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SOME HERMITE–HADAMARD AND OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES VIA NEW GENERALIZED m-TYPE CONVEXITY

ABDUL WAKIL BAIDAR AND MEHMET KUNT

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Abstract. A new generalized class of m-type convex functions is presented. Some Hermite–Hadamard and Ostrowski type inequalities via Riemann–Liouville fractional integrals are obtained for these functions. Some particular cases are discussed as applications of the findings acquired in this study.

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1. INTRODUCTION

Let Δ be an interval of real numbers. If the following inequality

$$\varphi\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\varphi(\tau)\,d\tau \leq \frac{\varphi(\nu)+\varphi(\mu)}{2} \tag{1.1}$$

holds for all $\mu,\nu \in \Delta$ with $\mu < \nu$, then φ is said to be a convex function on Δ. In the literature, inequality (1.1) is famed as the Hermite–Hadamard inequality for convex functions (see [13]).

A number of researchers have used fractional integrals, fractional quantum integrals, and other forms of integrals to establish the above inequality for various classes of convex functions. Sarikaya et al. [16], proposed the fractional integral version of (1.1), which is as follows

$$\varphi\left(\frac{\mu+\nu}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right) \leq \frac{\varphi(\nu)+\varphi(\mu)}{2}. \tag{1.2}$$

In addition, depending on (1.2), the authors have presented a trapezoid type inequality for convex functions in [16]. In the previous decade, many authors generalized inequality (1.2) and used it to derive a number of well-known fractional integral inequalities for different types of convex functions. For further detail on this, we refer the reader to [1,3,4,8,12,14,17,19–21,24,26–28].

G. Toader [22], defined a new class of m-convex functions, which gave rise to the definition of other classes of convex functions such as (α,m)-convex functions, (s,m)-convex functions and etc. [2,5,6,9,15,18] contains more information on extended classes of m-convex functions as well as certain well-known fractional integral inequalities for these functions, such as Hermite–Hadamard, Ostrowski type, and so forth.

Kashuri et al. [7], recently introduced a new generalized class of convex functions as well as several integral inequalities for these functions, leading to a number of known and unknown findings in the area of the modern theory of inequalities.

In this study, we define a new generalized class of m-type convex functions and derive some Hermite–Hadamard and Ostrowski type inequalities for these functions, based on the results and approaches in [7,16,18,22]. The previously known findings for numerous classes of convex functions as a particular case are included in our results.

2. Preliminaries

In this part, we revisit several previous definitions and lemmas which help us in proving our results.

Definition 1 ([13])**.** Let Δ be an interval of real numbers. A function $\varphi\colon \Delta \to \mathbb{R}$ is said to be convex on Δ if

$$\varphi(\mu\tau + (1-\tau)\nu) \leq \mu\varphi(\tau) + (1-\tau)\varphi(\nu) \tag{2.1}$$

holds for all $\mu,\nu \in \Delta$, and $\tau \in [0,1]$.

Ω represents the convex set $\{\psi_1,\psi_2,\dots,\psi_k,\phi_1,\phi_2,\dots,\phi_l\}$ in the following definition, (see [7]).

Definition 2 ([7])**.** Let $\psi_{r_1},\phi_{r_2}\colon [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\dots,k$ and $r_2 = 1,2,\dots,l$ where $k,l \in \mathbb{N}$ and $1 \leq k \leq l$. If a function $\varphi\colon \Delta \subseteq \mathbb{R} \to [0,\infty)$ satisfies the following inequality

$$\varphi(\mu\tau + (1-\tau)\nu) \leq \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi(\mu) + \left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi(\nu) \tag{2.2}$$

for all $\mu,\nu \in \Delta$ and $\tau \in [0,1]$, then it is said to be (k,l)-generalized convex with respect to Ω.

Definition 3 ([22])**.** Let $b > 0$. A function $\varphi\colon [0,b] \to \mathbb{R}$ is called m-convex on $[0,b]$ if

$$\varphi(\tau\mu + m(1-\tau)\nu) \leq \tau\varphi(\mu) + m(1-\tau)\varphi(\nu) \tag{2.3}$$

holds for all $\mu,\nu \in [0,b]$, and $\tau \in [0,1]$.

Definition 4 ([5])**.** Let $(s,m) \in (0,1]^2$ and $b > 0$. A function $\varphi\colon [0,b] \to \mathbb{R}$ is said to be (s,m)-convex in the second sense on $[0,b]$ if

$$\varphi(\mu\tau + m(1-\tau)\nu) \leq \tau^s \varphi(\mu) + m(1-\tau)^s \varphi(\nu) \tag{2.4}$$

holds for all $\mu,\nu \in [0,b]$ and $\tau \in [0,1]$.

Definition 5 ([9])**.** Let $(\alpha,m) \in (0,1]^2$ and $b > 0$. A function $\varphi\colon [0,b] \to \mathbb{R}$ is said to be (α,m)-convex on $[0,b]$ if

$$\varphi(\mu\tau + m(1-\tau)\nu) \leq \tau^\alpha \varphi(\mu) + m(1-\tau^\alpha)\varphi(\nu) \tag{2.5}$$

is true for all $\mu,\nu \in \Delta$ and $\tau \in [0,1]$.

Definition 6 ([14])**.** Let $s \in [0,1]$, $n \in \mathbb{N}$ and Δ is an interval of real numbers. A function $\varphi\colon \Delta \to \mathbb{R}$ is called n-polynomial s-type convex if

$$\varphi(\mu\tau + (1-\tau)\nu) \leq \frac{1}{k}\sum_{r=1}^{k}\left[1-(s(1-\tau))^r\right]\varphi(\mu) + \frac{1}{k}\sum_{r=1}^{k}\left[1-(s\tau)^r\right]\varphi(\nu) \tag{2.6}$$

holds for all $\mu,\nu \in \Delta$ and $\tau \in [0,1]$.

Definition 7 ([25])**.** Let $h\colon [0,1] \to \mathbb{R}$ be a positive function and Δ is an interval of real numbers. A function $\varphi\colon \Delta \subseteq [0,\infty) \to \mathbb{R}$ is said to be h-convex, if

$$\varphi(\mu\tau + (1-\tau)\nu) \leq h(\tau)\varphi(\mu) + h(1-\tau)\varphi(\nu) \tag{2.7}$$

holds for all $\mu,\nu \in \Delta$ and $\tau \in [0,1]$.

In the following, we state the definition of Riemann–Liouville integrals, which are used in this study. For further information, one can see [10].

