda^n t)\big).
\end{aligned} \tag{3.18}$$

Taking the limit as n tends to $+\infty$ in (3.18), we can get

$$\lim_{n\to\infty} \eta\big(\mathcal{M}_F(\sigma_n, \sigma_{n+1}, t)\big) = 0.$$

And thus $\lim_{n\to\infty} \mathcal{M}_F(\sigma_n, \sigma_{n+1}, t) = 1$.

In the sequel, we show by contradiction that $(\sigma_n)_{n\in\mathbb{N}}$ is $\mathcal{M}_F$-Cauchy. If $(\sigma_n)_{n\in\mathbb{N}}$ is not $\mathcal{M}_F$-Cauchy, then there exist $\varepsilon \in (0,1)$, $t_0 > 0$ and two subsequences $(\sigma_{n_k})_{k\in\mathbb{N}}$, $(\sigma_{m_k})_{k\in\mathbb{N}}$ of $(\sigma_n)_{n\in\mathbb{N}}$, such that

$$m_k > n_k \geq k \text{ and } \mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, t_0) \leq 1 - \varepsilon \text{ for every } k \in \mathbb{N}.$$

From (2) in Remark 1, we have

$$\mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, \frac{t_0}{2}) \leq 1 - \varepsilon. \tag{3.19}$$

The smallest integer exceeding m_k and satisfying inequality (3.19) will be assumed equal n_k, then we can write

$$\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2})>1-\varepsilon.$$

By inequality (3.16), (2) in Remark 1 and from condition (iii), we obtain

$$\begin{aligned}\eta\big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},t_0)\big)&<\eta\big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{\alpha(\sigma_{m_k-1},\sigma_{n_k},t_0)})\big)\\&\leq\eta\big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\lambda t_0)\big)\\&\leq\eta\big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\lambda t_0)\big).\end{aligned}$$

It remains to show Chauchyness that it can be proved as in Theorem 2. Now, since $(\Sigma,\mathcal{M}_F,*)$ is $\mathcal{M}_F$-complete, then there exists $\sigma\in\Sigma$ such that

$$\lim_{n\to\infty}\mathcal{M}_F(\sigma_n,x,t)=1,\quad t>0. \tag{3.20}$$

By condition (ii) and (3.16), we have

$$0\leq\eta\big(\mathcal{M}_F(Q\sigma_n,Q\sigma,t)\big)\leq\eta\big(\mathcal{M}_F(Q\sigma_n,Q\sigma,\frac{t}{\alpha(Q\sigma_n,Q\sigma,t)})\big)\leq\eta\big(\mathcal{M}_F(\sigma_n,x,\lambda t)\big).$$

By passing to the limit, letting n tend to infinity in the last inequality, using (3.20) and the fact that η is continuous, we conclude

$$\lim_{n\to\infty}\eta\big(\mathcal{M}_F(\sigma_{n+1},Q\sigma,t)\big)=0,\quad t>0.$$

That implies

$$\lim_{n\to\infty}\mathcal{M}_F(\sigma_{n+1},Q\sigma,t)=1,\quad t>0.$$

By the fact that the limit is unique, we get $Q\sigma=\sigma$, that is, σ is a fixed point of Q. □

Definition 11. Let $(\Sigma,\mathcal{M}_F,*)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A mapping $Q\colon\Sigma\to\Sigma$ is said to be fuzzy α-$\mathcal{H}$-weak contractive, if there exist $\eta\in\mathcal{H}$ and $\lambda>1$ such that

$$\eta\big(\mathcal{M}_F(Q\sigma,Q\rho,\frac{t}{\alpha(\sigma,\rho,t)})\big)\leq\eta(\mathcal{N}(\sigma,\rho,\lambda t)),\ \text{for all}\ \sigma,\rho\in\Sigma,\ t>0.$$

where $\mathcal{N}(\sigma,\rho,t)=\max\big\{\eta(\mathcal{M}_F(\sigma,\rho,t)),\eta(\mathcal{M}_F(x,Q\sigma,t)),\eta(\mathcal{M}_F(y,Q\rho,t))\big\}$.

