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AN ANALYTICAL APPROACH FOR SOLVING SYSTEM OF RICCATI EQUATIONS

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Abstract. In this paper, we investigate the solution of fractional system of Riccati equations. This system is important since it appears in several applications in science such as control theory. We use the operational matrix method to solve this system. The block-pulse operational matrices will initially assist in the reduction of the nonlinear fractional order Riccati-differential problem into an algebraic system. Benefits of this approach include inexpensive setup costs for the equations without the use of projection techniques like Galerkin, collocation, etc. In addition, we prove the convergence of the approximate solution using operational matrix method to the exact solution. Finally, we present two examples to provide numerical evidences to the efficiency of numerical approach used in this paper. We notice that the error is close to zero. Also, the approximate solutions are convergent to the exact solutions for different values of fractional derivative. Moreover, the approximate solutions approach to the solution of system of first order when the derivative order approaches to one.

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1. INTRODUCTION

Day after day the fractional order calculus is gradually replacing the classical calculus. Due to their complexity compared to their integer order counterparts, fractional models necessitate more sophisticated computations, both analytically and numerically. Numerous researchers from around the world have recently investigated various aspects of fractional models using theoretical and numerical methods. Taking the Caputo fractional derivative as an example, Shah et al. [12] studied the Cauchy type dynamical system under piece-wise equations with fractional order Caputo derivative. A discrete analogue for the composition of the fractional integral and Caputo derivative has been proved by Płociniczak [11]. Fractional derivatives with non-singular kernels are also discussed by several researchers. For example, Baleanu et

al. [2] used this type of derivative to find a new model in optimal control of a tumor-immune surveillance. Also, Caputo and Fabrizio [5] and Atangana and Baleanu [1] found more definitions for fractional derivatives with non-singular kernels. One of the most interesting nonlinear differential equations is the Riccati equation. The Riccati equation is applied in both physics and other fields of mathematics, such as algebraic geometry and conformal mapping theory. It also shows up in lots of real-world problems. Many researchers have been interested in various methods for determining estimate solutions to the Riccati differential equations. Recently, a variety of numerical techniques have been established to solve the fractional differential equations. Fathy and Abdelgaber [6] have been discussed and used the Galerkin method to provide fractional order semi-analytical solutions to the quadratic Riccati and Bagley-Torvik differential equations. Also, they compared the proposed method with other methods to prove the efficiency and reliability of Galerkin method. Khashan and Syam [8] used the the fractional series method to solve a special type of Riccati equation. In addition, Kashkari and Syam [7] solved this type of equations using the the implicit hybrid block method. Operational matrix method is one of the useful tools to use in solving fractional differential equations. Baleanu et al. [3] used Bernstein operational matrices to solve fractional differential equations using the Riemann-Liouville derivative. More researchers used different types of operation matrices to solve fractional initial value problems such as Bhrawy et al. [4] and Shiri and Baleanu [13]. A set of nonlinear Riccati-differential equations can be solved by using the operational block-pulse functions. The block-pulse operational will initially assist in the reduction of the nonlinear fractional order Riccati-differential problem into an algebraic system. Benefits of this approach include inexpensive setup costs for the equations without the use of projection techniques like Galerkin, collocation, etc. In this article, we study the following system of Riccati equations

$$D^{\gamma}G(t) = \Lambda_{21} + \Lambda_{22}G(t) - G(t)\Lambda_{11} - G(t)\Lambda_{12}G(t) \tag{1.1}$$

with

$$G(0) = Q, \tag{1.2}$$

where $\Lambda_{11}, \Lambda_{12}, \Lambda_{21}$, and Λ_{22} are $1\times 1, 1\times k_2, k_2\times 1$, and $k_2\times k_2$ real matrices, respectively. Also, $G(t)$ and G_0 are of size $k_2\times 1$ matrices. We should note that $\Lambda_{ij}, 1\le i,j\le 2$, are matrices depend on the variable t in most of the cases.

This article will be organized as follows. In the next section, we give the main definitions and theorems which we use in this article. In Section 3, we derive the operational matrices which we need to solve the system of Riccati equations. Then, we discuss the convergence analysis in Section 4 and some numerical results will be given in Section 5. Finally, we give some conclusions and discussion for our results in the last section.

2. PRELIMINARIES

Two important concepts which we use in our paper are fractional derivative and block pulse functions (BPF). For this reason, we start this section by the definition of the Caputo derivative and its properties [10, 13].

