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MILD SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS IN COLOMBEAU ALGEBRA

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Abstract. The objective of this study is to investigate the unique existence of a generalized solution to an abstract Cauchy problem where the initial data is singular and a generalized real number. Our main theorems were established using the Laplace transform for Caputo fractional derivatives. In addition, we demonstrate the applicability of these results with an illustrative example.

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1. INTRODUCTION

The theory of fractional differential equations has become an interesting area of exploration in recent years, the reader can see, for example [4–7, 19], and the references therein. Currently, in the mathematical literature, several authors study the existence and uniqueness of solutions to nonlinear fractional problems, which are more significant than those involving classical derivatives, [3, 8–11, 14]. The algebra of generalized functions $\mathcal{G}$ was introduced by Colombeau in the early 1980s to address the issue of multiplication distributions, as explained in references [1, 2]. This algebra serves as a differential over the inclusive Schwartz distributions space $\mathcal{D}^{'}$, and it allows for more general non-linear operations than just multiplication. As a result, $\mathcal{G}$ is a valuable tool for discovering and examining solutions to nonlinear differential equations that involve singular data and coefficients. It is instrumental in determining the multiplication of distributions, as described in references [12] and [18]. Furthermore, this algebra is an extension of distribution theory that deals with nonlinearities and singularities in PDE theory [18]. The algebra contains the space of distributions $\mathcal{D}'$ as a subspace, with an embedding achieved through convolution with an appropriate mollifier. The elements of $\mathcal{G}$ are classes of smooth functions, known as moderate functions, in relation to a set of negligible functions. This regularity is necessary for solving nonlinear problems that involve derivatives of arbitrary real order and singularities. In this paper, we investigate the existence and uniqueness of the solution to the Cauchy problem given by

$$\begin{cases} D^{\gamma}\nu(t)+\mathcal{A}\nu(t)=\Phi(t,\nu(t)), & 0<\gamma<1, \\ \nu(0)=\nu_0, \end{cases} \tag{1.1}$$

in the setting of Colombeau algebra of generalized functions, where $\nu_0 \in (\tilde{\mathbb{R}})^n$ is a generalized real number, $-\mathcal{A}$ be the infinitesimal generator of an analytic generalized semigroup $(Q(t))_{t\geq 0}$ of uniformly bounded linear operators on a a class of Colombeau algebras $\mathcal{G}(\mathbb{R}^n)$. In addition, regarding works on the Colombeau semigroup, we refer to [16, 17] and the references therein. Our idea is inspired by the one presented in [21], where the author proved the existence of the Cauchy problem (1.1) under two hypotheses concerning the infinitesimal generator $-\mathcal{A}$ of an analytic semigroup, also we have shown without making assumptions on the generator $-\mathcal{A}$ that the problem (1.1) admits a unique solution in the extension $\mathcal{G}^e(\mathbb{R}^n)$ of Colombeau's algebra.

The organization of the paper is as follows. In Section 2, we recall some fundamental properties of the generalized functions theory. The new notion of generalized semigroup and the proof of the existence and uniqueness of generalized solution in Colombeau algebra to the problem given in (1.1) take place in Section 3. In Section 4 we have introduced an example to illustrate our results.

2. Preliminaries

This section provides a review of the essential characteristics of generalized functions theory in the Colombeau framework. The regularization approaches based on the Colombeau method involve the approximation of nonsmooth things using nets of smooth functions with moderate asymptotic bounds. The objective is to identify regularizing nets whose differences, relative to the moderateness scale, can be neglect. These regularizations are grouped into equivalence classes based on their similarity to negligible nets. Elements of Colombeau generalized functions refer to these equivalence classes, which consist of sequences of smooth functions that meet the asymptotic conditions for the regularization parameter epsilon. Let $n \in \mathbb{N}^*$, as in [6], we define the set

$$\mathcal{E}(\mathbb{R}^n)=\left(\mathcal{C}^{\infty}(\mathbb{R}^n)\right)^{(0,1)}.$$

The set of moderate functions is given as follows

$$\mathcal{E}_M(\mathbb{R}^n)=\Big\{(\nu_\varepsilon)_{\varepsilon>0}\subset\mathcal{E}(\mathbb{R}^n): \\ \forall K\subset\subset\mathbb{R}^n, \forall\gamma\in\mathbb{N}_0^n, \exists N\in\mathbb{N}, \sup_{x\in K}|\partial^{\gamma}\nu_\varepsilon(x)|=O_{\varepsilon\to 0}(\varepsilon^{-N})\Big\}.$$

The ideal of negligible functions is defined by

$$\mathcal{N}(\mathbb{R}^n) = \Big\{(\nu_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n) : \forall K \subset\subset \mathbb{R}^n, \forall \gamma \in \mathbb{N}_0^n, \forall p \in \mathbb{N}, \sup_{x\in K} |\partial^\gamma \nu_\varepsilon(x)| = O_{\varepsilon\to 0}(\varepsilon^p)\Big\},$$

with operations defined component wise, e.g., $(\nu_\varepsilon)+(\mu_\varepsilon) = (\nu_\varepsilon + \mu_\varepsilon)$. The Colombeau algebra is defined as a factor set

$$\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n).$$

The ring of all generalized real numbers is given by the following set

$$\widetilde{\mathbb{R}} = \mathcal{E}(\mathbb{R})\,/\,I(\mathbb{R}),$$

where

$$\mathcal{E}(\mathbb{R}) = \Big\{(\nu_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1)} : \exists m \in \mathbb{N},\ |\nu_\varepsilon| = O_{\varepsilon\to 0}(\varepsilon^{-m})\Big\}$$

and

$$I(\mathbb{R}) = \Big\{(\nu_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1)} : \forall m \in \mathbb{N},\ |\nu_\varepsilon| = O_{\varepsilon\to 0}(\varepsilon^{m})\Big\}.$$