Theorem 5. *Let $Q\colon\Sigma\to\Sigma$ is a mapping as described in Definition 11, such that*

(i) *there is $\sigma_0\in\Sigma$ with $\alpha(\sigma_0,Q\sigma_0,t)\geq1,\quad t\geq0$;*

(ii) *Q is fuzzy α-admissible;*

(iii) *for each subsequence $(\sigma_{n_k})_{k\in\mathbb{N}}\subset(\sigma_n)_{n\in\mathbb{N}}=(Q^n\sigma_0)_{n\in\mathbb{N}}$, we have*

$$\alpha(\sigma_{n_k},\sigma_{n_l},t)\geq1,\quad \text{where } l<k,\ \ l,k\in\mathbb{N},\ \text{and } t>0;$$

(iv) *if $(\sigma_n)_{n\in\mathbb{N}}\subset\Sigma$ with $\alpha(\sigma_n,\sigma_{n+1},t)\geq1$, and $\lim_{n\to\infty}\sigma_n=\sigma$, then*

$$\alpha(\sigma_n,\sigma_{n+1},t)\geq1,\ t>0,\ n\in\mathbb{N}.$$

Then, Q possesses a unique fixed point.

Proof. By (i), there is $\sigma_0 \in \Sigma$ with $\alpha(\sigma_0, Q\sigma_0, t) \geq 1$, $t \geq 0$. We define a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in Σ by $\sigma_n = Q\sigma_{n-1}$, $\forall n \in \mathbb{N}$. Since Q is fuzzy α-admissible, then

$$\alpha(\sigma_0,\sigma_1,t) = \alpha(\sigma_0, Q\sigma_0, t) \geq 1 \text{ implies } \alpha(Q\sigma_0, Q\sigma_1, t) = \alpha(\sigma_1,\sigma_2,t) \geq 1,\ t > 0.$$

By induction, we get

$$\alpha(\sigma_n,\sigma_{n+1},t) \geq 1, \quad t > 0,\ n \in \mathbb{N}. \tag{3.21}$$

Since Q is a fuzzy α-$\mathcal{H}$-weak contractive and thanks to (3.21), we get

$$\begin{aligned}\eta\big(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)\big) &= \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, t)\big)\\ &\leq \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, \frac{t}{\alpha(\sigma_{n-1},\sigma_n,t)})\big)\\ &\leq \eta\big(\mathcal{N}(\sigma_{n-1},\sigma_n,\lambda t)\big),\end{aligned}$$

where $\mathcal{N}(\sigma,\rho,t) = \max\big\{\eta(\mathcal{M}_F(\sigma,\rho,t)), \eta(\mathcal{M}_F(x, Q\sigma, t)), \eta(\mathcal{M}_F(y, Q\rho, t))\big\}$. □

We can show the Chauchyness of the sequence as in Theorem 3.

Let $\Phi = \Big\{\psi\colon [0,1] \to [0,1] /\ (\Phi_1), (\Phi_2) \text{ hold } \Big\}$, where

(Φ_1) ψ is nondecreasing;
(Φ_2) $\psi(t) > t$, for all $t \in (0,1)$.

Definition 12. Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A mapping $Q\colon \Sigma \to \Sigma$ is called fuzzy α-$\mathcal{H}$-Φ-contractive, if there are $\eta \in \mathcal{H}$, $\psi \in \Phi$ and $\lambda > 1$ such that

$$\eta\big(\mathcal{M}_F(Q\sigma, Q\rho, \frac{t}{\alpha(\sigma,\rho,t)})\big) \leq \eta\big(\psi(\mathcal{M}_F(\sigma,\rho,\lambda t))\big), \text{ for all } \sigma,\rho \in \Sigma, t > 0.$$

Theorem 6. *Let $Q\colon \Sigma \to \Sigma$ be a fuzzy α-$\mathcal{H}$-Φ-contraction mapping over the space $(\Sigma, \mathcal{M}_F, *)$, such that*