Definition 1 ([14, 15])**.** Let $\iota \in (0,1)$ be a positive real number and λ be a real number. Then,

$$D^{\iota}\alpha(\lambda) = \frac{1}{\Gamma(1-\iota)} \int_0^{\lambda} (\lambda - q)^{-\iota} \alpha'(q) \mathrm{d}q \tag{2.1}$$

is called the Caputo derivative of order $\iota \in (0,1)$. The fractional integral operator is given as

$$I^{\iota}\alpha(\lambda) = \frac{1}{\Gamma(\iota)} \int_0^{\lambda} (\lambda - q)^{\iota-1} \alpha(q) \mathrm{d}q. \tag{2.2}$$

Let $\xi = \lambda q$. Simple calculations yield to

$$D^{\iota} q^{\mu} = \begin{cases} 0, & \mu < \iota, \mu \in \{0,1,2,\ldots\}, \\ \frac{\Gamma(\mu+1)}{\Gamma(\mu-\iota+1)} q^{\mu-\iota}, & \text{otherwise.} \end{cases}, \iota > 0. \tag{2.3}$$

The relations between the two operators in Definition 1 are given as follows:

$$D^{\iota} I^{\iota} \alpha(q) = \alpha(q) \tag{2.4}$$

and

$$I^{\iota} D^{\iota} \alpha(q) = \alpha(q) - \alpha(0). \tag{2.5}$$

The second concept which we need is the BPF which is given by the following definition.

Definition 2 ([9, 14, 15])**.** Let L be a natural number and η be a real number. Then, for $l \in \{0,1,\ldots,L-1\}$, the l^{th} BPF is given as

$$\varsigma_l(q) = \begin{cases} 1, & lz \le q < (l+1)z, \\ 0, & q \in [0,\eta) - [lz,(l+1)z) \end{cases} \tag{2.6}$$

with $z = \frac{\eta}{L}$.

We can write these functions in vector form as

$$\varsigma(q) = \begin{pmatrix} \varsigma_0(q) \\ \varsigma_1(q) \\ \ldots \\ \varsigma_{L-1}(q) \end{pmatrix}. \tag{2.7}$$

Here are two important properties of the BPF which are given as

$$\varsigma_{l_1}(q)\varsigma_{l_2}(q) = \begin{cases} \varsigma_{l_1}(q), & l_1 = l_2, \\ 0, & l_1 \neq l_2 \end{cases} \tag{2.8}$$

and

$$\int_0^{\eta} \varsigma_{l_1}(q)\varsigma_{l_2}(q)\mathrm{d}q = \begin{cases} z, & l_1 = l_2, \\ 0, & l_1 \neq l_2 \end{cases} \tag{2.9}$$

for $0 \leq l_1, l_2 \leq L-1$. The third property is the completeness which is given as follows.

Theorem 1. *If α is square integrable function on $[0,\eta)$, then*

$$\alpha(q) = \sum_{l_1=0}^{\infty} \alpha_{l_1}\varsigma_{l_1}(q), \tag{2.10}$$

where

$$\alpha_{l_1} = \frac{1}{z}\int_{l_1 z}^{(l_1+1)z} \alpha(q)\mathrm{d}q. \tag{2.11}$$

Proof. The result will follow directly from equations (2.8) and (2.9) after multiplying equation (2.10) by $\varsigma_{l_2}(q)$. □

If we take the first L terms of equations (2.10), then we can approximate $\alpha(q)$ as

$$\alpha(q) \approx \alpha\varsigma(q), \tag{2.12}$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{L-1})$.

3. Operational matrices for system of Riccati equations

Operational matrices are the main components of the proposed method. Hence, we start by deriving them in this section. Let us approximate the coefficient matrices and the function $G(t)$ of equation (1.1) in terms of the BPFs as

$$\Lambda_{ij}(t) = \sum_{l_1=0}^{L-1} \Lambda_{ij,l_1}\varsigma_{l_1}(t), \tag{3.1}$$

$$G(t) = \sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t), \tag{3.2}$$

where Λ_{ij,l_1} are $1 \times k_2$ matrices and G_{l_1} are $k_2 \times 1$ matrices for $i, j = 1, 2$. Since

$$\varsigma_{l_1}(0) = \begin{cases} 1, & l_1 = 0, \\ 0, & l_1 \neq 0, \end{cases}$$

then $G_0 = G(0) = Q$. Then, system (1.1)-(1.2) becomes

$$D^{\gamma}G(t) = \sum_{l_1=0}^{L-1} \Lambda_{21,l_1}\varsigma_{l_1}(t) + \left(\sum_{l_1=0}^{L-1} \Lambda_{22,l_1}\varsigma_{l_1}(t)\right)\left(\sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t)\right) - \left(\sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t)\right)\left(\sum_{l_1=0}^{L-1} \Lambda_{11,l_1}\varsigma_{l_1}(t)\right)$$

$$-\left(\sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t)\right)\left(\sum_{l_1=0}^{L-1}\Lambda_{12,l_1}\varsigma_{l_1}(t)\right)\left(\sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t)\right),$$

which can be written using equation (2.8) as

$$D^{\gamma}G(t)=\sum_{l_1=0}^{L-1}\Lambda_{21,l_1}\varsigma_{l_1}(t)+\sum_{l_1=0}^{L-1}\Lambda_{22,l_1}G_{l_1}\varsigma_{l_1}(t)$$
$$-\sum_{l_1=0}^{L-1}G_{l_1}\Lambda_{11,l_1}\varsigma_{l_1}(t)-\sum_{l_1=0}^{L-1}G_{l_1}\Lambda_{12,l_1}G_{l_1}\varsigma_{l_1}(t).$$