We note that $\widetilde{\mathbb{R}}$ is a ring obtained by factoring moderate families of real numbers with respect to negligible families. The space $\mathcal{E}(\mathbb{R})$ is an algebra, and $I(\mathbb{R})$ is an ideal of $\mathcal{E}(\mathbb{R})$. The extended Colombeau algebras of generalized functions $\mathcal{G}^e(\Omega)$ on open subset Ω of $\mathbb{R}^n$ is defined in the sense of the extension of the entire derivatives to be a fractional ones, which introduced for the first time by Stojanovic (see [20], for more detail). Let $\mathcal{E}(\Omega)$ be the algebra of all nets $(\nu_\varepsilon)_{\varepsilon>0}$ of real valued functions $\nu_\varepsilon \in \mathcal{C}^\infty(\Omega)$, the algebra of extended moderate functions is given by

$$\mathcal{E}_M^e(\Omega) = \Big\{(\nu_\varepsilon)_{\varepsilon>0} \in (\mathcal{E}(\Omega))^{(0,1)} : \forall K \subset\subset \mathbb{R}, \forall \gamma \in R_+ \cup \{0\}, \exists N \in \mathbb{N}, \sup_{x\in K} |D^\gamma \nu_\varepsilon(x)| = O_{\varepsilon\to 0}(\varepsilon^{-N})\Big\}$$

and the set of negligible functions is defined this time by

$$\mathcal{N}^e(\Omega) = \Big\{(\nu_\varepsilon)_{\varepsilon>0} \in (\mathcal{E}(\Omega))^{(0,1)} : \forall K \subset\subset \mathbb{R}, \forall \gamma \in R_+ \cup \{0\}, \forall q \in \mathbb{N}, \sup_{x\in K} |D^\gamma \nu_\varepsilon(x)| = O_{\varepsilon\to 0}(\varepsilon^{q})\Big\}.$$

$\mathcal{G}^e(\Omega)$ is given by the factor algebras $\mathcal{G}^e(\Omega) = \mathcal{E}_M^e(\Omega)/\mathcal{N}^e(\Omega)$. Here, D^γ is the Caputo fractional derivative, where $m-1 \le \gamma < m$, $m \in \mathbb{N}$, for the fractional derivatives and fractional integral we can see [19] and the references therein. Colombeau algebras for tempering generalized functions, $\mathcal{G}\tau(\mathbb{R}^n)$ which was introduced by Colombeau to develop the Fourier transform theory in the algebra of generalized functions. We first define

$$O_M(\mathbb{R}^n) = \Big\{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \forall \gamma \in \mathbb{N}_0^n,\ \exists N \in \mathbb{N},\ \sup_{x\in\mathbb{R}^n} \{<x>^{-N} |\partial^\gamma f(x)|\} < \infty\Big\},$$

where $< x >= (1+\|x\|)^{2N}$. The Colombeau algebra of tempered generalized functions is given by

$$\mathcal{G}\tau(\mathbb{R}^n) = \mathcal{E}^e_\tau(\mathbb{R}^n)/\mathcal{N}^e_\tau(\mathbb{R}^n)$$

with

$$\mathcal{E}^e_\tau(\mathbb{R}^n) = \Big\{(\nu_\varepsilon)_{\varepsilon>0} \in \big(O_M(\mathbb{R}^n)\big)^{(0,1)} : \\ \forall\gamma \in R_+ \cup \{0\}, \exists N \geq 0, \sup_{x\in\mathbb{R}^n} |D^\gamma \nu_\varepsilon(x)| = O_{\varepsilon\to 0}(\varepsilon^{-N})\Big\}$$

and

$$\mathcal{N}^e_\tau(\mathbb{R}^n) = \Big\{(\nu_\varepsilon)_{\varepsilon>0} \in \big(O_M(\mathbb{R}^n)\big)^{(0,1)} : \\ \forall\gamma \in R_+ \cup \{0\}, \forall q \geq 0, \sup_{x\in\mathbb{R}^n} |D^\gamma \nu_\varepsilon(x)| = O_{\varepsilon\to 0}(\varepsilon^{q})\Big\},$$

where D^γ is the Caputo fractional derivative, $m-1 \leq \gamma < m$, $m \in \mathbb{N}_0$. Embedding of the Shwartz distributions space $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{G}^e_\tau(\mathbb{R}^n)$ is given by $\nu \to [(\nu * \phi_\varepsilon)_{\varepsilon\in(0,1)}]$, such that $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{\nu}{\varepsilon})$, where $\phi \in \mathcal{C}^\infty_0(\mathbb{R})$ and