(i) *there is $\sigma_0 \in \Sigma$ with $\alpha(\sigma_0, Q\sigma_0, t) \geq 1$, $t \geq 0$;*
(ii) *Q is fuzzy α-admissible;*
(iii) *for each subsequence $(\sigma_{n_k})_{k\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}} = (Q^n\sigma_0)_{n\in\mathbb{N}}$, we have*

$$\alpha(\sigma_{n_k},\sigma_{n_l},t) \geq 1, \quad \text{where } l < k,\ l,k \in \mathbb{N}, \text{ and } t > 0;$$

(iv) *if $(\sigma_n)_{n\in\mathbb{N}} \subset \Sigma$ with $\alpha(\sigma_n,\sigma_{n+1},t) \geq 1$, and $\lim_{n\to\infty} \sigma_n = \sigma$, then*

$$\alpha(\sigma_n,\sigma_{n+1},t) \geq 1, \text{ for all } t > 0, n \in \mathbb{N}.$$

Then, Q possesses a unique fixed point.

Proof. From condition (i), there is $\sigma_0 \in X$ with $\alpha(\sigma_0, Q\sigma_0, t) \geq 1$, $t \geq 0$. Define the sequence $(\sigma_n)_{n\in\mathbb{N}}$ in Σ by $\sigma_n = Q\sigma_{n-1}$, for all $n \in \mathbb{N}$. If there exist $n_0 \in \mathbb{N}$ such

that $\sigma_{n_0} = \sigma_{n_0+1}$, then $v = \sigma_0$ is a fixed point of Q. So, we can suppose that $\sigma_n \neq \sigma_{n+1}$ for every $n \in \mathbb{N}$. Since Q is fuzzy α-admissible, then the following implication holds:

$$\alpha(\sigma_0,\sigma_1,t) = \alpha(\sigma_0, Q\sigma_0,t) \geq 1, \text{ implies } \alpha(Q\sigma_0, Q\sigma_1,t) = \alpha(\sigma_1,\sigma_2,t) \geq 1,\ t > 0.$$

Inductively, we get

$$\alpha(\sigma_n,\sigma_{n+1},t) \geq 1, \quad t > 0,\ n \in \mathbb{N}. \tag{3.22}$$

Since Q is a fuzzy α-$\mathcal{H}$-Φ-contractive mapping and by (3.22), we get

$$\begin{aligned}
\eta\big(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)\big) &= \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t)\big) \\
&\leq \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, \frac{t}{\alpha(\sigma_{n-1},\sigma_n,t)})\big) \\
&\leq \eta\big(\psi\big(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)\big)\big) \\
&\leq \eta\big(\psi\big(\mathcal{M}_F(\sigma_{n-1},\sigma_n,t)\big)\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{n-1},\sigma_n,t)\big).
\end{aligned} \tag{3.23}$$

Then,

$$\mathcal{M}_F(\sigma_n,\sigma_{n+1},t) \geq \psi\big(\mathcal{M}_F(\sigma_{n-1},\sigma_n,t)\big) \text{ for all } t > 0. \tag{3.24}$$

And thus,

$$\mathcal{M}_F(\sigma_n,\sigma_{n+1},t) \geq \mathcal{M}_F(\sigma_{n-1},\sigma_n,t). \tag{3.25}$$

Then, the sequence $\{\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)\}$ is nondescreasing. So, $\{\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)\}$ is convergent and denote

$$\lim_{n\to+\infty} \mathcal{M}_F(\sigma_n,\sigma_{n+1},t) = r \in (0,1].$$

By formula (3.24) and the continuity of ψ, it follows

$$r \geq \psi(r).$$

This implies $r = 1$, and thus

$$\lim_{n\to+\infty} \mathcal{M}_F(\sigma_n,\sigma_{n+1},t) = 1, \quad \text{for all } t > 0. \tag{3.26}$$

If $(\sigma_n)_{n\in\mathbb{N}}$ is not $\mathcal{M}_F$-Cauchy, there exist $t_0 > 0$ and $\varepsilon \in (0,1)$ such that for all $k \in \mathbb{N}$ there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ and

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_n},t_0) \leq 1-\varepsilon. \tag{3.27}$$

If n_k will be assumed to be the least integer exceeding m_k satisfying (3.27), then

$$\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_n},t_0) > 1-\varepsilon.$$