Definition 8. Assume that $\mu \geq 0$, $\alpha > 0$ and $\varphi \in L[\mu,\nu]$. The Riemann–Liouville integrals $J^\alpha_{\mu^+}\varphi$ and $J^\alpha_{\nu^-}\varphi$ of order α are defined by

$$J^\alpha_{\mu^+}\varphi(\xi) = \frac{1}{\Gamma(\alpha)}\int_\mu^\xi (\xi-\tau)^{\alpha-1}\varphi(\tau)\,d\tau\,,\ \xi > \mu$$

and

$$J^\alpha_{\nu^-}\varphi(\xi) = \frac{1}{\Gamma(\alpha)}\int_\xi^\nu (\tau-\xi)^{\alpha-1}\varphi(\tau)\,d\tau\,,\ \xi < \nu$$

respectively, where $\Gamma(\alpha) = \int_0^1 e^\tau \tau^{\alpha-1}\ d\tau$ and $J^0_{\mu^+}\varphi(\xi) = J^0_{\nu^-}\varphi(\xi) = \varphi(\xi)$.

The fractional integral is reduced to the classical integral when $\alpha = 1$.

Lemma 1 ([16])**.** *Assume that $\varphi\colon \Delta \to \mathbb{R}$ is a differentiable function on Δ°. If $\varphi' \in L[\mu,\nu]$, where $\mu,\nu \in \Delta$ with $\mu < \nu$, then the following equality*

$$\frac{\varphi(\mu)+\varphi(\nu)}{2} - \frac{\Gamma(1+\alpha)}{2(\nu-\mu)^\alpha}\left(J^\alpha_{\mu^+}\varphi(\nu) + J^\alpha_{\nu^-}\varphi(\mu)\right)$$

$$= \frac{(\nu-\mu)}{2}\int_0^1 \left[(1-\tau)^{\alpha}-\tau^{\alpha}\right]\varphi'(\mu\tau+(1-\tau)\nu)\,d\tau \tag{2.8}$$

is true for all $\alpha > 0$.

Lemma 2 ([18]). *Assume that* $\varphi\colon \Delta \to \mathbb{R}$ *is a differentiable function on* Δ°. *If* $\varphi' \in L[\mu,\nu]$, *where* $\mu,\nu \in \Delta$ *with* $\mu<\nu$, *then the following equality*

$$\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu+}\varphi(\nu)+J^{\alpha}_{\nu-}\varphi(\mu)\right) \tag{2.9}$$

$$=\frac{1}{\nu-\mu}\left((\xi-\mu)^{\alpha+1}\int_0^1\tau^{a}\varphi'(\tau\xi+(1-\tau)\mu)\,d\tau-(\nu-\xi)^{\alpha+1}\int_0^1\tau^{a}\varphi'(\tau\xi+(1-\tau)\nu)\,d\tau\right)$$

is valid for all $\xi \in [\mu,\nu]$ *and* $\alpha>0$.

The Euler beta and incomplete beta functions are defined by

$$\beta(\omega,\rho)=\int_0^1\tau^{\omega-1}(1-\tau)^{\rho-1}\,d\tau,$$

$$\beta(\xi;\omega,\rho)=\int_0^{\xi}\tau^{\omega-1}(1-\tau)^{\rho-1}\,d\tau,$$

respectively, where $\omega,\rho>0$ and $0<\xi<1$, (see [11]).

Now, we merge Definition 3 and 2 to obtain the class of generalized m-type convex functions as follows.

Definition 9. Let $\psi_{r_1},\phi_{r_2}\colon [0,1]\to[0,\infty)$ are continuous functions for all $r_1=1,2,\dots,k$ and $r_2=1,2,\dots,l$ where $k,l\in\mathbb{N}$ and $1\le k\le l$. If a function $\varphi\colon[0,b]\to[0,\infty)$, $b>0$, satisfies the following inequality

$$\varphi(\mu\tau+m(1-\tau)\nu)\le\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi(\mu)+m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi(\nu) \tag{2.10}$$

for all $\mu,\nu\in[0,b]$; $\tau\in[0,1]$, and $m\in(0,1]$, then it is said to be (k,l)-generalized m-type convex with respect to Ω.

Remark 1.

(1) Definition 9 is reduced to Definition 3 when $\psi_{r_1}(\tau)=\tau$, $\phi_{r_2}=1-\tau$ and $k=l=1$ in (2.10), and $\varphi\colon[0,b]\to[0,\infty)$, $b>0$.
(2) Definition 9 is reduced to Definition 4 when $\psi_{r_1}(\tau)=\tau^{s}$, $\phi_{r_2}=(1-\tau)^{s}$, and $k=l=1$ in (2.10), and $\varphi\colon[0,b]\to[0,\infty)$, $b>0$.
(3) Definition 9 is reduced to Definition 5 when $\psi_{r_1}(\tau)=\tau^{\alpha},\phi_{r_2}=1-\tau^{\alpha}$ and $k=l=1$ in (2.10), and $\varphi\colon[0,b]\to[0,\infty)$, $b>0$.
(4) Definition 9 is reduced to Definition 6 when $\psi_{r_1}(\tau)=1-(s(1-\tau))^{r_1}$, $\phi_{r_2}=1-(s\tau)^{r_2}$, $m=1$ and $r_1=r_2=1,2,\dots,k$ in (2.10), $\varphi\colon[0,b]\to[0,\infty)$, $b>0$.
(5) Definition 9 is reduced to Definition 7 when $\psi_{r_1}(\tau)=h(\tau)$, $\phi_{r_2}=h(1-\tau)$ and $m=k=l=1$ in (2.10).

By appropriately selecting the functions ψ_{r_1} and ϕ_{r_2}, the reader can obtain a number of known and unknown classes of m-convex functions.

3. Main results

In this section, we describe some algebraic properties of generalized m-type convex functions and investigate fractional integral inequalities of the Hermite–Hadamard and Ostrowski types for these functions.

Theorem 1. *Let $\varphi, \varphi_1, \varphi_2 \colon [0,b] \to [0,\infty)$, $b>0$, are (k,l)-generalized m-type convex functions with respect to Ω. Then*

(1) *$\varphi_1 + \varphi_2$ is a (k,l)-generalized m-type convex with respect to Ω.*
(2) *$\gamma\varphi$ is a $(k,l))$-generalized m-type convex with respect to Ω, where γ is a positive real number.*

Proof. We ignore the proof of the theorem since it is straightforward. □

Theorem 2. *Let $\psi_{r_1}, \phi_{r_2} \colon [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\ldots,k$ and $r_2 = 1,2,\ldots,l$ where $k,l \in \mathbb{N}$ and $1 \le k \le l$ and assume that $\varphi_2 \colon [0,b] \to [0,\infty)$, $b>0$, is an increasing and (k,l)-generalized m-type convex function with respect to Ω and $\varphi_1 : \varphi_2([0,b]) \to [0,\infty)$ is an m-convex function, then $\varphi_1 \circ \varphi_2$ is a (k,l)-generalized m-type convex function with respect to Ω.*