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon}), \phi(x) \geq 0, \int_{\mathbb{R}} \phi(x)\,\mathrm{d}x = 1, \int_{\mathbb{R}} x^\gamma \phi(x)\,\mathrm{d}x = 0, \forall\gamma \in \mathbb{N}^n, |\gamma| > 0. \quad (2.1)$$

2.1. *Caputo derivative*

The Caputo fractional integral is defined as follows

$$I^\gamma_0 x(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} x(s)\,\mathrm{d}s, \quad (2.2)$$

with $m-1 < \gamma < m$ and $m \in \mathbb{N}^*$. Fractional derivative in the Caputo sense of order γ of a function x is defined by

$$D^\gamma_0 x(t) = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} x^{(m)}(s)\,ds, \quad (2.3)$$

where $m-1 < \gamma < m$ and $m \in \mathbb{N}_0$, provided that this integral convergent which is the case when x belongs to the class of absolutely continuous functions and its of class $\mathcal{C}^m$, see [13].

2.2. *Embedding of the Caputo fractional derivative into Colombeau algebra*

Inspired by Colombeau's classical theory for entire derivatives, to prove the embedding of the distribution in the extended Colombeau algebra, we must show that all derivatives are moderate, including fractional derivatives, i.e. we have to show that

$\tilde{D}^{\gamma}\nu_{\varepsilon}(t) = D^{\gamma}\nu_{\varepsilon} * \varphi_{\varepsilon}(t)$ is moderate. For $\gamma \in (0,1)$, we have

$$\begin{aligned}
\tilde{D}^{\gamma}\nu_{\varepsilon} &= \mathrm{D}^{\gamma}\nu_{\varepsilon} * \varphi_{\varepsilon}(t) \\
&= \frac{1}{\Gamma(1-\gamma)} \left(\int_0^t \frac{\nu_{\varepsilon}'(s)\,\mathrm{d}s}{(t-s)^{\gamma}} \right) * (\varphi_{\varepsilon}(t)) \\
&\le \frac{1}{\Gamma(1-\gamma)} \sup_{t\in[0,T)} \left| \int_0^t \frac{\nu_{\varepsilon}'(s)\,\mathrm{d}s}{(t-s)^{\gamma}} \right| \|(\varphi_{\varepsilon}(t))\|_{L^1} \\
&\le \frac{C}{\Gamma(1-\gamma)} \sup_{t\in[0,T)} \left|\nu_{\varepsilon}'(t)\right| \frac{T^{1-\gamma}}{1-\gamma} \le C_{\gamma,T}\varepsilon^{-N}.
\end{aligned}$$

Then there exists $N > 0$, such that

$$\tilde{D}^{\gamma}\nu_{\varepsilon} \le C_{\gamma,T}\varepsilon^{-N}. \tag{2.4}$$

Hence

$$\left| \left(\tilde{D}^{\gamma}\nu_{\varepsilon}\right)' \right| = \left|\mathrm{D}^{\gamma}\nu_{\varepsilon}(t) * \varphi_{\varepsilon}'(t)\right| \le \frac{C}{\varepsilon} \sup_{t\in[0,T)} |\mathrm{D}^{\gamma}\nu_{\varepsilon}(t)|,$$

where the last expression is given with (2.4). Let $\beta \in (0,1)$. For higher fractional derivatives we use the semigroup property of fractional differentiation: $\mathrm{D}^{\beta}\left(\mathrm{D}^{\gamma}\nu_{\varepsilon}\right) = \mathrm{D}^{\beta+\gamma}\nu_{\varepsilon}$. We have

$$\mathrm{D}^{\beta}\left(\tilde{D}^{\gamma}\nu_{\varepsilon}\right) = \mathrm{D}^{\beta}\left(\mathrm{D}^{\gamma}\nu_{\varepsilon} * \varphi_{\varepsilon}\right) = \mathrm{D}^{\beta+\gamma}\nu_{\varepsilon} * \varphi_{\varepsilon}.$$

Then, there exist $N > 0$ and $t \in [0,T)$, $T > 0$ such that

$$\begin{aligned}
\left(\mathrm{D}^{\beta+\gamma}\nu_{\varepsilon}\right) * \varphi_{\varepsilon}(t) &= \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\nu_{\varepsilon}'(s)}{(t-s)^{\gamma+\beta}}\,\mathrm{d}s \\
&\le \frac{1}{\Gamma(1-\gamma)} \sup_{t\in[0,T)} \left|\nu_{\varepsilon}'(t)\right| \left| \frac{t^{1-(\gamma+\beta)}}{1-(\gamma+\beta)} * \varphi_{\varepsilon}(t) \right| \\
&\le \frac{1}{\Gamma(1-\gamma)} \sup_{t\in[0,T)} \left|\nu_{\varepsilon}'(t)\right| \frac{C}{\varepsilon} \frac{T^{2-(\gamma+\beta)}}{2-(\gamma+\beta)} \le C_{T,\gamma,\beta}\varepsilon^{-N}.
\end{aligned}$$

3. Main results

Let us consider the Cauchy problem in the framework of Colombeau algebras

$$\begin{cases} D^{\gamma}\nu(t) + \mathcal{A}\nu(t) = \Phi(t,\nu(t)), \\ \nu(0) = \nu_0 \in \tilde{\mathbb{R}}, \end{cases} \tag{3.1}$$

where $\nu \in (\mathcal{G}^e(\mathbb{R}))^n$, $-\mathcal{A}$ is an infinitesemal generator of a generalized Colombeau semigroup $(Q(t))_{t\ge 0} = [((Q_{\varepsilon}(t))_{t\ge 0})_{\varepsilon}]$, $\Phi \in (\mathcal{G}^e(\mathbb{R}))^{n+1}$.