Using equation (2.5), we get

$$G(t)-G(0)=\sum_{l_1=0}^{L-1}\left(\Lambda_{21,l_1}+\Lambda_{22,l_1}G_{l_1}-G_{l_1}\Lambda_{11,l_1}-G_{l_1}\Lambda_{12,l_1}G_{l_1}\right)I^{\gamma}\varsigma_{l_1}(t). \tag{3.3}$$

For any $t\in[0,\eta)$, there exists non-negative integer l_1 such that $t\in[l_1z,(l_1+1)z)$. Thus,

$$\varsigma_{l_2}(t)=\begin{cases}1, & l_2=l_1,\\ 0, & l_2\neq l_1.\end{cases}$$

Then,

$$\sum_{l_1=0}^{L-1}\varsigma_{l_1}(t)=1. \tag{3.4}$$

Now, $G(0)=Q=G_0$ and can be written using equations (2.8) and (2.9) as

$$G(0)=G_0\sum_{l_1=0}^{L-1}\varsigma_{l_1}(t)=G_0. \tag{3.5}$$

Hence, equations (3.3) and (3.5) imply that

$$\sum_{l_1=0}^{L-1}\left(G_{l_1}-G_0I_{k_1}\right)\varsigma_{l_1}(t)=\sum_{l_1=0}^{L-1}\left(\Lambda_{21,l_1}+\Lambda_{22,l_1}G_{l_1}-G_{l_1}\Lambda_{11,l_1}-G_{l_1}\Lambda_{12,l_1}G_{l_1}\right)I^{\gamma}\varsigma_{l_1}(t). \tag{3.6}$$

Let

$$A_{11}=\begin{pmatrix}\Lambda_{11,0}\\ \Lambda_{11,1}\\ \cdots\\ \Lambda_{11,L-1}\end{pmatrix},\ A_{12}=\begin{pmatrix}\Lambda_{12,0}\\ \Lambda_{12,1}\\ \cdots\\ \Lambda_{12,L-1}\end{pmatrix},\ A_{21}=\begin{pmatrix}\Lambda_{21,0} & 0 & 0 & \ldots & 0\\ 0 & \Lambda_{21,1} & 0 & \ldots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \ldots & \Lambda_{21,L-1}\end{pmatrix},$$

$$A_{22}=\begin{pmatrix}\Lambda_{22,0}\\ \Lambda_{22,1}\\ \dots\\ \Lambda_{22,L-1}\end{pmatrix},\ \Omega(t)=\begin{pmatrix}\varsigma_0(t)\\ \varsigma_1(t)\\ \dots\\ \varsigma_{L-1}(t)\end{pmatrix},\ G=\begin{pmatrix}G_0\\ G_1\\ \dots\\ G_{L-1}\end{pmatrix},\ G_0=\begin{pmatrix}G_0\\ G_0\\ \dots\\ G_0\end{pmatrix}.$$

We define the operator $*$ on the matrix of block matrices as follows

$$A^*=\begin{pmatrix}A_0\\ A_1\\ \dots\\ A_{L-1}\end{pmatrix}^*=\begin{pmatrix}A_0 & A_1 & \dots & A_{L-1}\end{pmatrix}. \tag{3.7}$$

Then, equations (3.2) and (3.5) imply that

$$G(t)=G^*\Omega(t), G_0=G_0^*\Omega(t). \tag{3.8}$$

Therefore, we can rewrite equation (3.6) as

$$(G^*-G_0^*)\,\Omega(t)=(A_{21}^*+A_{22}^*G-G^*A_{11}-G^*A_{12}G)\,I^\gamma\Omega(t). \tag{3.9}$$

To complete the derivation of the proposed method, we need to compute the operational matrix of the operator $I^\gamma\omega(t)$ which is given in the following theorem.

Theorem 2. *The operational matrix of the operator $I^\gamma\Omega$ is*

$$\Pi=\frac{z^\alpha}{\Gamma(\gamma+2)}\begin{pmatrix}1 & \eta_1 & \eta_2 & \dots & \eta_{L-2} & \eta_{L-1}\\ 0 & 1 & \eta_1 & \dots & \eta_{L-3} & \eta_{L-2}\\ 0 & 0 & 1 & \dots & \eta_{L-4} & \eta_{L-3}\\ \dots & \dots & \dots & \ddots & \ddots & \dots\\ 0 & 0 & 0 & \dots & 1 & \eta_1\\ 0 & 0 & 0 & \dots & 0 & 1\end{pmatrix}, \tag{3.10}$$

where $\eta_r=(r+1)^{\gamma+1}-2r^{\gamma+1}+(r-1)^{\gamma+1}$, $r=1,2,...,L-1$.

