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UNIT GROUP OF INTEGRAL GROUP RING $\mathbb{Z}(G\times C_3)$

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Abstract. Presenting an explicit descryption of unit group in the integral group ring of a given non-abelian group is a classical and open problem. Let S_3 be a symmetric group of order 6 and C_3 be a cyclic group of order 3. In this study, we firstly explore the commensurability in unit group of integral group ring $\mathbb{Z}(S_3\times C_3)$ by showing the existence of a subgroup as $(F_{55}\rtimes F_3)\rtimes(S_3^*\times C_2)$ where F_ρ denotes a free group of rank ρ. Later, we introduce an explicit structure of the unit group in $\mathbb{Z}(S_3\times C_3)$ in terms of semi-direct product of torsion-free normal complement of S_3 and the group of units in RS_3 where $R=\mathbb{Z}[\omega]$ is the complex integral domain since ω is the primitive 3rd root of unity. At the end, we give a general method that determines the structure of the unit group of $\mathbb{Z}(G\times C_3)$ for an arbitrary group G depends on torsion-free normal complement $V(G)$ of G in $U(\mathbb{Z}(G\times C_3))$ in an implicit form. As a consequence, a conjecture which is found in [21] is solved.

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1. INTRODUCTION

We denote the integral group ring of a given finite group G over the ring of integers by $\mathbb{Z}G$. Its elements are all finite sums of the form $\sum_{g\in G}\alpha_g g$ where $\alpha_g\in\mathbb{Z}$. The ring epimorphism defined by $\varepsilon:\mathbb{Z}G\longrightarrow\mathbb{Z}$, $\varepsilon(g)=1$, is called the *augmentation map*. Hence, naturally kernel of ε which is named by *augmentation ideal* and denoted by $\Delta_{\mathbb{Z}}(G)$ is obtained as a $\mathbb{Z}$-module based on the set $S=\{g-1:g\in G\}$.

The group of all units in $\mathbb{Z}G$ is generally shown by $U(\mathbb{Z}G)$ and the subgroup of units whose augmentations are 1 denoted by $U_1(\mathbb{Z}G)$. It is well known that $\pm U_1(\mathbb{Z}G)=U(\mathbb{Z}G)$ and the set $\pm G$ is said to be *trivial units* in $U(\mathbb{Z}G)$ [19].

Describing units of integral group rings is a classical hard problem for various type of groups. The subject matter has captured the interest of researchers in the fields of algebra, number theory, and algebraic topology throughout the years. As stated in [16], most descriptions of $U(\mathbb{Z}G)$ in the mathematical literature either give an explicit description of unit group, the general structure of $U(\mathbb{Z}G)$, or a subgroup of finite index of the unit group $U(\mathbb{Z}G)$. Results were often attained by using techniques from representation theory and algebraic number theory [22].

In 1940, a crucial work on the unit problem was done by Graham Higman who displayed that if $U(\mathbb{Z}G) = \pm G$, then $U(\mathbb{Z}(G \times C_2)) = \pm(G \times C_2)$ [6, 7]. Using this, he showed that $U(\mathbb{Z}G) = \pm G$ if and only if G is an abelian group of exponent 2, 3, 4, or 6 or $G = E \times K_8$ where K_8 is the quaternion group of order 8, E is an elementary abelian 2–group and moreover, Higman gave a general structure theorem for $U(\mathbb{Z}A)$, where A is a finite abelian group [6, 7]. Some additional results have been provided for these groups: A_4 and S_4 by Allen-Hobby [1, 2], D_{2p} written by Passman-Smith [18], $G = C_p \rtimes C_q$, such that q is a prime dividing $p-1$ made by Galovitch-Reiner-Ullom [5], $|G| = p^3$ given in [20], and a version of $U(\mathbb{Z}S_3)$ can be found at [8]. In [10], Jespers and Parmenter submitted an explicit description of $U(\mathbb{Z}S_3)$ in terms of bicyclic units as follows:

Theorem 1. *[10] In $U_1(\mathbb{Z}S_3)$, S_3 has a torsion-free normal complement which is generated by bicyclic units as $U_1(\mathbb{Z}S_3) = V \rtimes S_3$ such that $V = \langle u_{b,a}, u_{ba,a}, u_{ba^2,a} \rangle$ where*

$$\begin{aligned} u_{b,a} &= 1 + (1-b)a(1+b) \\ u_{ba,a} &= 1 + (1-ba)a(1+ba) \\ u_{ba^2,a} &= 1 + (1-ba^2)a(1+ba^2) \end{aligned}$$