Proof. The m-convexity of φ_2 and (k,l)-generalized m-type convexity of φ_1 give that

$$\begin{aligned}
\varphi_1 \circ \varphi_2 (\mu\tau + m(1-\tau)\nu) &= \varphi_1 (\varphi_2 (\mu\tau + m(1-\tau)\nu)) \\
&\le \varphi_1 (\tau\varphi_2(\mu) + m(1-\tau)\varphi_2(\nu)) \\
&\le \left(\frac{1}{k}\sum_{r_1=1}^{k} \psi_{r_1}(\tau)\right) \varphi_1(\varphi_2(\mu)) + m\left(\frac{1}{l}\sum_{r_2=1}^{l} \phi_{r_2}(\tau)\right)\varphi_1(\varphi_2(\nu)) \\
&= \left(\frac{1}{k}\sum_{r_1=1}^{k} \psi_{r_1}(\tau)\right) \varphi_1\circ\varphi_2(\mu) + m\left(\frac{1}{l}\sum_{r_2=1}^{l} \phi_{r_2}(\tau)\right)\varphi_1\circ\varphi_2(\nu)
\end{aligned}$$

is true for all $\mu,\nu \in [0,b]$ and $\tau \in [0,1]$. This ends the proof. □

Theorem 3. *Assume that $\psi_{r_1}, \phi_{r_2} \colon [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\ldots,k$ and $r_2 = 1,2,\ldots,l$ where $k,l \in \mathbb{N}$ and $1 \le k \le l$ and let $(\varphi_i)_{i\in I}$ be a family, where $\varphi_i \colon [0,b] \to [0,\infty)$, $b>0$, is a (k,l)-generalized m-type convex functions with respect to Ω for any $i \in I$.Then $\varphi(\omega) := \sup_i \varphi_i(\omega)$ is a (k,l)-generalized m-type convex function with respect to Ω and $\Upsilon = \{\omega \in [0,b] \colon \varphi(\omega) < +\infty\}$ is an interval.*

Proof. The (k,l)-generalized m-type convexity of φ_i with respect to Ω for any $i \in I$, give that

$$\varphi(\mu\tau + m(1-\tau)\nu) = \sup_i \varphi_i(\mu\tau + m(1-\tau)\nu)$$

$$\leq \sup_i \left(\left(\frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}(\tau) \right) \varphi_i(\mu) + m \left(\frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}(\tau) \right) \varphi_i(\nu) \right)$$
$$\leq \left(\frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}(\tau) \right) \sup_i \varphi_i(\mu) + m \left(\frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}(\tau) \right) \sup_i \varphi_i(\nu)$$
$$= \left(\frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}(\tau) \right) \varphi(\mu) + m \left(\frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}(\tau) \right) \varphi(\nu)$$

is valid for all $\mu, \nu \in \Upsilon$ and $\tau \in [0,1]$.

This ends the proof. □

Theorem 4. *Suppose that $\psi_{r_1}, \phi_{r_2} \colon [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\ldots,k$ and $r_2 = 1,2,\ldots,l$ where $k,l \in \mathbb{N}$ and $1 \leq k \leq l$ and let $\varphi \colon [0,b] \to [0,\infty)$, $b > 0$, be a (k,l)-generalized convex function with respect to Ω. If $\mu, \nu \in [0,b]$ with $\mu < \nu$ and $\varphi \in L[\mu,\nu]$, then for any $\alpha > 0$ the following inequality holds*

$$\varphi\left(\frac{\mu+\nu}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(\nu-\mu)} \left(\Psi\left(\frac{1}{2}\right) J^{\alpha}_{\mu^+} \varphi(\nu) + \Phi\left(\frac{1}{2}\right) J^{\alpha}_{\nu^-} \varphi(\mu) \right)$$
$$\leq \left[\left(\frac{\alpha \Psi\left(\frac{1}{2}\right)}{k} \sum_{r_1=1}^{k} \Psi_{r_1} + \frac{\alpha \Phi\left(\frac{1}{2}\right)}{l} \sum_{r_2=1}^{l} \Phi_{r_2} \right) \varphi(\mu) \right.$$
$$\left. + \left(\frac{\alpha \Phi\left(\frac{1}{2}\right)}{k} \sum_{r_1=1}^{k} \Psi_{r_1} + \frac{\alpha \Psi\left(\frac{1}{2}\right)}{l} \sum_{r_2=1}^{l} \Phi_{r_2} \right) \varphi(\nu) \right], \tag{3.1}$$

where

$$\Psi_{r_1} = \int_0^1 \tau^{\alpha-1} \psi_{r_1}(\tau) d\tau, \quad r_1 = 1,2,\ldots,k,$$
$$\Phi_{r_2} = \int_0^1 \tau^{\alpha-1} \phi_{r_2}(\tau) d\tau, \quad r_2 = 1,2,\ldots,l,$$
$$\Psi\left(\frac{1}{2}\right) = \frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}\left(\frac{1}{2}\right),$$
$$\Phi\left(\frac{1}{2}\right) = \frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}\left(\frac{1}{2}\right).$$

Proof. Since φ is a (k,l)-generalized m-type convex function with respect to Ω, then for all $\mu, \nu \in [0,b]$ with $\mu < \nu$, we get

$$\varphi(\mu\tau + (1-\tau)\nu) \leq \left(\frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}(\tau) \right) \varphi(\mu) + \left(\frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}(\tau) \right) \varphi(\nu). \tag{3.2}$$

Hence, we have that

$$\varphi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\varphi(\zeta_1) + \left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\varphi(\zeta_2)$$

is valid for any $\zeta_1, \zeta_2 \in \Delta$ with $\zeta_1 < \zeta_2$. If we set $\zeta_1 = \tau\mu + (1-\tau)\nu$ and $\zeta_2 = \nu\tau + (1-\tau)\mu$, where $\tau \in [0,1]$, then we obtain

$$\begin{aligned}\varphi\left(\frac{\mu+\nu}{2}\right) \leq &\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\varphi(\tau\mu+(1-\tau)\nu) \\ &+ \left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\varphi(\nu\tau+(1-\tau)\mu).\end{aligned} \tag{3.3}$$