Proof. Let $r\in\{0,1,...,L-1\}$. Then,

$$I^\gamma\varsigma_r(t)=\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}\varsigma_r(s)ds=\begin{cases}0, & t<rz,\\ \frac{(t-rz)^\gamma}{\Gamma(\gamma+1)}, & rz\le t<(r+1)z,\\ \frac{(t-rz)^\gamma-(t-rz-z)^\gamma}{\Gamma(\gamma+1)}, & (r+1)z\le t<\eta.\end{cases}$$

If

$$I^\gamma\varsigma_r(t)=\sum_{l1=0}^{L-1}a_{l_1,r}\varsigma_r(t),$$

then

$$a_{l_1,r}=\frac{1}{z}\int_0^\eta\left(I^\gamma\varsigma_{l_1}(s)\right)\varsigma_r(s)\mathrm{d}s=\frac{1}{z}\int_{rz}^{(r+1)z}\left(I^\gamma\varsigma_{l_1}(s)\right)\mathrm{d}s$$

$$= \begin{cases} \frac{z^{\gamma}}{\Gamma(\gamma+2)}, & 0 \le l_1 = r \le L-1, \\ \frac{z^{\gamma}\left(r-l_1+1\right)^{\gamma+1}-2(r-l_1)^{\gamma+1}+(r-l_1-1)^{\gamma+1}\right)}{\Gamma(\gamma+2)}, & 0 \le l_1 < r \le L-1, \\ 0, & 0 \le r < l_1 \le L-1, \end{cases}$$

which completes the proof of the theorem. □

Now, equations (3.9) and (3.10) imply that

$$(G^* - G_0^*)\,\Omega(t) = (A_{21}^* + A_{22}^* G - G^* A_{11} - G^* A_{12} G)\,\Pi\Omega(t). \tag{3.11}$$

Since $\{\varsigma_0(t), \varsigma_1(t), \ldots, \varsigma_{L-1}(t)\}$ are linearly independent, then

$$(A_{21}^* + A_{22}^* G - G^* A_{11} - G^* A_{12} G)\,\Pi + G_0^* - G^* = 0. \tag{3.12}$$

We use Mathematica to solve the algebraic system (3.12).

4. Convergence Analysis

Let $G\colon [0,\eta) \longrightarrow \Re^{k_2\times 1}$ be a square integrable function with $\eta \ge 1$. Define the following norm

$$\|G\| = \sqrt{\int_0^{\eta} \|G\|_E^2 \mathrm{d}t}, \tag{4.1}$$

where $\|G\|_E$ is the Euclidean norm. Using equation (3.2), we have

$$G_L(t) = \sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t) \tag{4.2}$$

For the first theorem, we want to prove that the mean squared error reaches a minimum when G_{l_1} is given by

$$G_{l_1} = \frac{1}{z}\int_0^{\eta} G(s)\varsigma_{l_1}(s)\mathrm{d}s. \tag{4.3}$$

Theorem 3. *Let G be a square integrable on $[0,\eta)$. Then, the least squares error*

$$E(G_0, G_1, \ldots, G_{L-1}) = \int_0^{\eta} \|G(s) - G_L(s)\|_E^2 \mathrm{d}s$$

achieves its minimum value at G_{l_1} given by formula (4.3) for $l_1 = 0, 1, \ldots, L-1$.

Proof. For any $l_1 \in \{0, 1, \ldots, L-1\}$, we have

$$\frac{\partial E}{\partial G_{l_1}} = -2\int_0^{\eta} (G(s) - G_L(s))\varsigma_{l_1}(s)\mathrm{d}s = -2\left(\int_0^{\eta} G(s)\varsigma_{l_1}(s)\mathrm{d}s - zG_{l_1}\right) = 0.$$

Hence,

$$G_{l_1} = \frac{1}{z}\int_0^{\eta} G(s)\varsigma_{l_1}(s)\mathrm{d}s.$$

Equations (2.8) and (2.9) imply that

$$\frac{\partial^2 E}{\partial G_{l_1}\partial G_{l_2}} = 2I_{k_2}\int_0^{\eta} \varsigma_{l_1}(s)\varsigma_{l_2}(s)\mathrm{d}s = \begin{cases} 2zI_{k_2}, & l_1 = l_2 \\ 0, & l_1 \neq l_2 \end{cases}, 0 \le l_1, l_2 \le L-1.$$