In 1993, Jespers and Parmenter described $U(\mathbb{Z}G)$ for all groups of order 16 [11]. In [9], Jespers introduced $U(\mathbb{Z}G)$ for the dihedral group of order 12 and for $G = D_8 \times C_2$. Kelebek and Bilgin gave the structure of $U(\mathbb{Z}(C_n \times K_4))$ where C_n is a cyclic group of order n and K_4 is a Klein-4 group [14]. Köklüce and Tüfekçi presented the symmetric unit groups whose rank is less than 4 of integral group rings of cyclic groups [15]. In [16, 21], a general algebraic framework was implicitly developed to study $U(\mathbb{Z}G^*)$, where $G^* = G \times C_p$, p is a prime integer and G is a finite group. It was stated that $U(\mathbb{Z}G^*)$ is an open problem, where $G^* = T \times C_2$ such that T is a binary dihedral group (or dicyclic group) of order 12 and C_2 is a cyclic group of order 2 was posed as a conjecture [21]. In [3], Bilgin, Küsmüş and Low studied this open problem and introduced some substantial proofs which explain the isomorphic form of unit group in integral group ring of direct product

$$T \times C_2 = \langle a, b, x : a^6 = x^2 = 1, a^3 = b^2, bab^{-1} = a^{-1}, ax = xa, bx = xb \rangle$$

in terms of semi-direct products of finitely generated free-groups. In [21], Low stated that succeeding a complete characterization of $U(\mathbb{Z}(G \times C_p))$ depends on clarifying the structure of $U(RC_p)$ where R is a complex integral domain for $p \geq 3$ which keeps its enigma. In this study, we focus on the case where $G = S_3$, $p = 3$ and we then study the open problem which related to characterization of unit group of integral group ring $\mathbb{Z}(G \times C_p)$ where G is a non-abelian group and $p = 3$ found in [21].

As a solution to this problem, we characterize the unit group in integral group ring $\mathbb{Z}(S_3\times C_3)$ of

$$S_3^* := S_3\times C_3 = \left\langle a,b,x : a^3=b^2=x^3=1, bab^{-1}=a^{-1}, ax=xa, bx=xb\right\rangle.$$

2. Decomposition of $U(\mathbb{Z}S_3^*)$

It is worth mentioning that the groups possessing a non-abelian torsion-free normal complement have been classified in [12]. Furthermore, various studies [9], [17], [21], [3] have described the free group structure of $U(\mathbb{Z}(G\times C_2))$ for $G=S_3, D_4, P$ and T, respectively. It is crucial to note that these characterizations heavily rely on the fact that $\omega^2=1$ implies $\mathbb{Z}[\omega]=\mathbb{Z}$ which provides an advantage according to the scheme presented in [3]. However, this is not the case for $\omega^n=1$ where $n\geq 3$, as we have $\mathbb{Z}\subsetneq\mathbb{Z}[\omega]$. In this section, we firstly concentrate on investigating a subgroup of $U(\mathbb{Z}S_3^*)$ that is commensurable [13]. To proceed further, we require some preliminary propositions.

Proposition 1. *Since* $S_3=\left\langle a,b : a^3=b^2=1, bab^{-1}=a^{-1}\right\rangle$,

$$U_1(\mathbb{F}_3S_3)\simeq(3_+^{1+2}\rtimes C_3)\rtimes K_4$$

where K_4 *is Klein 4-group,* C_3 *is a cyclic group of order 3 and* 3_+^{1+2} *extraspecial group of order 27.*

Proof. Let $N=\langle a\rangle$. We can define the projection $\varphi : S_3\longrightarrow S_3/N$ by $\varphi(g)=gN$. Linearity of φ over the ring $\mathbb{F}_3$ follows that $\overline{\varphi}:\mathbb{F}_3S_3\longrightarrow\mathbb{F}_3(S_3/N)$ is a projection at ring level by

$$\overline{\varphi}(\alpha_i a^i+\beta_i ba^i)=(\alpha_0+\alpha_1+\alpha_2)N+(\beta_0+\beta_1+\beta_2)bN \tag{2.1}$$

Let $\tilde{\varphi}$ be the restriction of $\overline{\varphi}$ on the unit level which is defined in the same as the rule in (2.1).