Multiplying both sides of (3.3) by $\tau^{\alpha-1}$, then integrating the resulting inequality in terms of τ on $[0,1]$ yields

$$\begin{aligned}\frac{1}{\alpha}\varphi\left(\frac{\mu+\nu}{2}\right) &\leq \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\int_0^1 \tau^{\alpha-1}\varphi(\tau\mu+(1-\tau)\nu)\,d\tau \\ &\quad + \left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\int_0^1 \tau^{\alpha-1}\varphi(\nu\tau+(1-\tau)\mu)\,d\tau \\ &= \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\frac{1}{(\mu-\nu)}\int_\nu^\mu \left(\frac{\nu-\omega}{\nu-\mu}\right)^{\alpha-1}\varphi(\omega)\,d\omega \\ &\quad + \left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\frac{1}{(\nu-\mu)}\int_\mu^\nu \left(\frac{\varpi-\mu}{\nu-\mu}\right)^{\alpha-1}\varphi(\varpi)\,d\varpi \\ &= \frac{\Gamma(\alpha)}{(\nu-\mu)}\left(\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\frac{1}{\Gamma(\alpha)}\int_\nu^\mu \left(\frac{\nu-\omega}{\nu-\mu}\right)^{\alpha-1}\varphi(\omega)\,d\omega\right. \\ &\quad \left. + \left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\frac{1}{\Gamma(\alpha)}\int_\mu^\nu \left(\frac{\varpi-\mu}{\nu-\mu}\right)^{\alpha-1}\varphi(\varpi)\,d\varpi\right) \\ &= \frac{\Gamma(\alpha)}{(\nu-\mu)}\left(\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)J_{\mu^+}^{\alpha}\varphi(\nu) + \left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)J_{\nu^-}^{\alpha}\varphi(\mu)\right) \\ &= \frac{\Gamma(\alpha)}{(\nu-\mu)}\left(\Psi\left(\frac{1}{2}\right)J_{\mu^+}^{\alpha}\varphi(\nu) + \Phi\left(\frac{1}{2}\right)J_{\nu^-}^{\alpha}\varphi(\mu)\right).\end{aligned}$$

Thus we have

$$\varphi\left(\frac{\mu+\nu}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(\nu-\mu)}\left(\Psi\left(\frac{1}{2}\right)J_{\mu^+}^{\alpha}\varphi(\nu) + \Phi\left(\frac{1}{2}\right)J_{\nu^-}^{\alpha}\varphi(\mu)\right). \tag{3.4}$$

Also, from the generalized convexity of φ on $[\mu,\nu]$, we have

$$\varphi(\mu\tau+(1-\tau)\nu) \leq \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi(\mu)+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi(\nu) \tag{3.5}$$

and

$$\varphi(\nu\tau+(1-\tau)\mu) \leq \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi(\nu)+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi(\mu). \tag{3.6}$$

Multiplying both sides of the inequalities (3.5) and (3.6) respectively by $\tau^{\alpha-1}\Psi\left(\frac{1}{2}\right)$ and $\tau^{\alpha-1}\Phi\left(\frac{1}{2}\right)$, then integrating the resulting inequalities in terms of τ on $[0,1]$, gives

$$\begin{aligned}\Psi\left(\frac{1}{2}\right)\int_0^1\tau^{\alpha-1}\varphi(\mu\tau+(1-\tau)\nu)\,d\tau \leq{}& \left(\frac{\Psi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\int_0^1\tau^{\alpha-1}\psi_{r_1}(\tau)d\tau\right)\varphi(\mu)\\ &+\left(\frac{\Psi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\int_0^1\tau^{\alpha-1}\phi_{r_2}(\tau)d\tau\right)\varphi(\nu)\end{aligned}$$

and

$$\begin{aligned}\Phi\left(\frac{1}{2}\right)\int_0^1\tau^{\alpha-1}\varphi(\nu\tau+(1-\tau)\mu)\,d\tau \leq{}& \left(\frac{\Phi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\int_0^1\tau^{\alpha-1}\psi_{r_1}(\tau)d\tau\right)\varphi(\nu)\\ &+\left(\frac{\Phi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\int_0^1\tau^{\alpha-1}\phi_{r_2}(\tau)d\tau\right)\varphi(\mu).\end{aligned}$$

Hence we find

$$\frac{\Psi\left(\frac{1}{2}\right)\Gamma(1+\alpha)}{(\nu-\mu)}J_{\mu^+}^{\alpha}\varphi(\nu) \leq \left(\frac{\alpha\Psi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\Psi_{r_1}\right)\varphi(\mu)+\left(\frac{\alpha\Psi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\Phi_{r_2}\right)\varphi(\nu) \tag{3.7}$$

and

$$\frac{\Phi\left(\frac{1}{2}\right)\Gamma(1+\alpha)}{(\nu-\mu)}J_{\nu^-}^{\alpha}\varphi(\mu) \leq \left(\frac{\alpha\Phi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\Psi_{r_1}\right)\varphi(\nu)+\left(\frac{\alpha\Phi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\Phi_{r_2}\right)\varphi(\mu). \tag{3.8}$$

Adding the inequalities (3.7) and (3.8) gives

$$\begin{aligned}&\frac{\Gamma(1+\alpha)}{(\nu-\mu)}\left[\Psi\left(\frac{1}{2}\right)J_{\mu^+}^{\alpha}\varphi(\nu)+\Phi\left(\frac{1}{2}\right)J_{\nu^-}^{\alpha}\varphi(\mu)\right]\\ &\leq\left[\left(\frac{\alpha\Psi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\Psi_{r_1}+\frac{\alpha\Phi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\Phi_{r_2}\right)\varphi(\mu)\right.\end{aligned}$$

$$+\left(\frac{\alpha\Phi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\Psi_{r_1}+\frac{\alpha\Psi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\Phi_{r_2}\right)\varphi(\nu)\Bigg]. \tag{3.9}$$

Combining the inequalities (3.4) and (3.9) gives the required result.

This completes the proof. □

Remark 2. Theorem 4 leads to the following specific cases:

(1) If $\psi_{r_1}(\tau)=\tau$, $\phi_{r_2}(\tau)=1-\tau$ for each r_1 and r_2, and $k=l=1$, then (3.1) reduces to (1.2).

(2) If $\psi_{r_1}(\tau)=\tau^s$, $\phi_{r_2}(\tau)=(1-\tau)^s$ for each r_1 and r_2, and $k=l=1$, then we get [20, Theorem 3].

(3) If $\psi_{r_1}(\tau)=h(\tau)$, $\phi_{r_2}(\tau)=h(1-\tau)$ for each r_1 and r_2, and $k=l=1$, then we have [24, Theorem 2.1].

(4) If $\psi_{r_1}(\tau)=1-(s(1-\tau))^{r_1}$, $\phi_{r_2}(\tau)=1-(s\tau)^{r_2}$, where $r_1=r_2=1,2,\ldots,k$, $k\in\mathbb{N}$, and $s\in[0,1]$, then (3.1) reduces to [14, Theorem 2.1], in case $\kappa=1$.