Thus, if $0 \le l_1 \le L-1$, we have

$$\begin{vmatrix} \frac{\partial^2 E}{\partial G_0^2} & \frac{\partial^2 E}{\partial G_0 \partial G_1} & \cdots & \frac{\partial^2 E}{\partial G_0 \partial G_{l_1}} \\ \frac{\partial^2 E}{\partial G_1 \partial G_0} & \frac{\partial^2 E}{\partial G_1^2} & \cdots & \frac{\partial^2 E}{\partial G_1 \partial G_{l_1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 E}{\partial G_{l_1} \partial G_0} & \frac{\partial^2 E}{\partial G_{l_1} \partial G_1} & \cdots & \frac{\partial^2 E}{\partial G_{l_1}^2} \end{vmatrix} = \begin{vmatrix} 2zI_{k_2} & 0 & \cdots & 0 \\ 0 & 2zI_{k_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 2zI_{k_2} \end{vmatrix} = (2z)^{l_1+1} > 0.$$

Thus, E achieves its minimum value at G_{l_1} given by formula (4.3) for $l_1 = 0, 1, \ldots, L-1$. □

Theorem 4. *If $G(s)$ is bounded continuous function in $[0,\eta)$, then $\{G_{l_1}(s)\}$ converges pointwise to $G(s)$ on $[0,\eta)$. In addition,*

$$\int_0^{\eta} \|G(s)\|_E^2 \mathrm{d}s = \sum_{l_1=0}^{\infty} \|G_{l_1}\|_E^2 \|\varsigma_{l_1}\|^2.$$

Proof. Theorem 2 implies that

$$G_{l_1} = \frac{L}{\eta}\int_{\frac{l_1\eta}{L}}^{\frac{(l_1+1)\eta}{L}} G(s)\mathrm{d}s, l_1 = 0, 1, \ldots, L-1. \tag{4.4}$$

For $l_1 = 0$, partition the interval $[0, \frac{\eta}{L}]$ into j uniform subintervals of length a. Then,

$$G_0 = \frac{L}{\eta}\int_0^{\frac{\eta}{L}} G(s)\mathrm{d}s = \frac{L}{\eta}\lim_{a\to 0}\sum_{l_1=0}^{j-1} G(s_{l_1})a = \frac{L}{\eta}\lim_{a\to 0}\sum_{l_1=0}^{j-1} G(l_1 a)\frac{\eta}{jL} = \lim_{a\to 0}\sum_{l_1=0}^{j-1}\frac{G(l_1 a)}{j}.$$

The continuity of the function G yields to

$$\lim_{L\to\infty} G_0 = \lim_{L\to\infty}\lim_{a\to 0}\sum_{l_1=0}^{j-1}\frac{G(l_1 a)}{j} = \lim_{a\to 0}\sum_{l_1=0}^{j-1}\frac{G(\lim_{L\to\infty}\frac{l_1\eta}{Lj})}{j} = \lim_{a\to 0}\sum_{l_1=0}^{j-1}\frac{G(0)}{j} = G(0).$$

Similarly, if $s_{l_1} \in [0,\eta)$, then

$$\lim_{L\to\infty} G_{l_1} = G(s_{l_1}).$$

Hence, $\{G_L(s)\}$ converges pointwise to $G(s)$ on $[0,\eta)$. Therefore,

$$\int_0^{\eta} \|G(s)\|_E^2 \mathrm{d}s = \lim_{L\to\infty}\int_0^{\eta} \|G_L\|_E^2 \mathrm{d}s = \lim_{L\to\infty}\sum_{l_1=0}^{L-1} \|G_{l_1}\|_E^2 \int_0^{\eta} \varsigma_{l_1}(s)^2 \mathrm{d}s$$

$$= \sum_{l_1=0}^{\infty} \|G_{l_1}\|_E^2 \|\varsigma_{l_1}\|^2.$$

□

In the next theorem, the order of the mean square error is investigated.

Theorem 5. *If $\xi > 0$ and $G(t)$ is differentiable function on $[0,\eta)$ with*

$$\|G'(t)\|_E \leq \xi, t \in [0,\eta), \tag{4.5}$$

then

$$\|G - G_L\|_E^2 \leq \alpha z^2. \tag{4.6}$$

Proof. If $s_{l_1} = l_1 z$, then the mean value theorem for integrals gives us

$$G_L(s) = G_{l_1}, s \in [s_{l_1}, s_{l_1+1}) \tag{4.7}$$

$$= \frac{1}{z}\int_{s_{l_1}}^{s_{l_1+1}} G(s)\mathrm{d}s \tag{4.8}$$

$$= G(\zeta_{l_1}), \zeta_{l_1} \in [s_{l_1}, s_{l_1+1}), l_1 = 0,1,\ldots,L-1. \tag{4.9}$$

Mean value theorem for integrals implies that

$$\|G(t) - G_L(t)\|^2 = \int_0^{\eta} \|G(s) - G_L(s)\|_E^2 \mathrm{d}s = \sum_{l_1=0}^{L-1} \int_{s_{l_1}}^{s_{l_1+1}} \|G(s) - G_L(s)\|_E^2 \mathrm{d}s$$

$$= \sum_{l_1=0}^{L-1} \int_{s_{l_1}}^{s_{l_1+1}} \|G(s) - G(\zeta_{l_1})\|_E^2 \mathrm{d}s = z\sum_{l_1=0}^{L-1} \|G(\omega_{l_1}) - G(\zeta_{l_1})\|_E^2,$$

where $\omega_{l_1}, \zeta_{l_1} \in [s_{l_1}, s_{l_1+1}), l_1 = 0,1,\ldots,L-1$. Therefore, mean value theorem yields to

$$\|G(t) - G_L(t)\|^2 \leq z\xi^2 \sum_{l_1=0}^{L-1} (\omega_{l_1} - \zeta_{l_1})^2 \leq z\xi^2 \sum_{l_1=0}^{L-1} z^2 = \alpha z^2,$$

where $\alpha = \xi^2\eta$. □

In the next theorem we study the pointwise convergence of the product of two sequences.