This indicates that $\tilde{\varphi}(U(\mathbb{F}_3S_3))=U(\mathbb{F}_3C_2)$. Furthermore, we have from [4] that $U(\mathbb{F}_{3^k}C_2)=C_{p^k-1}\times C_{p^k-1}$. Therefore $U(\mathbb{F}_3C_2)=K_4$ where K_4 is Klein 4-group and thus $Im\tilde{\varphi}=K_4$.

On the other hand,

$$Ker\,\tilde{\varphi}=\left\{\sum_{i=0}^{2}\alpha_i a^i+\beta_i ba^i : (\sum_{i=0}^{2}\alpha_i,\sum_{i=0}^{2}\beta_i)=(1,0), \forall\alpha_i,\beta_i\in\mathbb{F}_3\right\}$$

is composed of 81 elements. Since $\langle a\rangle\subset Ker\tilde{\varphi}$, consider the quotient $G^{'}=Ker\tilde{\varphi}/\langle a\rangle$. One can discern that $\left\langle -1+a+a^2\right\rangle$ is a cyclic group of order 3 and moreover it is central in $G^{'}$.

As a result, we conclude that $G^{'}/\left\langle -1+a+a^2\right\rangle$ is an elementary abelian 3-group and hence $G^{'}$ is an extraspecial 3-group which can generally be denoted as 3_+^{1+2}. This manifests that $Ker\tilde{\varphi}$ is isomorphic to $3_+^{1+2} \rtimes C_3$. Therefore, it is possible to achieve an exact sequence as

$$Ker\tilde{\varphi} \xrightarrow{\iota} U_1(\mathbb{F}_3S_3) \xrightarrow{\tilde{\varphi}} Im\tilde{\varphi} = U_1(\mathbb{F}_3C_2) \simeq K_4$$

Reminding the identity map is the reverse direction of embedding i, we can phrase that the sequence splits as $U_1(\mathbb{F}_3S_3) = Ker\tilde{\varphi} \rtimes Im\tilde{\varphi}$ which substantiates the proof. □

Proposition 2. *Let $\sigma_3 : \mathbb{Z}G \to \mathbb{Z}_3G$ be a surjective ring homomorphism that reduces the coefficients modulo 3 and $V = \langle u_{b,a}, u_{ba,a}, u_{ba^2,a} \rangle$ be the torsion free normal complement of S_3. Then, $\sigma_3(U_1(\mathbb{Z}S_3)) = \sigma_3(V) \rtimes S_3$.*

Proof. Clearly, $\sigma_3(U_1(\mathbb{Z}S_3)) = \sigma_3(V \rtimes S_3) = \sigma_3(V) \cdot S_3$. In addition, $\sigma_3(V)$ is normalized by S_3 due to the following equalities which can be easily shown:

$$a\sigma_3(u_{b,a})a^2 = \sigma_3(u_{ba,a})$$
$$a\sigma_3(u_{ba,a})a^2 = \sigma_3(u_{ba^2,a})$$
$$a\sigma_3(u_{ba^2,a})a^2 = \sigma_3(u_{b,a})$$
$$b\sigma_3(u_{b,a})b = \sigma_3(u_{b,a})^{-1}$$
$$b\sigma_3(u_{ba,a})b = \sigma_3(u_{ba^2,a})^{-1}$$
$$b\sigma_3(u_{ba^2,a})b = \sigma_3(u_{ba,a})^{-1}$$

On the other hand, the fact that $\sigma_3(V) \cap S_3 = 1$ infers that $\sigma_3(U_1(\mathbb{Z}S_3))$ is indeed a semi-direct product as claimed. □

Proposition 3. $\sigma_3(V) \simeq C_3 \times C_3 \times C_3$.