From (3.25), (3.27) and (C4), the following estimations hold:

$$\begin{aligned}
1-\varepsilon &\geq \mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) \\
&\geq \mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)
\end{aligned}$$

$$\geq \mathcal{M}_F(\sigma_{m_k-1}, \sigma_{n_k}, \frac{t_0}{2}) * \mathcal{M}_F(\sigma_{n_k}, \sigma_{n_k-1}, \frac{t_0}{2})$$
$$\geq (1-\varepsilon) * \mathcal{M}_F(\sigma_{n_k}, \sigma_{n_k-1}, \frac{t_0}{2}),$$

which by passing to the limit as k tends to $+\infty$ and using (3.26) with (C3) yields

$$\lim_{k\to\infty} \mathcal{M}_F(\sigma_{m_k-1}, \sigma_{n_k-1}, t_0) = 1-\varepsilon = \lim_{k\to\infty} \mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, t_0).$$

On the other hand, since Q is a fuzzy α-$\mathcal{H}$-Φ-contractive mapping and thanks to (3.22) we have

$$\begin{aligned}
\eta\big(\mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, t)\big) &= \eta\big(\mathcal{M}_F(Q\sigma_{m_k-1}, Q\sigma_{n_k-1}, t)\big) \\
&\leq \eta\Big(\mathcal{M}_F(Q\sigma_{m_k-1}, Q\sigma_{n_k-1}, \frac{t}{\alpha(Q\sigma_{m_k-1}, Q\sigma_{n_k-1}, t)})\Big) \\
&\leq \eta\big(\psi\big(\mathcal{M}_F(\sigma_{m_k-1}, \sigma_{n_k-1}, \lambda t)\big)\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{m_k-1}, \sigma_{n_k-1}, \lambda t)\big) \\
&\quad \vdots \\
&\leq \eta\big(\mathcal{M}_F(\sigma_1, \sigma_0, \lambda^{n_k} t)\big).
\end{aligned}$$

Since η is continuous, we get

$$\eta(\lim_{k\to\infty} \mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, t_0)) \leq \eta(\lim_{k\to\infty} \mathcal{M}_F(\sigma_0, \sigma_1, \lambda^{n_k} t_0)).$$

This implies $\eta(1-\varepsilon) \leq \eta(1)$, which gives an absurdity, and thus $(\sigma_n)_{n\in\mathbb{N}}$ is an $\mathcal{M}_F$-Cauchy in Σ. Now, since $(\Sigma, \mathcal{M}_F, *)$ is a $\mathcal{M}_F$-complete, then there is $\sigma \in \Sigma$ such that

$$\lim_{n\to\infty} \mathcal{M}_F(\sigma_n, \sigma, t) = 1, \quad t > 0.$$

By using (3.23), we can get

$$\lim_{n\to\infty} \mathcal{M}_F(\sigma_{n+1}, Q\sigma, t) = 1, \quad t > 0.$$

Since the limit is unique, we conclude that $Q\sigma = \sigma$, that is, σ is a fixed point of Q. To show the uniqueness of σ, let σ_1, σ_2 be two different fixed points of Q. Then using (3.7), we have

$$\begin{aligned}
\eta(\mathcal{M}_F(\sigma_1, \sigma_2, t)) &= \eta(\mathcal{M}_F(Q\sigma_1, Q\sigma_2, t)) \\
&\leq \eta\Big(\mathcal{M}_F(Q\sigma_1, Q\sigma_2, \frac{t}{\alpha(\sigma_1, \sigma_0, t)})\Big) \\
&\leq \eta\big(\psi\big(\mathcal{M}_F(\sigma_1, \sigma_0, \lambda t)\big)\big) \\
&\leq \eta(\mathcal{M}_F(\sigma_1, \sigma_2, \lambda t)) \\
&\leq \eta(\mathcal{M}_F(\sigma_1, \sigma_2, t)). \qquad (3.28)
\end{aligned}$$

If $\mathcal{M}_F(\sigma_1,\sigma_2,t) < 1$, then $\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t)) > 0$ and hence (3.28) we have

$$\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t)) \leq \eta(\mathcal{M}_F(\sigma_1,\sigma_2,\lambda t)) < \eta(\mathcal{M}_F(\sigma_1,\sigma_2,t)),$$

which is a contradiction. Therefore, $\mathcal{M}_F(\sigma_1,\sigma_2,t) = 1$ yielding $\sigma_1 = \sigma_2$, which completes the proof. □

ACKNOWLEDGEMENTS

The authors would like to express their sincere appreciation to the referees for their very helpful suggestions and many comments.