Theorem 6. *Let $G_1(t), G_2(t)$, and $G_3(t)$ be continuous bounded functions in $[0,\eta)$ such that G_1G_2 and $G_1G_2G_3$ are well-defined. Then,*

1: *$\{G_{1,L}(t)G_{2,L}(t)\}$ converges pointwise to $G_1(t)G_2(t)$,*
2: *$\{G_{1,L}(t)G_{2,L}(t)G_{3,L}(t)\}$ converges pointwise to $G_1(t)G_2(t)G_3(t)$.*

Proof. Let $G_1(t)$ and $G_2(t)$ be continuous bounded functions in $[0,\eta)$. Then, there exists $\xi_1, \xi_2 \in \mathbb{R}$ such that

$$\|G_1(t)\| \leq \xi_1, \|G_2(t)\| \leq \xi_2. \tag{4.10}$$

Let $0 < \alpha \leq \{3\xi_1\xi_2\}$. Theorem 4 implies that $\{G_{1,L}(t)\}$ and $\{G_{2,L}(t)\}$ converges pointwise to $G_1(t)$ and $G_2(t)$, respectively. Thus, there exists positive integer $M(\alpha)$ such that

$$\|G_{1,L}(t) - G_1(t)\| < \frac{\alpha}{3\xi_2}, \|G_{2,L}(t) - G_2(t)\| < \frac{\alpha}{3\xi_1}, L \geq M(\alpha).$$

Then, for $L \geq M(\alpha)$, we have

$$\begin{aligned}\|G_{1,L}(t)G_{2,L}(t) - G_1(t)G_2(t)\| &\leq \|(G_{1,L}(t) - G_1(t))(G_{2,L}(t) - G_2(t))\| \\ &\quad + \|G_1(t)(G_{2,L}(t) - G_2(t))\| \\ &\quad + \|G_2(t)(G_{1,L}(t) - G_1(t))\| \\ &\leq \frac{\alpha^2}{9\xi_1\xi_2} + \frac{\xi_1\alpha}{3\xi_1} + \frac{\xi_2\alpha}{3\xi_2} \leq \alpha. \end{aligned} \tag{4.11}$$

Hence, $\{G_{1,L}(t)G_{2,L}(t)\}$ converges pointwise to $G_1(t)G_2(t)$. Part (2) follows directly from the first part. □

Now we prove the convergence theorem for problem (1.1)-(1.2).

Theorem 7. *Let $G(t)$ and $G_L(t)$ be the exact and approximate solutions of equations (1.1) and (1.2). If $G(t)$ is continuous bounded function in $[0,\eta)$, then, $\{G_L(t)\}$ converges pointwise to $G(t)$.*

Proof. Let $G(t)$ and $G_L(t)$ be the exact and approximate solutions of equations (1.1) and (1.2), respectively. Then,

$$\begin{aligned} D^{\gamma}G(t) &= \Lambda_{21} + \Lambda_{22}G(t) - G(t)\Lambda_{11} - G(t)\Lambda_{12}G(t), \\ D^{\gamma}G_L(t) &= \Lambda_{21} + \Lambda_{22}G_L(t) - G_L(t)\Lambda_{11} - G_L(t)\Lambda_{12}G_L(t). \end{aligned}$$

By taking the fractional integral operator for both sides of the last two equations, we get

$$\begin{aligned} G(t) - Q &= \frac{1}{\Gamma(\gamma)}\int_0^t (\Lambda_{21} + \Lambda_{22}G(s) - G(s)\Lambda_{11} - G(s)\Lambda_{12}G(s))\,\mathrm{d}s, \\ G_L(t) - Q &= \frac{1}{\Gamma(\gamma)}\int_0^t (\Lambda_{21} + \Lambda_{22}G_L(s) - G_L(s)\Lambda_{11} - G_L(s)\Lambda_{12}G_L(s))\,\mathrm{d}s. \end{aligned}$$

Then,

$$\begin{aligned} G(t) - G_L(t) = \frac{1}{\Gamma(\gamma)}\int_0^t &(\Lambda_{22}(G(s) - G_L(s)) - (G(s) - G_L(s))\Lambda_{11} \\ &- (G(s)\Lambda_{12}G(s) - G_L(s)\Lambda_{12}G_L(s)))\mathrm{d}s. \end{aligned}$$