Proof. Since $V = \langle u_{b,a}, u_{ba,a}, u_{ba^2,a} \rangle$, check that

$$\sigma_3(u_{b,a})^3 = \sigma_3(u_{ba,a})^3 = \sigma_3(u_{ba^2,a})^3 = 1.$$

We can conclude by the previous proposition that each of $\sigma_3(u_{b,a}), \sigma_3(u_{ba,a})$ and $\sigma_3(u_{ba^2,a})$ gives a normal subgroup. Moreover, as the bicyclic units in V and the powers of images under σ_3 of them are distinct, the product is direct. □

Now, we can introduce the following commutative diagram by taking inspration from [16]:

$$\begin{array}{ccccc} K & \xrightarrow{\iota} & U(\mathbb{Z}S_3^*) & \xrightarrow{\pi} & U(\mathbb{Z}S_3) \\ f\downarrow & & f\downarrow & & \sigma_3\downarrow \\ M & \xrightarrow{\iota} & U(\mathbb{Z}[\omega]S_3) & \xrightarrow{\alpha} & U(\mathbb{Z}_3S_3) \end{array}$$

Lemma 1. *K and M are isomorphic to each other.*

Proof. Using the definitions of π, α, f and σ_3 defined on $\mathbb{Z}S_3^*$, we have the following diagram:

$$\begin{array}{ccccc} K^* & \xrightarrow{\iota} & \mathbb{Z}S_3^* & \xrightarrow{\pi} & \mathbb{Z}S_3 \\ f\big\downarrow & & f\big\downarrow & & \sigma_3\big\downarrow \\ M^* & \xrightarrow{\iota} & \mathbb{Z}[\omega]S_3 & \xrightarrow{\alpha} & \mathbb{Z}_3S_3 \end{array}$$

Let $(x-1)c_1+(x^2-1)c_2\in K^*$. Then, linearity of f follows that

$$f((x-1)c_1+(x^2-1)c_2)=(\omega-1)c_1+(\omega^2-1)c_2.$$

Due to the fact that ω is the primitive root of unity, rearranging the terms, we have

$$(\omega-1)c_1+(\omega^2-1)c_2=(\omega-1)(c_1-c_2)-3c_2.$$

It requires that $f((x-1)c_1+(x^2-1)c_2)=0$ if and only if $(x-1)c_1+(x^2-1)c_2=0$. Thus, f is injective. Taking the pre-image of ω as x, it is easy to perceive that f is surjective. We deduce from [16] that $K=U(1+K^*)$ and $M=U(1+M^*)$ are isomorphic groups of units in multiplicative monoids K^* and M^* respectively. $\square$

Corollary 1.

$$\begin{array}{ccccc} K & \xrightarrow{\iota} & U(\mathbb{Z}S_3^*) & \xrightarrow{\pi} & U(\mathbb{Z}S_3) \\ \simeq\big\downarrow & & f\big\downarrow & & \sigma_3\big\downarrow \\ M & \xrightarrow{\iota} & U(\mathbb{Z}[\omega]S_3) & \xrightarrow{\sigma_3} & U(\mathbb{Z}_3S_3) \\ \iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow \\ M^+ & \xrightarrow{\iota} & U_1(\mathbb{Z}S_3) & \xrightarrow{onto} & \sigma_3(V)\rtimes S_3 \\ \iota\big\uparrow & & \iota\big\uparrow & & \iota\big\uparrow \\ M^+\cap V & \xrightarrow{\iota} & V & \xrightarrow{onto} & \sigma_3(V) \end{array}$$

Corollary 2. *M^+ is a free group of rank 55.*

Proof. Let $w(\sigma_3(u_{b,a}),\sigma_3(u_{ba,a}),\sigma_3(u_{ba^2,a}))$ be a word in $\sigma_3(V)$ and $g\in S_3$. As $\sigma_3(V)\cap S_3=\{1\}$, observe that if $w(\sigma_3(u_{b,a}),\sigma_3(u_{ba,a}),\sigma_3(u_{ba^2,a}))\cdot g=1$, then

$$w(\sigma_3(u_{b,a}),\sigma_3(u_{ba,a}),\sigma_3(u_{ba^2,a}))=g^{-1}\in\sigma_3(V)\cap S_3.$$

This implies that $g=1$. This means that M^+ consists of words of the form

$$u_{b,a}^{i_1}\cdot u_{ba,a}^{i_2}\cdot u_{ba^2,a}^{i_3}$$

where $0\le i_1,i_2,i_3\le 2$. This shows that $M^+=M^+\cap V$. Besides $\frac{V}{M^+\cap V}\simeq\sigma_3(V)$ implies that

$$[V:M^+]=|\sigma_3(V)|=27$$

Since V is a free group of rank 3, we conclude by Schreier's index formula and Schreier's Theorem that M^+ is a free group of rank 55 via Schreier's index formula. □