Theorem 5. *Suppose that $\psi_{r_1},\phi_{r_2}\colon [0,1]\to[0,\infty)$ are continuous functions for all $r_1=1,2,\ldots,k$ and $r_2=1,2,\ldots,l$ where $k,l\in\mathbb{N}$ and $1\le k\le l$, and let for $0\le\mu<\nu$ and $m\in(0,1]$, $\varphi\colon[\mu,\frac{\nu}{m}]\to[0,\infty)$ be a differentiable function on $(\mu,\frac{\nu}{m})$ such that $\varphi'\in L[\mu,\nu]$. If $\left|\varphi'\right|$ is (k,l)-generalized m-type convex with respect to Ω, on $[\mu,\frac{\nu}{m}]$, then for any $\alpha>0$, we have*

$$\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{\Gamma(1+\alpha)}{2(\nu-\mu)^{\alpha}}\left(J^{\alpha}_{\mu^+}\varphi(\nu)+J^{\alpha}_{\nu^-}\varphi(\mu)\right)\right|$$
$$\le\frac{(\nu-\mu)}{2}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\left(\Psi^1_{r_1}+\Psi^2_{r_1}\right)+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\left(\Phi^1_{r_1}+\Phi^2_{r_1}\right)\right) \tag{3.10}$$

where

$$\Psi^1_{r_1}=\int_0^{\frac{1}{2}}[(1-\tau)^{\alpha}-\tau^{a}]\psi_{r_1}(\tau)d\tau,$$
$$\Psi^2_{r_1}=\int_{\frac{1}{2}}^{1}[\tau^{a}-(1-\tau)^{\alpha}]\psi_{r_1}(\tau)d\tau,$$
$$\Phi^1_{r_1}=\int_0^{\frac{1}{2}}[(1-\tau)^{\alpha}-\tau^{a}]\phi_{r_2}(\tau)d\tau,$$
$$\Phi^2_{r_1}=\int_{\frac{1}{2}}^{1}[\tau^{a}-(1-\tau)^{\alpha}]\phi_{r_2}(\tau)d\tau.$$

Proof. From Lemma 1 and the fact that the $\left|\varphi'\right|$ is (k,l)-generalized m-type convex with respect to Ω on $[\mu,\frac{\nu}{m}]$, we have

$$\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{\Gamma(1+\alpha)}{2(\nu-\mu)^{\alpha}}\left(J^{\alpha}_{\mu^+}\varphi(\nu)+J^{\alpha}_{\nu^-}\varphi(\mu)\right)\right|$$

$$
\begin{aligned}
&\leq \frac{(\nu-\mu)}{2}\int_0^1 \left|(1-\tau)^{\alpha}-\tau^{a}\right|\left|\varphi'\left(\mu\tau+(1-\tau)\nu\right)\right|d\tau\\
&= \frac{(\nu-\mu)}{2}\int_0^1 \left|(1-\tau)^{\alpha}-\tau^{a}\right|\left|\varphi'\left(\mu\tau+m(1-\tau)\frac{\nu}{m}\right)\right|d\tau\\
&\leq \frac{(\nu-\mu)}{2}\int_0^1 \left|(1-\tau)^{\alpha}-\tau^{a}\right|\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\left|\varphi'(\mu)\right|+m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\left|\varphi'\left(\frac{\nu}{m}\right)\right|\right)d\tau\\
&= \frac{(\nu-\mu)}{2}\left[\int_0^{\frac{1}{2}}[(1-\tau)^{\alpha}-\tau^{a}]\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\left|\varphi'(\mu)\right|+m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\left|\varphi'\left(\frac{\nu}{m}\right)\right|\right)d\tau\right.\\
&\quad\left.+\int_{\frac{1}{2}}^1[\tau^{a}-(1-\tau)^{\alpha}]\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\left|\varphi'(\mu)\right|+m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\left|\varphi'\left(\frac{\nu}{m}\right)\right|\right)d\tau\right]\\
&= \frac{(\nu-\mu)}{2}\left[\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\int_0^{\frac{1}{2}}[(1-\tau)^{\alpha}-\tau^{a}]\psi_{r_1}(\tau)d\tau\right.\\
&\quad+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\int_0^{\frac{1}{2}}[(1-\tau)^{\alpha}-\tau^{a}]\phi_{r_2}(\tau)d\tau\\
&\quad\left.+\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\int_{\frac{1}{2}}^1[\tau^{a}-(1-\tau)^{\alpha}]\psi_{r_1}(\tau)d\tau+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\int_{\frac{1}{2}}^1[\tau^{a}-(1-\tau)^{\alpha}]\phi_{r_2}(\tau)d\tau\right]\\
&= \frac{(\nu-\mu)}{2}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}(\Psi_{r_1}^1+\Psi_{r_1}^2)+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}(\Phi_{r_1}^1+\Phi_{r_1}^2)\right).
\end{aligned}
$$

This ends the proof. □

Remark 3.

(1) If $\psi_{r_1}(\tau)=\tau$, $\phi_{r_2}(\tau)=1-\tau$ for each r_1 and r_2,and $k=l=m=1$, then inequality (3.10) reduces to the trapezoid type fractional integral inequality for convex functions as

$$
\begin{aligned}
&\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{\Gamma(1+\alpha)}{2(\nu-\mu)^{\alpha}}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|\\
&\leq \frac{(\nu-\mu)(1-2^{-\alpha})}{2(\alpha+1)}\left\{\left|\varphi'(\mu)\right|+\left|\varphi'(\nu)\right|\right\}.
\end{aligned}
$$

(2) If $\psi_{r_1}(\tau)=\tau^{s}$, $\phi_{r_2}(\tau)=(1-\tau)^{s}$ for each r_1 and r_2,and $k=l=m=1$ where $s\in(0,1]$, then inequality (3.10) reduces to the trapezoid type fractional integral inequality for s-convex functions as

$$
\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{\Gamma(1+\alpha)}{2(\nu-\mu)^{\alpha}}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|
$$

$$\leq \frac{(\nu-\mu)\left(\left|\varphi^{'}(\mu)\right|+\left|\varphi^{'}(\nu)\right|\right)}{2}$$
$$\times\left\{\left(\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\beta\left(\frac{1}{2};\alpha+1,s+1\right)+\frac{1-2^{-\alpha-s}}{\alpha+s+1}\right)\right\}.$$

(3) If $\psi_{r_1}(\tau)=1-(1-\tau)^{r_1}$, $\phi_{r_2}(\tau)=1-(\tau)^{r_2}$, where $r_1=r_2=1,2,\ldots,k$, $k,\in \mathbb{N}$, and $\alpha=m=1$, then inequality (3.10) reduces to the trapezoid type integral inequality for n-polynomial convex functions as

$$\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{1}{(\nu-\mu)^{\alpha}}\int_{\mu}^{\nu}\varphi(\tau)\,d\tau\right|$$
$$\leq \frac{(\nu-\mu)\left(\left|\varphi^{'}(\mu)\right|+\left|\varphi^{'}(\nu)\right|\right)}{2k}\sum_{r_1=1}^{k}\left[\frac{\left(r_1^2+r_1+2\right)2^{r_1}-2}{(r_1+1)(r_1+2)2^{r_1+1}}\right].$$

See also [16, 20, 23].