Then,

$$\begin{aligned} \|G - G_L\|_E &\leq \frac{1}{\Gamma(\gamma)}\int_0^t (\|G - G_L\|_E(\|\Lambda_{22}\|_E + \|\Lambda_{11}\|_E) + \|G\Lambda_{12}G - G_L\Lambda_{12}G_L\|_E)\,\mathrm{d}s \\ &= \frac{\eta}{\Gamma(\gamma)}(\|G - G_L\|_E(\|\Lambda_{22}\|_E + \|\Lambda_{11}\|_E) + \|G\Lambda_{12}G - G_L\Lambda_{12}G_L\|_E). \end{aligned}$$

We can simplify the last equation as

$$\|G - G_L\|_E \leq \frac{\upsilon\eta}{\Gamma(\gamma)}(\|G\Lambda_{12}G - G_L\Lambda_{12}G_L\|_E),$$

where $\upsilon = \frac{\eta}{\Gamma(\gamma)-\eta\|\Lambda_{22}\|_E-\eta\|\Lambda_{11}\|_E}$. Using the second part of Theorem 6, for any $\xi > 0$, there exists $N(\xi) \in \mathbb{N}$ such that

$$\|G\Lambda_{12}G - G_L\Lambda_{12}G_L\|_E \leq \xi\frac{\Gamma(\gamma)}{\upsilon\eta}, L \geq N(\xi).$$

Therefore, $\|G - G_L\|_E \leq \xi$, which gives the result of the theorem. □

5. Numerical results

In this section, two examples will be given to show the efficiency of the operational matrix method. In this section, we use $L = 20$.

Example 1. Consider the following system of Riccati equations

$$D^\gamma G(t) = \Lambda_{21} + \Lambda_{22}G(t) - G(t)\Lambda_{11} - G(t)\Lambda_{12}G(t), G(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

where $h_1(t) = 2t^\gamma - 2t^{2\gamma} + 1, h_2(t) = t^\gamma - 4t^{2\gamma} + t^{3\gamma} - 1, \Lambda_{22} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \Lambda_{11} = 2,$

$$\Lambda_{21} = \begin{pmatrix} 2\Gamma(\gamma+1) - \frac{2\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}t^\gamma + h_1^2(t) + 2h_1(t)h_2(t) - h_2(t) \\ \frac{\Gamma(3\gamma+1)}{\Gamma(2\gamma+1)}t^{2\gamma} - \frac{4\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}t^\gamma + \Gamma(\gamma+1) + 2h_2^2(t) + h_1(t)h_2(t) - 3h_1(t) \end{pmatrix},$$

$\Lambda_{12} = \begin{pmatrix} 1 & 2 \end{pmatrix}$. Then, the exact solution is $G(t) = \begin{pmatrix} 2t^\gamma - 2t^{2\gamma} + 1 \\ t^\gamma - 4t^{2\gamma} + t^{3\gamma} - 1 \end{pmatrix}$.

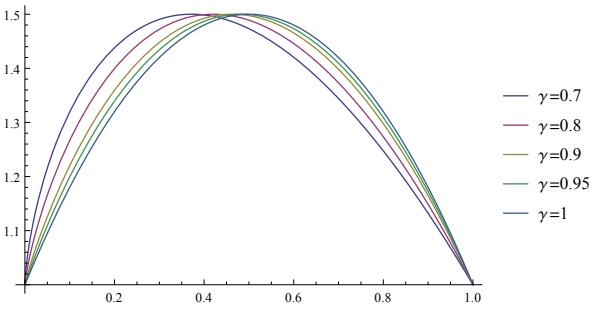

FIGURE 1. The approximate solution $G_1(t)$ for different values of γ

Then, the approximate solutions of $G_1(t)$ and $G_2(t)$ for different values of γ are given in Figures 1 and 2, respectively.

Let us define the L_2-error as follows:

$$\phi(\gamma) = \sqrt{\int_0^1 \|G(t) - G_{app}(t)\|_E^2 \mathrm{d}t}.$$

Then, the L_2-error for $\gamma = 0.5, 0.6, 0.7, 0.8., 0.9$ and 1 are reported in Table 1.