Definition 1. Consider the one-sided stable probability density given in [15] by

$$\psi_\gamma(\xi)=\frac{1}{\pi}\sum(-1)^{n-1}\xi^{-\gamma n-1}\frac{\Gamma((n+1)\gamma+1)}{\Gamma(n+1)}\sin(n\pi\gamma),\ \ \xi\in(0,\infty) \tag{3.2}$$

and the probability density

$$\phi_\gamma(\xi)=\frac{1}{\gamma}\xi^{1-\frac{1}{\gamma}}\psi_\gamma(\xi^{-\frac{1}{\gamma}}).$$

Theorem 1. *Assume that* $\Phi\in(\mathcal{G}^e_\tau(\mathbb{R}))^{n+1}$*, and* $|\nabla_x\Phi|\leq C|\ln(\varepsilon)|$, $\varepsilon\in(0,1)$. *Then the Cauchy problem* (3.1) *has a unique solution in the extended Colombeau algebra* $(\mathcal{G}^e(\mathbb{R}))^n$.

Let us consider the following problem

$$\begin{cases}D^\gamma \nu_\varepsilon(t)+\mathcal{A}_\varepsilon\nu_\varepsilon(t)=\Phi_\varepsilon(t,\nu_\varepsilon(t)),\\ \nu_\varepsilon(0)=\nu_{0\varepsilon}\in\mathcal{E}(\mathbb{R}).\end{cases} \tag{3.3}$$

According to Definitions 2.2 and 2.3, we write the Cauchy problem in the integral equation

$$\begin{cases}\nu_\varepsilon(t)=\nu_{0\varepsilon}+\frac{1}{\Gamma(\gamma)}\displaystyle\int_0^t(t-s)^{\gamma-1}[-\mathcal{A}_\varepsilon\nu_\varepsilon(s)+\Phi_\varepsilon(s,\nu_\varepsilon(s))]\,\mathrm{d}s,\\ \nu_\varepsilon(0)=\nu_{0\varepsilon}.\end{cases} \tag{3.4}$$

The proof of the theorem requires the two lemmas below.

Lemma 1. *If* (3.4) *holds, then there is a probability density function* ϕ_γ *defined on* $(0,+\infty)$ *such that*

$$\begin{cases}\nu_\varepsilon(t)=\displaystyle\int_0^\infty\phi_\gamma Q_\varepsilon(t^\gamma\xi)\nu_{0\varepsilon}\,\mathrm{d}\xi\\ \qquad\quad+\gamma\displaystyle\int_0^t\int_0^\infty\xi(t-s)^{\gamma-1}\phi_\gamma(\xi)Q_\varepsilon((t-s)^\gamma\xi)\Phi_\varepsilon(s,\nu_\varepsilon(s))\,\mathrm{d}\xi\,\mathrm{d}s,\\ \nu_\varepsilon(0)=\nu_{0\varepsilon}.\end{cases}$$

Proof. Applying the Laplace transform to the first equation in (3.4), we can get

$$\mathcal{L}\nu_\varepsilon(\lambda)=\frac{1}{\lambda}\nu_{0\varepsilon}+\frac{1}{\lambda^\gamma}A_\varepsilon\mathcal{L}(\nu_\varepsilon(\lambda))+\frac{1}{\lambda^\gamma}\mathcal{L}(\Phi_\varepsilon(\lambda,\nu_\varepsilon(\lambda)).$$

On the other hand, we have

$$\begin{aligned}\lambda^\gamma R(\lambda^\gamma,-A_\varepsilon)\mathcal{L}\nu_\varepsilon(\lambda)&=\lambda^{\gamma-1}(\lambda^{\gamma-1}I+A_\varepsilon)^{-1}\nu_{0\varepsilon}+(\lambda^{\gamma-1}I+A_\varepsilon)^{-1}\\&=\lambda^{\gamma-1}(\lambda^\gamma I+A_\varepsilon)^{-1}\nu_{0\varepsilon}\\&\quad+(\lambda^\gamma I+A_\varepsilon)^{-1}\mathcal{L}(e^{-\lambda s}\Phi_\varepsilon(s,\nu_\varepsilon(s)))(\lambda)\\&=\lambda^{\gamma-1}\int_0^\infty e^{-\lambda^\gamma s}Q_\varepsilon(s)\nu_{0\varepsilon}\,\mathrm{d}s+\int_0^\infty e^{-\lambda^\gamma s}Q_\varepsilon(s)\omega(\lambda)\,\mathrm{d}s,\end{aligned}$$

where I is the identity operator, and $\omega(\lambda)$ is the Laplace transform of $\Phi_\varepsilon(s,\nu_\varepsilon(s))$. The probability density given in (3.2) has the Laplace transform given by

$$\mathcal{L}\psi_\gamma(\lambda)=\int_0^\infty e^{-\lambda\xi}\psi_\gamma(\xi)\,\mathrm{d}\xi=e^{-\lambda^\gamma},\ \text{with } \gamma\in(0,1).$$