Theorem 6. *Assume that $\psi_{r_1},\phi_{r_2}\colon[0,1]\to[0,\infty)$ are continuous functions for all $r_1=1,2,\ldots,k$ and $r_2=1,2,\cdots,l$ where $k,l\in\mathbb{N}$ and $1\leq k\leq l$, and let for $0\leq\mu<\nu$ and $m\in(0,1]$, $\varphi\colon[\mu,\frac{\nu}{m}]\to[0,\infty)$ be a differentiable function on $(\mu,\frac{\nu}{m})$ such that $\varphi^{'}\in L[\mu,\nu]$. If $\left|\varphi^{'}\right|$ is (k,l)-generalized m-type convex with respect to Ω, on $[\mu,\frac{\nu}{m}]$, then for any $\xi\in[\mu,\nu]$ and $\alpha>0$, we have*

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^{+}}^{\alpha}\varphi(\nu)+J_{\nu^{-}}^{\alpha}\varphi(\mu)\right)\right|$$
$$\leq\frac{1}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi^{'}(\mu)\right|}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{3}+\frac{m\left|\varphi^{'}\left(\frac{\xi}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{3}\right)\right.$$
$$\left.+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi^{'}(\xi)\right|}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{4}+\frac{m\left|\varphi^{'}\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{4}\right)\right],\tag{3.11}$$

where

$$\Psi_{r_1}^{3}=\int_0^1(1-\tau)^{\alpha}\psi_{r_1}(\tau)d\tau,$$
$$\Phi_{r_2}^{3}=\int_0^1(1-\tau)^{\alpha}\phi_{r_2}(\tau)d\tau,$$
$$\Psi_{r_1}^{4}=\int_0^1\tau^{\alpha}\psi_{r_1}(\tau)d\tau,$$
$$\Phi_{r_2}^{4}=\int_0^1\tau^{\alpha}\phi_{r_2}(\tau)d\tau.$$

Proof. From Lemma 2 and using the (k,l)-generalized m-type convexity of $\left|\varphi'\right|$, we get

$$\begin{aligned}
&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^{+}}^{\alpha}\varphi(\nu)+J_{\nu^{-}}^{\alpha}\varphi(\mu)\right)\right| \\
&\leq \frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\int_{0}^{1}\tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|d\tau+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\int_{0}^{1}\tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|d\tau \\
&= \frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\int_{0}^{1}(1-\tau)^{a}\left|\varphi'(\mu\tau+(1-\tau)\xi)\right|d\tau+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\int_{0}^{1}\tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|d\tau \\
&\leq \frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\int_{0}^{1}(1-\tau)^{\alpha}\psi_{r_1}(\tau)d\tau+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\int_{0}^{1}(1-\tau)^{\alpha}\phi_{r_2}(\tau)d\tau\right) \\
&\quad+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\left(\frac{\left|\varphi'(\xi)\right|}{k}\sum_{r_1=1}^{k}\int_{0}^{1}\tau^{\alpha}\psi_{r_1}(\tau)d\tau+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\int_{0}^{1}\tau^{\alpha}\phi_{r_2}(\tau)d\tau\right) \\
&= \frac{1}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{3}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{3}\right)\right. \\
&\quad\left.+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{4}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{4}\right)\right].
\end{aligned}$$

□

Corollary 1. *If we choose $\xi=\frac{\mu+\nu}{2}$ in Theorem 6, then we have*

$$\begin{aligned}
&\left|\varphi\left(\frac{\mu+\nu}{2}\right)-\frac{2^{\alpha-1}\Gamma(1+\alpha)}{(\nu-\mu)^{\alpha}}\left(J_{\mu^{+}}^{\alpha}\varphi(\nu)+J_{\nu^{-}}^{\alpha}\varphi(\mu)\right)\right| \\
&\leq \frac{\nu-\mu}{4}\left[\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{3}+\frac{m\left|\varphi'\left(\frac{\mu+\nu}{2m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{3}\right)\right. \\
&\quad\left.+\left(\frac{\left|\varphi'\left(\frac{\mu+\nu}{2}\right)\right|}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{4}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{4}\right)\right],
\end{aligned}$$

where

$$\Psi_{r_1}^{3}=\int_{0}^{1}(1-\tau)^{\alpha}\psi_{r_1}(\tau)d\tau,$$

$$\Phi_{r_2}^{3}=\int_{0}^{1}(1-\tau)^{\alpha}\phi_{r_2}(\tau)d\tau,$$

$$\Psi_{r_1}^{4}=\int_{0}^{1}\tau^{\alpha}\psi_{r_1}(\tau)d\tau,$$

$$\Phi_{r_2}^{4}=\int_0^1 \tau^{\alpha}\phi_{r_2}(\tau)d\tau.$$

Remark 4.

(1) If $\psi_{r_1}(\tau)=\tau^s$, $\phi_{r_2}(\tau)=(1-\tau)^s$ for each r_1 and r_2, and $k=l=m=1$ where $s\in(0,1]$, then inequality (3.11) reduces to the Ostrowski fractional integral inequality for s-convex functions as

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|$$
$$\leq\frac{1}{\nu-\mu}\left\{\begin{array}{c}(\xi-\mu)^{\alpha+1}\left(\left|\varphi'(\mu)\right|\beta(s+1,\alpha+1)+\left|\varphi'(\xi)\right|\beta(1,\alpha+s+1)\right)\\+(\nu-\xi)^{\alpha+1}\left(\left|\varphi'(\xi)\right|\beta(\alpha+s+1,1)+\left|\varphi'(\nu)\right|\beta(\alpha+1,s+1)\right)\end{array}\right\}.$$

(2) If $\psi_{r_1}(\tau)=1-(s(1-\tau))^{r_1}$, $\phi_{r_2}(\tau)=1-(s\tau)^{r_2}$, where $r_1=r_2=1,2,\ldots,k$, $k,\in\mathbb{N}$, $m=1$ and $s\in[0,1]$, then inequality (3.11) reduces to the Ostrowski fractional integral inequality for n-polynomial s-type convex functions as

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|$$
$$\leq\frac{1}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(1,\alpha+r_1+1)\right)\\+\frac{\left|\varphi'(\xi)\right|}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(r_2+1,\alpha+1)\right)\end{array}\right)\right.$$
$$\left.+(\nu-\xi)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\xi)\right|}{k}\sum_{r_1=1}^{k}\left(\beta(\alpha+1,1)-s^{r_1}\beta(\alpha+1,r_1+1)\right)\\+\frac{\left|\varphi'(\nu)\right|}{k}\sum_{r_1=1}^{k}\left(\beta(\alpha+1,1)-s^{r_1}\beta(\alpha+r_2+1,1)\right)\end{array}\right)\right].$$

See also [14, 18].