Theorem 2. *$U(\mathbb{Z}S_3^*)$ is commensurable which contains a subgroup of finite index as*

$$(F_{55} \rtimes F_3) \rtimes (S_3^* \times C_2)$$

Proof. As $U(\mathbb{Z}[\omega]S_3) = U_1(\mathbb{Z}[\omega]S_3) \rtimes U(\mathbb{Z}[\omega])$ and $U(\mathbb{Z}[\omega]) = \langle -1 \rangle \times \langle \omega \rangle \simeq C_6$, we can recognize that $U_1(\mathbb{Z}S_3) \hookrightarrow U_1(\mathbb{Z}[\omega]S_3) \times C_6$ and hence $M^+ \hookrightarrow M^+ \times C_6 \subsetneq M$. We know from the diagram in Corollary 1 and Lemma 1 that

$$U_1(\mathbb{Z}S_3^*) = K \rtimes (V \rtimes S_3) \simeq M \rtimes (V \rtimes S_3)$$

Since,

$$(M^+ \times C_6) \rtimes (V \rtimes S_3) \subsetneq M \rtimes (V \rtimes S_3)$$

we conclude that

$$(M^+ \times C_6) \rtimes (V \rtimes S_3) = (M^+ \rtimes V) \rtimes (S_3 \times C_3) \times C_2 = (F_{55} \rtimes F_3) \rtimes (S_3^* \times C_2)$$

as claimed. □

Let $R = \mathbb{Z}[\omega]$ and $\{u_{ba^i,a}\}_{i=1}^{3}$ be the set of generators in torsion-free normal complement of S_3. Note that $\mathbb{Z}[\omega]S_3$ can be decomposed as $\mathbb{Z}S_3 \oplus \omega\mathbb{Z}S_3$. Defining a ring epimorphism $\psi : \mathbb{Z}S_3 \oplus \omega\mathbb{Z}S_3 \longrightarrow \mathbb{Z}S_3$ by $\psi(\alpha, \omega\beta) = \alpha$ follows that $Ker\psi = \omega\mathbb{Z}S_3$ and the following exact sequence

$$\omega\mathbb{Z}S_3 \xrightarrow{\ \iota\ } \mathbb{Z}S_3 \oplus \omega\mathbb{Z}S_3 \xrightarrow{\ \psi\ } \mathbb{Z}S_3$$

At unit group level, this sequence can be extended as

$$(1+\omega\mathbb{Z}S_3) \cap U(\mathbb{Z}S_3) \xrightarrow{\ \iota\ } U(\mathbb{Z}[\omega]S_3) \xrightarrow{\ \psi\ } U(\mathbb{Z}S_3)$$

Let $u = 1 + \omega r \in (1+\omega\mathbb{Z}S_3) \cap U(\mathbb{Z}S_3)$. So there exists $v = 1 + \omega s$ where $r, s \in \mathbb{Z}S_3$ such that $uv = 1$. Notice that u and v are inverses of each other if and only if

$$uv = (1 - rs) + \omega(r + s - rs) = 1$$

and so r and s are either trivial or nilpotent elements of index 2 in $\mathbb{Z}S_3$. It follows that either $1 - r = 1$ or $1 - r \in V$ due to the fact that every unipotent unit is lifted from a nilpotent element of index 2. Hence we can take $r = u_{ba^i,a} - 1$. Accordingly,

$$(1+\omega\mathbb{Z}S_3) \cap U(\mathbb{Z}S_3) = F$$

and

$$U(\mathbb{Z}[\omega]S_3) = F \rtimes U(\mathbb{Z}S_3) \tag{2.2}$$

where

$$F = \left\langle 1 + \omega(u_{ba^i,a} - 1) : u_{ba^i,a} \in V, i = 0,1,2 \right\rangle$$

Theorem 3.
$$U(RS_3)\simeq\frac{F\rtimes V\rtimes S_3}{(3^{1+2}_{+}\rtimes C_3)\rtimes K_4}$$
where 3^{1+2}_{+} denotes extraspecial group of order 27.