Then,

$$\begin{aligned}
\lambda^{\gamma-1}(\lambda^{\gamma-1}I+A_\varepsilon)^{-1}\nu_{0\varepsilon}&=\lambda^{\gamma-1}\int_0^\infty e^{-\lambda^\gamma s}Q_\varepsilon(s)\nu_{0\varepsilon}\,\mathrm{d}s\\
&=\int_0^\infty \gamma(\lambda t)^{\gamma-1}e^{-(\lambda t)^\gamma}Q_\varepsilon(t^\gamma)\nu_{0\varepsilon}\,\mathrm{d}s\\
&=\int_0^\infty \frac{-1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-(\lambda t)^\gamma}\right]Q_\varepsilon(t^\gamma)\nu_{0\varepsilon}\,\mathrm{d}s\\
&=\int_0^\infty\left[\int_0^\infty \xi\psi_\gamma(\xi)e^{-(\lambda t\xi)}Q_\varepsilon(t^\gamma)\nu_{0\varepsilon}\,\mathrm{d}\xi\right]\mathrm{d}t\\
&=\int_0^\infty e^{-\lambda t}\left[\int_0^\infty \psi_\gamma(\xi)Q_\varepsilon\Big(\frac{t^\gamma}{\xi^\gamma}\Big)\nu_{0\varepsilon}\,\mathrm{d}\xi\right]\mathrm{d}t.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
\int_0^\infty e^{-\lambda^\gamma s}Q_\varepsilon(s)\omega(\lambda)\,\mathrm{d}s&=\int_0^\infty\left[\int_0^\infty \gamma t^{\gamma-1}e^{-(\lambda t)^\gamma}Q_\varepsilon(t^\gamma)e^{-\lambda s}\Phi_\varepsilon(s,\nu_\varepsilon(s))\,\mathrm{d}s\right]\mathrm{d}t\\
&=\int_0^\infty\int_0^\infty\int_0^\infty \gamma\psi_\gamma(\xi)e^{-\lambda t\xi}Q_\varepsilon(t^\gamma)e^{-\lambda s}t^{-\gamma-1}\Phi_\varepsilon(s,\nu_\varepsilon(s))\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t\\
&=\int_0^\infty e^{-\lambda t}\left[\gamma\int_0^t\int_0^\infty \psi_\gamma(\xi)Q_\varepsilon\Big(\frac{(t-s)^\gamma}{\xi^\gamma}\Big)\Phi_\varepsilon(s,\nu_\varepsilon(s))\frac{(t-s)^\gamma}{\xi^\gamma}\,\mathrm{d}\xi\,\mathrm{d}s\right]\mathrm{d}t.
\end{aligned}$$

According to the last equalities, we obtain

$$\begin{aligned}
\mathcal{L}\nu_\varepsilon(\lambda)=\int_0^\infty\Bigg[&e^{-\lambda t}\int_0^\infty \phi_\gamma(\xi)Q_\varepsilon(t^\gamma\xi)\nu_{0\varepsilon}\,\mathrm{d}\xi\\
&+\gamma\int_0^t\int_0^\infty \xi(t-s)^{\gamma-1}\phi_\gamma(\xi)Q_\varepsilon((t-s)^\gamma\xi)\Phi_\varepsilon(s,\nu_\varepsilon(s))\,\mathrm{d}\xi\,\mathrm{d}s\Bigg]\mathrm{d}t,
\end{aligned}$$

where $\phi_\gamma(\xi)=\frac{1}{\gamma\xi}\psi_\gamma(\beta(\xi))$ and β is the reciprocal function of the function $x\mapsto x^\gamma$. And the integral solution of (3.3) becomes

$$\begin{aligned}
\nu_\varepsilon(t)=&\int_0^\infty \phi_\gamma(\xi)Q_\varepsilon(t^\gamma\xi)\nu_{0\varepsilon}\,\mathrm{d}\xi\\
&+\gamma\int_0^t\int_0^\infty \xi(t-s)^{\gamma-1}\phi_\gamma(\xi)Q_\varepsilon((t-s)^\gamma\xi)\Phi_\varepsilon(s,\nu_\varepsilon(s))\,\mathrm{d}\xi\,\mathrm{d}s.
\end{aligned}$$

Now, we define a representative semigroup $(Q_\varepsilon^\gamma)_{t\in\mathbb{R}_+}$ by

$$Q_\varepsilon^\gamma(t)\nu_\varepsilon = \gamma\int_0^\infty \xi\phi_\gamma(\xi)Q_\varepsilon(t^\gamma\xi)\nu_\varepsilon\,\mathrm{d}\xi. \tag{3.5}$$