Theorem 7. *Suppose that $\psi_{r_1},\phi_{r_2}\colon[0,1]\to[0,\infty)$ are continuous functions for all $r_1=1,2,\ldots,k$ and $r_2=1,2,\ldots,l$ where $k,l\in\mathbb{N}$ and $1\leq k\leq l$, and let for $0\leq\mu<\nu$ and $m\in(0,1]$, $\varphi\colon[\mu,\frac{\nu}{m}]\to[0,\infty)$ be a differentiable function on $(\mu,\frac{\nu}{m})$ such that $\varphi'\in L[\mu,\nu]$. If $\left|\varphi'\right|^q$ is (k,l)-generalized m-type convex with respect to Ω, on $[\mu,\frac{\nu}{m}]$ for $q>1$; $\frac{1}{p}+\frac{1}{q}=1$, then for any $\xi\in[\mu,\nu]$ and $\alpha>0$, the following inequality holds*

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|$$
$$\leq\frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{5}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{5}\right)^{\frac{1}{q}}\right.$$

$$+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|}{k}\sum_{r_1=1}^{k}\Psi^5_{r_1}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi^5_{r_2}\right)^{\frac{1}{q}}\Bigg], \tag{3.12}$$

where

$$\Psi^5_{r_1}=\int_0^1\psi_{r_1}(\tau)\mathrm{d}\tau,$$

$$\Phi^5_{r_2}=\int_0^1\phi_{r_2}(\tau)\mathrm{d}\tau.$$

Proof. From Lemma 2, Hölder's inequality and the fact that $|\varphi'|^q$ is (k,l)-generalized m-type convex with respect to Ω on $[\mu,\frac{\nu}{m}]$, we have

$$\begin{aligned}
&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^+}\varphi(\nu)+J^{\alpha}_{\nu^-}\varphi(\mu)\right)\right|\\
&\le\frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\int_0^1\tau^a\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|\mathrm{d}\tau+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\int_0^1\tau^a\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|\mathrm{d}\tau\\
&\le\left[\frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\left(\int_0^1\tau^{p\alpha}\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|^q\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\left(\int_0^1\tau^{p\alpha}\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|^q\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&=\frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\int_0^1\left|\varphi'(\tau\mu+(1-\tau)\xi)\right|^q\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+(\nu-\xi)^{\alpha+1}\left(\int_0^1\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|^q\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&\le\frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^{k}\int_0^1\psi_{r_1}(\tau)\mathrm{d}\tau+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^q}{l}\sum_{r_2=1}^{l}\int_0^1\phi_{r_2}(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|^q}{k}\sum_{r_1=1}^{k}\int_0^1\psi_{r_1}(\tau)\mathrm{d}\tau+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^{l}\int_0^1\phi_{r_2}(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&=\frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^{k}\Psi^5_{r_1}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^q}{l}\sum_{r_2=1}^{l}\Phi^5_{r_2}\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|^{q}}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{5}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^{q}}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{5}\right)^{\frac{1}{q}}\Bigg].$$

This completes the proof. □

Corollary 2. *If we take $\xi=\frac{\mu+\nu}{2}$ in Theorem 7, then we get*

$$\begin{aligned}&\left|\varphi\left(\frac{\mu+\nu}{2}\right)-\frac{2^{\alpha-1}\Gamma(1+\alpha)}{(\nu-\mu)^{\alpha}}\left(J_{\mu^{+}}^{\alpha}\varphi(\nu)+J_{\nu^{-}}^{\alpha}\varphi(\mu)\right)\right|\\&\leq\frac{(\nu-\mu)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{4}\left[\left(\frac{\left|\varphi'(\mu)\right|^{q}}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{5}+\frac{m\left|\varphi'\left(\frac{\mu+\nu}{2m}\right)\right|^{q}}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{5}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{\left|\varphi'\left(\frac{\mu+\nu}{2}\right)\right|^{q}}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{5}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^{q}}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{5}\right)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$\Psi_{r_1}^{5}=\int_0^1\psi_{r_1}(\tau)d\tau,$$
$$\Phi_{r_2}^{5}=\int_0^1\phi_{r_2}(\tau)d\tau.$$

Remark 5.

(1) If $\psi_{r_1}(\tau)=\tau^{s}$, $\phi_{r_2}(\tau)=(1-\tau)^{s}$ for each r_1 and r_2, and $k=l=m=1$, where $s\in(0,1]$, then inequality (3.12) reduces to the Ostrowski fractional integral inequality for s-convex functions as

$$\begin{aligned}&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^{+}}^{\alpha}\varphi(\nu)+J_{\nu^{-}}^{\alpha}\varphi(\mu)\right)\right|\\&\leq\frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}}{(\nu-\mu)}\left\{\begin{array}{c}(\xi-\mu)^{\alpha+1}\left(\left|\varphi'(\mu)\right|^{q}+\left|\varphi'(\xi)\right|^{q}\right)^{\frac{1}{q}}\\+(\nu-\xi)^{\alpha+1}\left(\left|\varphi'(\xi)\right|^{q}+\left|\varphi'(\nu)\right|^{q}\right)^{\frac{1}{q}}\end{array}\right\}\end{aligned}$$

(2) If $\psi_{r_1}(\tau)=1-(s(1-\tau))^{r_1}$, $\phi_{r_2}(\tau)=1-(s\tau)^{r_2}$ for each r_1 and r_2, where $r_1=r_2=1,2,\ldots,k$, $k\in\mathbb{N}$, $m=1$, and, $s\in[0,1]$, then inequality (3.12) reduces to the Ostrowski fractional integral inequality for n-polynomial s-type convex functions as

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^{+}}^{\alpha}\varphi(\nu)+J_{\nu^{-}}^{\alpha}\varphi(\mu)\right)\right|$$

$$\leq \frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\sum_{r_1=1}^{k}\frac{r_1+1-s^{r_1}}{r_1+1}\right)^{\frac{1}{q}}}{(\nu-\mu)}\left\{\begin{array}{c}(\xi-\mu)^{\alpha+1}\left(\left|\varphi'(\mu)\right|+\left|\varphi'(\xi)\right|\right)^{\frac{1}{q}}\\+(\nu-\xi)^{\alpha+1}\left(\left|\varphi'(\xi)\right|+\left|\varphi'(\nu)\right|\right)^{\frac{1}{q}}\end{array}\right\}.$$

See also [14, 18].

Theorem 8. *Assume that $\psi_{r_1},\phi_{r_2}\colon [0,1]\to[0,\infty)$ are continuous functions for all $r_1=1,2,\ldots,k$ and $r_2=1,2,\ldots,l$ where $k,l\in\mathbb{N}$ and $1\leq k\leq l$, and let for $0<\mu<\nu$ and $m\in(0,1]$, $\varphi\colon[\mu,\frac{\nu}{m}]\to[0,\infty)$ be a differentiable function on $(\mu,\frac{\nu}{m})$ such that $\varphi'\in L[\mu,\nu]$. If $\left|\varphi'\right|^{q}$ is (k,l)-generalized m-type convex with respect to Ω, on $[\mu,\frac{\nu}{m}]$ for $q>1$,then for any $\xi\in[\mu,\nu]$ and $\alpha>0$, we have*

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|$$
$$\leq\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|^{q}}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{3}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^{q}}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{3}\right)^{\frac{1}{q}}\right.$$
$$\left.+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|^{q}}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{4}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^{q}}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{4}\right)^{\frac{1}{q}}\right] \tag{3.13}$$

where $\Psi_{r_1}^{3},\Phi_{r_2}^{3},\Psi_{r_1}^{4}$ and $\Phi_{r_2}^{4}$ are defined in Theorem 6.