Proof. In Corollary 1, consider the second row of the diagram. As the map i is an embedding, this row can be split into two parts: That is $U(RS_3)$ is isomorphic to the semidirect product of M and $U(\mathbb{Z}_3S_3)$. Additionally, equation (2.2) tells us that $U(RS_3)$ is isomorphic to the direct product of F and $U(\mathbb{Z}S_3)$. Combining these different forms of $U(RS_3)$, we can conclude that F and $U(\mathbb{Z}S_3)$ together form a group which is isomorphic to the semidirect product of M and $U(\mathbb{Z}_3S_3)$ that is
$$U(RS_3)=F\rtimes U(\mathbb{Z}S_3)\simeq M\rtimes U(\mathbb{Z}_3S_3)$$
Using this result, we can infer that M is isomorphic to the quotient group obtained by dividing the direct product of F and $U(\mathbb{Z}S_3)$ by the normal subgroup $U(\mathbb{Z}_3S_3)$. In other words,
$$M\simeq\frac{F\rtimes U(\mathbb{Z}S_3)}{U(\mathbb{Z}_3S_3)}$$
Furthermore, Proposition 1 tells us that $U(\mathbb{Z}_3S_3)$ is an extraspecial 3-group of order 27, denoted by 3^{1+2}_{+}. This establishes the proof. □

An alternative way to describe $U(RS_3)$ is through the use of cyclic groups instead of 3^{1+2}_{+} as follows.

Theorem 4.
$$U(RS_3)\simeq\frac{F\rtimes V\rtimes S_3}{[(C_3^3\rtimes C_3)\rtimes C_2]\times C_2}$$
where $C_3^3=C_3\times C_3\times C_3$.

Proof. To enhance clarity of the proof, it is adequate to take into account $k=1$ in the main result of [4] concerning $U(\mathbb{Z}_3S_3)$. □

Lemma 2. *F is a free group on the generator set $S=\{v_{b,a},v_{ba,a},v_{ba^2,a}\}$ where $v_{ba^i,a}=1+\omega(u_{ba^i,a}-1)$.*

Proof. Let $v_{ba^i,a}$ denote $1+\omega(u_{ba^i,a}-1)$. Then F can be stated as
$$F=\left\langle v_{b,a},v_{ba,a},v_{ba^2,a}\right\rangle$$
Assume that F is not a free group. Then any unit in F can be written in different ways as a product of finitely many elements of $S=\{v_{b,a},v_{ba,a},v_{ba^2,a}\}$ and their inverses disregarding trivial variations. Let any element $v_{ba^i,a}$ in F be phrased as
$$v_{ba^i,a}=v_{ba^{i_1},a}v_{ba^{i_2},a}\tag{2.3}$$
Then the equation (2.3) hold if and only if
$$v_{ba^i,a}=1+(u_{ba^{i_1},a}+u_{ba^{i_2},a}-u_{ba^{i_1},a}u_{ba^{i_2},a}-1)+\omega(2(u_{ba^{i_1},a}+u_{ba^{i_2},a})-3)$$

and so

$$2u_{ba^{i_1},a}u_{ba^{i_2},a} = u_{ba^i,a}$$

which is a contradiction due to the fact that V is free. Besides, $T : V \longrightarrow F$ defined by

$$T(u_{ba^i,a}) = v_{ba^i,a}$$

is a bijective map. Therefore, F is a free group as well. □

We are now prepared to establish our principal structural theorem in this section, as follows.

Theorem 5. *$U(\mathbb{Z}S_3^*) = M \rtimes V \rtimes S_3$ where $M \simeq \frac{F \rtimes V \rtimes S_3}{(3_+^{1+2} \rtimes C_3) \rtimes K_4}$.*

Proof. Remind that we have the split extension

$$U(\mathbb{Z}S_3^*) = K \rtimes U(\mathbb{Z}S_3)$$

from the commutative diagram of maps in Corollary 1. We also know from Lemma 1 that $K \simeq M$. Besides this,

$$U(RS_3) = M \rtimes (3_+^{1+2} \rtimes C_3) \rtimes K_4 \tag{2.4}$$

because of Theorem 3. On the other hand, equation (2.2) and [10] follows that

$$U(RS_3) = F \rtimes V \rtimes S_3 \tag{2.5}$$

Let

$$\delta : M \rtimes (3_+^{1+2} \rtimes C_3) \rtimes K_4 \longrightarrow M$$

by $\delta(m.(a.b.c)) = m$ where $(a.b).c \in (3_+^{1+2} \rtimes C_3) \rtimes K_4$. Then, it is clear that