Finally the integral solution of the Cauchy (3.1) becomes

$$\nu_\varepsilon(t) = S_\varepsilon^\gamma(t)\nu_{0\varepsilon} + \int_0^t (t-s)^{\gamma-1}Q_\varepsilon^\gamma(t-s)\Phi_\varepsilon(s,\nu_\varepsilon(s))\,\mathrm{d}s, \tag{3.6}$$

where $(S_\varepsilon^\gamma(t))_{t\in\mathbb{R}_+}$ by

$$S_\varepsilon^\gamma(t)\nu_\varepsilon = \int_0^\infty \phi_\gamma(\xi)Q_\varepsilon(t^\gamma\xi)\nu_\varepsilon\,\mathrm{d}\xi. \tag{3.7}$$

□

Lemma 2. *For any fixed $t\in[0,T]$, $T>0$, $(Q_\varepsilon^\gamma(t))_{t\geq 0}$, $(Q_\varepsilon^\gamma(t))_{t\geq 0}$ are linear and bounded operators for every $\varepsilon\in(0,1)$.*

Proof. We will give the proof for (Q_ε^γ) because that of (S_ε^γ) is similar. For fixed $t>0$, $Q_\varepsilon^\gamma(t)$ is linear operator since $Q_\varepsilon(t)$ is a linear operator. Let $\eta\in[0,1]$ we have the following property (see [15])

$$\int_0^\infty \frac{1}{\xi^\eta}\psi_\gamma(\xi)\,\mathrm{d}\xi = \frac{\Gamma(1+\eta/\gamma)}{\Gamma(1+\eta)},$$

then we have,

$$\int_0^\infty \xi^\eta\phi_\gamma(\xi)\,\mathrm{d}\xi = \int_0^\infty \frac{1}{\xi^{\gamma\eta}}\psi_\gamma(\xi)\,\mathrm{d}\xi = \frac{\Gamma(1+\eta)}{\Gamma(1+\gamma\eta)}.$$

Since $Q_\varepsilon(t)$ has a moderate bounds, then there exists a positive real number a such that

$$\sup_{t\in[0,T]}\|Q_\varepsilon(t)\| = O(\varepsilon^{-a}) \text{ as } \varepsilon\to 0, \tag{3.8}$$

then there exists $c>0$ such that for all $t\in[0,T]$, $\nu_\varepsilon\in\mathcal{E}_M^e(\mathbb{R})$, we have $\|Q_\varepsilon(t)\nu_\varepsilon\|\leq c\varepsilon^{-a}$, then

$$\begin{aligned}\left|Q_\varepsilon^\gamma(t)\nu_\varepsilon\right| &= \left|\gamma\int_0^\infty \xi\phi_\gamma(\xi)Q_\varepsilon(t^\gamma\xi)\nu_\varepsilon\mathrm{d}\xi\right| \\ &\leq \sup_{t\in[0,T]}\left|Q_\varepsilon(t^\gamma\xi)\nu_\varepsilon\right|\times\gamma\frac{1}{\Gamma(1+\gamma)},\end{aligned} \tag{3.9}$$

since $(Q_\varepsilon(t))_\varepsilon$ is a representative of the generalized semigroup $((Q(t))_t$ and by using (3.8), we get

$$\left|Q_\varepsilon^\gamma(t)\nu_\varepsilon\right| \leq \left(C\varepsilon^{-a}\times\gamma\frac{1}{\Gamma(1+\gamma)}\right)\left|\nu_\varepsilon\right| = C_\gamma\left|\nu_\varepsilon\right|,$$

where $C_\gamma = C\varepsilon^{-a}\times\gamma\frac{1}{\Gamma(1+\gamma)}$, which proves that $Q_\varepsilon^\gamma(t)$ is a linear and bounded operator.

□

Proof of Theorem 1 For any $\varepsilon \in (0,1)$ and $\gamma \in (0,1)$, we have to show that the integral solution $(\nu_\varepsilon)_\varepsilon$ given in (3.6) of Lemma 1 is an element of $\mathcal{E}^e_M(\mathbb{R})$. First we have the estimation

$$\begin{aligned}|\nu_\varepsilon(t)| &= \left| S^\gamma_\varepsilon(t)\nu_{0\varepsilon} + \int_0^t (t-s)^{\gamma-1} Q^\gamma_\varepsilon(t-s)\Phi_\varepsilon(s,\nu_\varepsilon(s))\,\mathrm{d}s \right| \\ &\le |S^\gamma_\varepsilon(t)\nu_{0\varepsilon}| + \int_0^t |(t-s)^{\gamma-1} Q^\gamma_\varepsilon(t-s)\Phi_\varepsilon(s,\nu_\varepsilon(s))|\,\mathrm{d}s \\ &\le |S^\gamma_\varepsilon(t)\nu_{0\varepsilon}| + \int_0^t (t-s)^{\gamma-1} |Q^\gamma_\varepsilon(t-s)\Phi_\varepsilon(s,\nu_\varepsilon(s))|\,\mathrm{d}s.\end{aligned}$$