Proof. Using Lemma 2 and the power-mean inequality, we obtain

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|$$
$$\leq\frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\int_0^1\tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|d\tau$$
$$+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\int_0^1\tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|d\tau$$
$$\leq\frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\left(\int_0^1\tau^{a}d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|^{q}d\tau\right)^{\frac{1}{q}}$$
$$+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\left(\int_0^1\tau^{a}d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|^{q}d\tau\right)^{\frac{1}{q}}$$
$$=\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\int_0^1(1-\tau)^{a}\left|\varphi'(\mu\tau+(1-\tau)\xi)\right|^{q}d\tau\right)^{\frac{1}{q}}\right.$$

$$+(\nu-\xi)^{\alpha+1}\left(\int_0^1 \tau^a \left|\varphi'(\tau\xi+(1-\tau)\nu)\right|^q d\tau\right)^{\frac{1}{q}}\Bigg].$$

Since $\left|\varphi'\right|^q$ is a (k,l)-generalized m-type convex function with respect to Ω, we gain

$$\left|\frac{(\xi-\mu)^\alpha+(\nu-\xi)^a}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^\alpha_{\mu^+}\varphi(\nu)+J^\alpha_{\nu^-}\varphi(\mu)\right)\right|$$
$$\leq \frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^k\int_0^1(1-\tau)^a\psi_{r_1}(\tau)d\tau\\ +\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\int_0^1(1-\tau)^a\phi_{r_2}(\tau)d\tau\end{array}\right)^{\frac{1}{q}}\right.$$
$$\left.+(\nu-\xi)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\xi)\right|^q}{k}\sum_{r_1=1}^k\int_0^1\tau^a\psi_{r_1}(\tau)d\tau\\ +\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\int_0^1(\tau)^a\phi_{r_2}(\tau)d\tau\end{array}\right)^{\frac{1}{q}}\right]$$
$$=\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^k\Psi^3_{r_1}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\Phi^3_{r_2}\right)^{\frac{1}{q}}\right.$$
$$\left.+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|^q}{k}\sum_{r_1=1}^k\Psi^4_{r_1}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\Phi^4_{r_2}\right)^{\frac{1}{q}}\right].$$

This ends the proof. □

Corollary 3. *If we select $\xi=\frac{\mu+\nu}{2}$ in Theorem 8, then we derive*

$$\left|\varphi\left(\frac{\mu+\nu}{2}\right)-\frac{2^{\alpha-1}\Gamma(1+\alpha)}{(\nu-\mu)^\alpha}\left(J^\alpha_{\mu^+}\varphi(\nu)+J^\alpha_{\nu^-}\varphi(\mu)\right)\right|$$
$$\leq\frac{(\nu-\mu)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{4}\left[\left(\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^k\Psi^3_{r_1}+\frac{m\left|\varphi'\left(\frac{\mu+\nu}{2m}\right)\right|^q}{l}\sum_{r_2=1}^l\Phi^3_{r_2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\varphi'\left(\frac{\mu+\nu}{2}\right)\right|^q}{k}\sum_{r_1=1}^k\Psi^4_{r_1}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\Phi^4_{r_2}\right)^{\frac{1}{q}}\right]$$

where $\Psi^3_{r_1},\Phi^3_{r_2},\Psi^4_{r_1}$ and $\Phi^4_{r_2}$ are defined in Theorem 6.

Remark 6.

(1) If $\psi_{r_1}(\tau)=\tau^s$, $\phi_{r_2}(\tau)=(1-\tau)^s$ for each r_1 and r_2, and $k=l=m=1$, where $s\in(0,1]$, then inequality (3.13) reduces to the Ostrowski fractional integral inequality for s-convex functions as

$$\begin{aligned}&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^+}\varphi(\nu)+J^{\alpha}_{\nu^-}\varphi(\mu)\right)\right|\\&\leq\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\left|\varphi'(\mu)\right|^q\beta(s+1,\alpha+1)+\left|\varphi'(\xi)\right|^q\beta(1,\alpha+s+1)\right)^{\frac{1}{q}}\right.\\&\left.+(\nu-\xi)^{\alpha+1}\left(\left|\varphi'(\xi)\right|^q\beta(\alpha+s+1,1)+\left|\varphi'(\nu)\right|^q\beta(s+1,\alpha+1)\right)^{\frac{1}{q}}\right].\end{aligned}$$

(2) If $\psi_{r_1}(\tau)=1-(s(1-\tau))^{r_1}$, $\phi_{r_2}(\tau)=1-(s\tau)^{r_2}$ for each r_1 and r_2, where $r_1=r_2=1,2,\cdots,k$, $k\in\mathbb{N}$, $m=1$, and, $s\in[0,1]$, then inequality (3.13) reduces to the Ostrowski fractional integral inequality for n-polynomial s-type convex functions as

$$\begin{aligned}&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^+}\varphi(\nu)+J^{\alpha}_{\nu^-}\varphi(\mu)\right)\right|\\&\leq\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\\&\times\left[(\xi-\mu)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(1,\alpha+r_1+1)\right)\\+\frac{\left|\varphi'(\xi)\right|^q}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(\alpha+1,r_1+1)\right)\end{array}\right)^{\frac{1}{q}}\right.\\&\left.+(\nu-\xi)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\xi)\right|^q}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(\alpha+1,r_1+1)\right)\\+\frac{\left|\varphi'(\nu)\right|^q}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(1,\alpha+r_1+1)\right)\end{array}\right)^{\frac{1}{q}}\right].\end{aligned}$$

See also [14, 18].

4. Conclusion

The aims of the present study were to define a new class of generalized m-type convex functions and derive some Hermite–Hadamard and Ostrowski type fractional integral inequalities for them. Using the findings of this work and the right selection of functions ψ_{r_1}, and ϕ_{r_2}, where $r_1=1,2,\ldots,k$, $r_2=1,2,\ldots,l$ and $k,l\in\mathbb{N}$ where $1\leq k\leq l$, as special cases, several new and existing classes of m-convex functions, as well as some of the stated fractional integral inequalities for them, can be obtained. This illustrates that the notion of new generalized m-type convex functions, as well as the findings presented in this paper, are broad and enhance the number of results from the literature of modern inequalities theory. Furthermore, some particular cases have been investigated. It is hoped that the ideas and approaches described in this paper will spur more study into the issue.

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Authors’ addresses

Abdul Wakil Baidar
Kabul University, Department of Mathematics, Kabul, Afghanistan
E-mail address: baidarmath87@ku.edu.af

Mehmet Kunt
(**Corresponding author**) Karadeniz Technical University, Department of Mathematics, 61080, Trabzon, Türkiye
E-mail address: mkunt@ktu.edu.tr