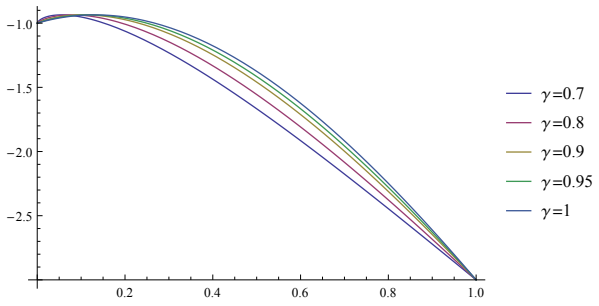

FIGURE 2. The approximate solution $G_2(t)$ for different values of γ

γ	$\phi(\gamma)$
0.5	$1.23*10^{-13}$
0.6	$2.34*10^{-13}$
0.7	$2.87*10^{-13}$
0.8	$3.16*10^{-13}$
0.9	$3.30*10^{-13}$
1	$3.98*10^{-13}$

TABLE 1. The L_2-error for different values of γ for Example 1

Example 2. Consider the following system of Riccati equations

$$D^{\gamma}G(t) = \Lambda_{21} + \Lambda_{22}G(t) - G(t)\Lambda_{11} - G(t)\Lambda_{12}G(t), G(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\Lambda_{22} = \begin{pmatrix} t^{\gamma} & 0 \\ 0 & t^{\gamma} \end{pmatrix}, \Lambda_{11} = t^{2\gamma}, \Lambda_{12} = \begin{pmatrix} -t^{\gamma} & t^{\gamma} \end{pmatrix},$

$$\Lambda_{21} = \begin{pmatrix} \frac{\Gamma(3\gamma+1)}{\Gamma(2\gamma+1)}t^{2\gamma} - t^{7\gamma} + 2t^{5\gamma} \\ \Gamma(\gamma+1) - t^{5\gamma} - t^{4\gamma} + 2t^{3\gamma} + 2t^{2\gamma} + t^{\gamma} - 1 \end{pmatrix}.$$

Then, the exact solution is $G(t) = \begin{pmatrix} t^{3\gamma} \\ t^{\gamma}+1 \end{pmatrix}$. Then, the approximate solutions of $G_1(t)$ and $G_2(t)$ for different values of γ are given in Figures 3 and 4, respectively. Now, the L_2-error for $\gamma = 0.5, 0.6, 0.7, 0.8., 0.9$ and 1 are reported in Table 2.

6. CONCLUSION AND CLOSING REMARKS

In this paper, we investigate the solution of the fractional system of Riccati equations. This system is important since it appears in several applications in science such as algebraic geometry, conformal mapping theory, and control theory. We use the operational matrix method to solve this system. The block-pulse operational

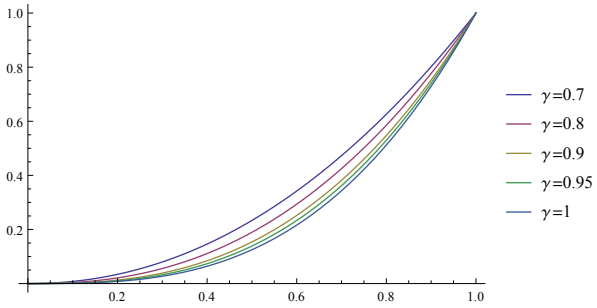

FIGURE 3. The approximate solution $G_1(t)$ for different values of γ

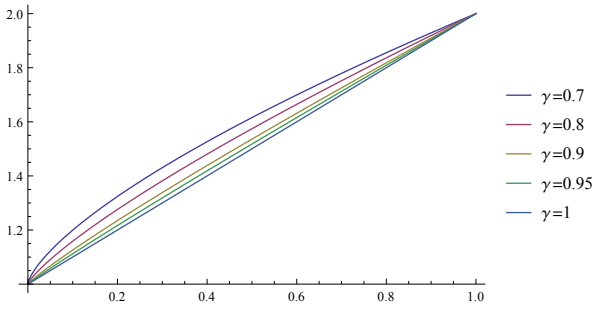

FIGURE 4. The approximate solution $G_2(t)$ for different values of γ

γ	$\phi(\gamma)$
0.5	$5.61 * 10^{-14}$
0.6	$5.69 * 10^{-14}$
0.7	$5.93 * 10^{-14}$
0.8	$6.01 * 10^{-14}$
0.9	$6.21 * 10^{-14}$
1	$6.40 * 10^{-14}$

TABLE 2. The L_2 error for different values of γ for Example 2

matrices will initially assist in the reduction of the nonlinear fractional order Riccati-differential problem into an algebraic system. Benefits of this approach include inexpensive setup costs for the equations without the use of projection techniques like Galerkin, collocation, etc. In addition, we prove the convergence of the approximate solution using the operational matrix method to the exact solution. We arrive to this conclusion after proving a series of theorems in Section four. Finally, we present

two examples to provide the reader with numerical evidences to the efficiency of numerical approach used in this paper. Based on these two examples, we notice the following.

1: The error is within 10^{-13} in Example 1 and within 10^{-14} in Example 2 as in Tables 1 and 2, respectively.

2: We notice that the approximate solutions are convergent to the exact solutions for different values of γ as reported in Tables 1 and 2.

3: We notice that the approximate solutions approach to the solution when $\gamma = 1$ as γ approaches to one as in Figures 1-4.

4: The operational matrix method works very efficiently for this type of problems. It is a useful tool to solve nonlinear fractional equations. We recommend this numerical approach for other problems in physics and engineering such as a system of integro-differential equations.

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