The approximation of the first order to Φ_ε yields

$$\Phi_\varepsilon(t,x(t)) = \Phi_\varepsilon(t,0) + |\nabla_x \Phi_\varepsilon| \nu_\varepsilon(t) + N_\varepsilon(t), \tag{3.10}$$

where $N_\varepsilon(t)$ is the negligible part of this approximation. By Lemma 2, and the fact that $(\nu_\varepsilon) \in \mathcal{E}^e_M(\mathbb{R})$ there are positives constants c, c_1, c_2, N_1 and N_2 such that

$$\begin{aligned}|\nu_\varepsilon(t)| &\le c\, c_2 \varepsilon^{-N_2} + \int_0^t (t-s)^{\gamma-1} \frac{\gamma\, c_1\, \varepsilon^{-N_1}}{\Gamma(1+\gamma)} |\Phi_\varepsilon(s,\nu_\varepsilon(s))|\,\mathrm{d}s \\ &\le c\, c_2 \varepsilon^{-N_2} + \int_0^t (t-s)^{\gamma-1} \frac{\gamma\, c_1\, \varepsilon^{-N_1}}{\Gamma(1+\gamma)} |\Phi_\varepsilon(s,0) + |\nabla_x \Phi_\varepsilon| \nu_\varepsilon(s) + N_\varepsilon(s)|\,\mathrm{d}s.\end{aligned}$$

By the Gronwall lemma, we obtain

$$|\nu_\varepsilon(t)| \le (c\, c_2 \varepsilon^{-N_2} + c_T \varepsilon^{-N_1}) \exp(-T \ln \varepsilon).$$

Hence, there are positive constants $\tilde{c}$, $\tilde{N}$ such that $|\nu_\varepsilon(t)| \le \tilde{c}\, \varepsilon^{-\tilde{N}}$ which proves the moderateness of the solution. Let us prove the uniqueness of the solution in $(\mathcal{G}^e(\mathbb{R}))^n$. Suppose that there are two solutions $\nu_{1,\varepsilon}$, $\nu_{2,\varepsilon}$ to the regularized problem (3.3). Let μ_ε their difference, we have

$$\mu_\varepsilon(t) = \int_0^t (t-s)^{\gamma-1} Q^\gamma_\varepsilon(t-s)[\Phi_\varepsilon(s,\nu_{1,\varepsilon}) - \Phi_\varepsilon(s,\nu_{2,\varepsilon})]\,\mathrm{d}s.$$

Now, using the approximation (3.10) of Φ_ε, yields

$$|\mu_\varepsilon(t)| \le \int_0^t \frac{T^\gamma}{\gamma} |Q^\gamma_\varepsilon(t-s)| \nabla_x \Phi_\varepsilon| [(\nu_{1,\varepsilon}(s) - \nu_{2,\varepsilon}(s)) + N_\varepsilon(s)]|\,\mathrm{d}s,$$

by using the boundedness of the linear operator $Q^\gamma_\varepsilon(t)$, $t \ge 0$ and the Gronwall lemma, and the fact that $\nu_{1,\varepsilon}(s) - \nu_{2,\varepsilon}(s)$ is of order $O(\varepsilon^N)$ and the same applies to the negligible part N_ε, it follows that every $N \ge 0$, we have $|\mu_\varepsilon(t)| = O_{\varepsilon\to 0}(\varepsilon^N)$, which proves the uniqueness of the solution in the algebra $(\mathcal{G}^e(\mathbb{R}))^n$.

4. EXAMPLE

Consider the following fractional evolution equation in the space $\mathrm{Y} = L^2([0,\pi])$:

$$\begin{cases} D^{\gamma}_{0^+}\nu(t,x) = \frac{\partial^2 \nu}{\partial x^2}(t,x) + \nu(t,x)D^{\gamma}H(t), & (t,x) \in [0,1]\times[0,\pi], \\ \nu(t,0) = \nu(t,\pi) = 0, & t \in [0,1], \\ \nu(0,x) = \delta(x), \end{cases} \tag{4.1}$$

where H is the Heaviside distribution and the initial data is the Dirac distribution. Let (ν_{ε}) be a representative of the generalized solution. We define the operator $\mathcal{A}\colon D(\mathcal{A}) \subset \mathrm{Y} \to \mathrm{Y}$ by

$$D(\mathcal{A})\colon = \Big\{\nu \in \mathrm{Y} : \nu, \nu' \text{ are absolutely continuous and } \nu'' \in \mathrm{Y}, \nu(0) = \nu(\pi) = 0\Big\}$$

and

$$\mathcal{A}\nu = \frac{\partial^2 \nu}{\partial x^2}.$$

It is well known that $\mathcal{A}$ has a discrete spectrum, with eigenvalues $-n^2$, $n \in \mathbb{N}$, and the corresponding normalized eigenvectors $e_n(z) = \sqrt{\frac{2}{\pi}}\sin(nz)$. It is an orthonormal basis for the Hilbert space $L^2_{\pi}(\mathbb{R})$ of $\pi-$periodic square integrable functions on $\mathbb{R}$, that can be naturally identified with the Hilbert space $L^2(\mathbb{T})$ of square integrable functions on the half unit circle $\mathbb{T}$ via the mapping $\Phi f(t) = f(e^{2i\pi t})$. It follows

$$\mathcal{A}\nu = \sum_{n=1}^{\infty} -n^2\langle \nu, e_n\rangle e_n, \quad \nu \in D(\mathcal{A}).$$