REFERENCES

[1] M. Abbas, M. Imdad, and D. Gopal, "ψ-weak contractions in fuzzy metric spaces," *Iranian Journal of Fuzzy Systems*, vol. 8, no. 5, pp. 141–148, 2011, doi: 10.22111/IJFS.2011.303.

[2] M. I. Abbas and M. A. Ragusa, "Nonlinear fractional differential inclusions with non-singular Mittag-Leffler kernel," *AIMS Mahtematics*, vol. 7, no. 11, pp. 20 328–20 340, 2022, doi: 10.3934/math.20221113.

[3] I. Beg, D. Gopal, T. Dosenovic, and D. Rakic, "α-type fuzzy *h*-contractive mappings in fuzzy metric spaces," *Fixed Point Theory* , vol. 19, no. 2, pp. 463–474, 2018, doi: 10.24193/fpt-ro.2018.2.37.

[4] K. C. Border, "Fixed point theorems with applications to economics and game theory," *Cambridge University Press*, 1985.

[5] S. Etemad, M. M. Matar, and M. A. Ragusa, "Tripled fixed points and existence study to a tripled impulsive fractional differential system via measures of noncompactness," *Mathematics*, vol. 10, no. 1, p. 25, 2022, doi: 10.3390/math10010025.

[6] C. Feblin, "Finite dimensional fuzzy normed linear space," *Fuzzy Sets and System*, vol. 48, no. 2, pp. 239–248, 1992, doi: 10.1016/0165-0114(92)90338-5.

[7] A. George and P. V. Veeramani, "On some results of fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395–399, 1994, doi: 10.1016/0165-0114(94)90162-7.

[8] D. Gobal and C. Vecto, "Some new fixed point theorems in fuzzy metric spaces," *Iranian Journal of Fuzzy Systems*, vol. 11, no. 3, pp. 95–107, 2014.

[9] M. Grabiec, "Fixed points in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 27, no. 3, pp. 385–389, 1988, doi: 10.1016/0165-0114(88)90064-4.

[10] V. Gregori and S. Romaguera, "Some properties of fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 115, no. 3, pp. 485–489, 2000, doi: 10.1016/S0165-0114(98)00281-4.

[11] E. Guariglia and R. C. Guido, "Chebyshev wavelet analysis," *Journal of Function Spaces*, vol. 2022, p. 5542054, 2022, doi: 10.1155/2022/5542054.

[12] E. Guariglia and S. Silvestrov, "Fractional-wavelet analysis of positive definite distributions and wavelets on $D'(C)$," in *Engineering Mathematics II*, S. Silvestrov and M. Rančić, Eds., doi: 10.1007/978-3-319-42105-6_16. Cham: Springer International Publishing, 2016, pp. 337–353.

[13] O. Hadzic and E. Pap, "A fixed point theorem for multivalued mapping in propbalitistic metric spaces and an application in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 127, no. 3, pp. 333–344, 2002, doi: 10.1016/S0165-0114(01)00144-0.

[14] O. Kaleva and S. Seikkala, "On fuzzy metric space," *Fuzzy Sets and Systems*, vol. 12, no. 3, pp. 215–229, 1984, doi: 10.1016/0165-0114(84)90069-1.

[15] F. Khojasteh, S. Shukla, and S. Radenović, "A new approach to the study of fixed point theory for simulation functions," *Filomat*, vol. 29, no. 6, pp. 1189–1194, 2015, doi: 10.2298/FIL1506189K.