$$Ker\delta = U_1(\mathbb{F}_3S_3) = (3_+^{1+2} \rtimes C_3) \rtimes K_4$$

Thus,

$$M \simeq U(RS_3)/U_1(\mathbb{F}_3S_3) \tag{2.6}$$

Using equations (2.4), (2.5) and (2.6), we conclude that $U(\mathbb{Z}S_3^*) = M \rtimes V \rtimes S_3$ where

$$M \simeq \frac{F \rtimes V \rtimes S_3}{(3_+^{1+2} \rtimes C_3) \rtimes K_4}$$

as required. □

3. Conclusion and Discussion

In this study, we have examined the commensurability of $U(\mathbb{Z}S_3^*)$, demonstrating the existence of a subgroup in $U(\mathbb{Z}S_3^*)$, one of whose factors is free-by-free group, specifically $F_{55} \rtimes F_3$. We then proceeded to investigate the whole unit group of the integral group ring $\mathbb{Z}S_3^*$ by characterizing it through the description of the unit group of RS_3, where $R = \mathbb{Z}[\omega]$, with ω being the primitive 3rd root of unity, and its torsion-free subgroup generated by units which are lifted from square-zero elements.

Regarding the characterization of the entire unit group in $\mathbb{Z}S_3^*$, we have established that the torsion-free normal complement of S_3^* is derived from the unipotent unit generators of $\mathbb{Z}S_3^*$. In fact, by modifying the definitions of maps π, σ_3 and f in the commutative diagram established in Corollary 1 over an arbitrary finite group G rather than specifically on the symmetric group S_3, we can derive the aforementioned diagram in the following manner:

$$\begin{array}{ccccc} K & \xrightarrow{\iota} & U(\mathbb{Z}G^*) & \xrightarrow{\pi} & U(\mathbb{Z}G) \\ \simeq\big\downarrow & & f\big\downarrow & & \sigma_3\big\downarrow \\ M & \xrightarrow{\iota} & U(\mathbb{Z}[\omega]G) & \xrightarrow{\alpha} & U(\mathbb{Z}_3G) \end{array} \tag{3.1}$$

where $G^* = G \times C_3$.

To achieve a comprehensive description of $U(\mathbb{Z}G^*)$ using (3.1), it is imperative to possess an explicit version of $U(\mathbb{Z}G)$ that either elucidates its general structure or identifies a subgroup of finite index. Moreover, as can be seen from the split extension of the second row of our diagram, since $U(\mathbb{Z}[\omega]G) = M \rtimes U(\mathbb{Z}_3G)$, another characterization of the unit group $U(\mathbb{Z}G^*)$ in terms of the unit group of the finite group algebra $\mathbb{Z}_3G$ can be given. The commutative diagram constructed on an arbitrary group G only yields an implicit version of $U(\mathbb{Z}G^*)$. Our diagram can extend the integral group ring $\mathbb{Z}G$, which has been specifically examined by relevant researchers to explore the unit group and provide an explicit description, to $U(\mathbb{Z}G^*)$. In particular, when working with a group G for which G has a torsion-free normal complement, denoted by $V(G)$ in $U(\mathbb{Z}G)$, one can establish split extension form of $U(\mathbb{Z}G^*)$ in the context of $V(G)$. We know that $U(\mathbb{Z}G^*) \simeq M \rtimes U(\mathbb{Z}G)$ and $U(RG) = F \rtimes U(\mathbb{Z}G)$. As another homomorphic image, we have $M \simeq U(RG)/U(\mathbb{Z}_3G)$. Therefore, we can implicitly characterize $U(\mathbb{Z}G^*)$ as

$$U(\mathbb{Z}G^*) \simeq \frac{F \rtimes V(G) \rtimes G}{U(\mathbb{Z}_3G)} \rtimes V(G) \rtimes G \tag{3.2}$$

The isomorphism established in (3.2) can be applied to provide a precise characterization for all groups G, for which the unit group in group algebra of G over $\mathbb{Z}_3$ is completely characterized and has a known torsion-free normal complement in $U(\mathbb{Z}G)$. From this perspective, this study may serve as a source of inspiration for future investigations.

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