Furthermore, $\mathcal{A}$ generates a uniformly bounded analytic semigroup $\{\mathcal{Q}(t)\}_{t\geq 0}$ in Y and it is given by

$$\mathcal{Q}(t)\nu = \sum_{n=1}^{\infty} e^{-n^2 t}\langle \nu, e_n\rangle e_n, \quad \nu \in \mathrm{Y},$$

where

$$\langle \nu, e_n\rangle = \int_0^{\pi} \nu(t)e_n(t)\,\mathrm{d}t.$$

Since $\|\mathcal{Q}(t)\| \leq e^{-t}$ for all $t \geq 0$, $M = 1$, implying

$$\sup_{t\in[0,\infty)} \|\mathcal{Q}(t)\| = 1.$$

The integral solution to the problem (4.1) can be given by

$$\begin{aligned} \nu_{\varepsilon}(t,x) = &\int_0^{\infty} \phi_{\gamma}(\xi)\mathcal{Q}_{\varepsilon}(t^{\gamma}\xi)\delta_{\varepsilon}(\xi)\,\mathrm{d}\xi \\ &+ \gamma\int_0^t\int_0^{\infty} \xi(t-s)^{\gamma-1}\phi_{\gamma}(\xi)\mathcal{Q}_{\varepsilon}((t-s)^{\gamma}\xi)\nu_{\varepsilon}(s,\xi)\tilde{D}^{\gamma}h_{\varepsilon}(s)\,\mathrm{d}\xi\,\mathrm{d}s, \end{aligned}$$

where $(h_\varepsilon)_\varepsilon$, $(\delta_\varepsilon)_\varepsilon$ are representatives of the Heaviside function H and the Dirac measure δ respectively, given by $h_\varepsilon(t) = \int_{-\infty}^{t} \varphi_\varepsilon(\xi)\,\mathrm{d}\xi$, $\delta_\varepsilon(\xi) = \varphi_\varepsilon(\xi) \quad \forall \varepsilon \in (0,1)$, where φ_ε is given as in (2.1) and $Q_\varepsilon = Q$. Using Definitions 3.5 and 3.7, we can write the solution with the following formula

$$\nu_\varepsilon(t,x) = S_\varepsilon^\gamma(t)\varphi_\varepsilon(x) + \int_0^t (t-s)^{\gamma-1} Q_\varepsilon^\gamma(t-s)\nu_\varepsilon(s,x)\tilde{D}^\gamma h_\varepsilon(s)\,\mathrm{d}s.$$

We have to establish the moderateness of $\nu_\varepsilon(t,x)$ with the sup-norm, we have the estimation

$$\begin{aligned}|\nu_\varepsilon(t,x)| &\le |S_\varepsilon^\gamma(t)\varphi_\varepsilon(x)| + \int_0^t (t-s)^{\gamma-1}|Q_\varepsilon^\gamma(t-s)\nu_\varepsilon(s,x)\tilde{D}^\gamma h_\varepsilon(s)|\,\mathrm{d}s \\ &\le |S_\varepsilon^\gamma(t)\varphi_\varepsilon(x)| + \int_0^t (t-s)^{\gamma-1}\frac{\gamma}{\Gamma(1+\gamma)}|\nu_\varepsilon(s,x)|\ |\tilde{D}^\gamma h_\varepsilon(s)|\,\mathrm{d}s.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{t\in[0,T)}|\nu_\varepsilon(t,x)| \le& \sup_{t\in[0,T)}|S_\varepsilon^\gamma(t)\varphi_\varepsilon(x)| \\ &+ \sup_{t\in[0,T)}\int_0^t (t-s)^{\gamma-1}\frac{\gamma}{\Gamma(1+\gamma)}|\nu_\varepsilon(s,x)|\ |\tilde{D}^\gamma h_\varepsilon(s)|\,\mathrm{d}s \\ &\le \|S_\varepsilon^\gamma(t)\varphi_\varepsilon\|_\infty + \int_0^t (t-s)^{\gamma-1}\frac{\gamma}{\Gamma(1+\gamma)}|\nu_\varepsilon(s,x)|\ \|\tilde{D}^\gamma h_\varepsilon(s)\|_\infty\,\mathrm{d}s,\end{aligned}$$

since we use the estimates (3.9) of the semigroups $S^\gamma(t)$ and $Q^\gamma(t)$, and the fact that $Q(t)$ is a contraction semigroup. According to the estimation (2.4) of $\tilde{D}^\gamma$, we can write $\|\tilde{D}^\gamma\nu_\varepsilon\| \le C_{\gamma,T}\varepsilon^{-N} \le C_\gamma\varepsilon^{-N}$ $\varepsilon \in (0,1)$, $N > 0$. By Gronwall inequality it yields

$$\begin{aligned}|\nu_\varepsilon(t,x)| &\le \|\varphi_\varepsilon\|_\infty + \int_0^t (t-s)^{\gamma-1}C_\gamma\varepsilon^{-N}\,\mathrm{d}s, \\ &\le \|\varphi_\varepsilon\|_\infty \times \int_0^t \exp\left((t-s)^{\gamma-1}C_\gamma\varepsilon^{-N}\right)\,\mathrm{d}s\end{aligned}$$

since the integral in the last inequality is convergent, there are $N > 0$, C_γ such that $|\nu_\varepsilon(t,x)| \le C_\gamma\varepsilon^{-N}$, which proves the moderateness of $(\nu_\varepsilon)_\varepsilon$.

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