[16] A. Latif, M. E. Gordji, and E. Karapinar, "Fixed point results for generalized (α, ψ)-Meir-Keeler contractive mappings and applications," *Journal of Inequalities and Applications*, vol. 2014, pp. 1–11, 2014, doi: 10.1186/1029-242X-2014-68.
[17] G. S. Mathews, "Partial metric topology. Reseach Report 212," *Dept. of Computer Science University of Warwick* , 1992.
[18] D. Mihet, "Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces," *Fuzzy Sets and Systems* , vol. 159, no. 6, pp. 739–744, 2008, doi: 10.1016/j.fss.2007.07.006.
[19] E. A. Ok, *Real Analysis with Economic Applications*. Princeton University Press, 2007, vol. 10.
[20] A. Perveen and M. Imdad, "Proving new fixed point results in fuzzy metric spaces employing simulation function," *Journal of Intelligent and Fuzzy Systems*, vol. 36, no. 6, pp. 6493–6501, 2019, doi: 10.3233/JIFS-182873.
[21] P. D. Proinov, "A generalization of the banach contraction principle with high order of convergence of successive approximations," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 67, no. 8, pp. 2361–2369, 2007, doi: 10.1016/j.na.2006.09.008.
[22] S. Romaguera and O. Valero, "A quantitative computational model for complete partial metric spaces via formal balls," *Mathematical Structures in Computer Science*, vol. 19, no. 3, pp. 541–563, 2009, doi: 10.1017/S0960129509007671.
[23] H. N. Saleh, I. A. Khan, M. Imdad, and W. M. Alfagih, "New fuzzy φ-fixed point results employing a new class of fuzzy contractive mappings," *Journal of Intelligent and Fuzzy Systems*, vol. 37, no. 4, pp. 5391–5402, 2019, doi: 10.3233/JIFS-190543.
[24] B. Samet, C. Vetro, and P. A. Vetrol, "Fixed point theorems for α-ψ contractive type mappings." *Nonlinear Analysis, Theory, Methods and Applications* , vol. 75, no. 4, pp. 2154–2165, 2012, doi: 10.1016/j.na.2011.10.014.
[25] B. Schweizar and A. Shklare, "Statistical metric spaces," *Pacific Journal of Mathematics*, vol. 10, no. 1, pp. 313–334, 1960.
[26] S. Sedghi, N. Shobkolaei, and I. Altun, "A new approach to Caristi's fixed point theorem on non-Archimedean fuzzy metric spaces," *Iranian Journal of Fuzzy Systems* , vol. 12, no. 2, pp. 137–143, 2015, doi: 10.22111/IJFS.2015.1989.
[27] S. Shukla, "Fuzzy H-weak contractions and fixed point theorems in fuzzy metric spaces," *Gulf Journal of Mathematics* , vol. 2, no. 1, pp. 67–75, 2013, doi: 10.56947/gjom.v1i2.220.
[28] A. Taqbibt, M. Elomari, and S. Melliani, "Nonlocal semilinear Φ-Caputo fractional evolution equation with a measure of noncompactness in Banach space," *Filomat*, vol. 37, no. 20, pp. 6877–6890, 2023, doi: 10.2298/FIL2320877T.
[29] D. Wardowsk, "Fuzzy contractive mappings and fixed points in fuzzy metric spaces," *Fuzzy Sets and Systems* , vol. 222, pp. 108–114, 2013, doi: 10.1016/j.fss.2013.01.012.

Authors' addresses

Abdellah Taqbibt
(**Corresponding author**) Sultan Moulay Slimane University, Laboratory of Applied Mathematics and Scientific Calculus, FST, PO Box 523, Beni Mellal, Morocco
E-mail address: abdellah.taqbibt@usms.ma

Mohamed Chaib
Sultan Moulay Slimane University, Laboratory of Applied Mathematics and Scientific Calculus, FST, PO Box 523, Beni Mellal, Morocco
E-mail address: mohamed.chaib@usms.ma

M'hamed Elomari
Sultan Moulay Slimane University, Laboratory of Applied Mathematics and Scientific Calculus, FST, PO Box 523, Beni Mellal, Morocco
E-mail address: m.elomari@usms.ma

Said Melliani
Sultan Moulay Slimane University, Laboratory of Applied Mathematics and Scientific Calculus, FST, PO Box 523, Beni Mellal, Morocco
E-mail address: said.melliani@gmail.com