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ON THE WEIGHTED MIDPOINT TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. The main object of this paper is to present some weighted midpoint type inequalities for convex functions.

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1. INTRODUCTION

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. Some famous results for such estimations consist of Hermite-Hadamard, trapezoid, midpoint, Simpson or Jensen inequalities etc.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequalities are well known in the literature as the Hermite-Hadamard inequalities [5]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of a convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions. These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics. The absolute value of the difference of the second part of the (1.1) inequalities is known as the trapezoidal inequality in the literature and was given by Dragomir and Agarwal in 1998 [2]. Then, in 2004, the absolute value of the difference of the first part of the (1.1) inequalities, known as the midpoint inequality by Kirmaci, was given in [7]. Thus, these two important inequalities have attracted the attention of many readers to date, and many studies have been carried out for different types of convex functions. For recent results and generalizations concerning Hermite-Hadamard's inequalities see [1, 3, 6, 9–11, 13–18] and the references given therein.

The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [8–12]). In [4], Fejér gave a weighted generalization of the inequalities (1.1) as the following:

Theorem 1. *Let $f\colon [a,b]\to\mathbb{R}$ be a convex function, then the inequalities*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \le \frac{1}{b-a}\int_a^b f(x)w(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x$$

hold, where $w\colon [a,b]\to\mathbb{R}$ is non-negative, integrable, and symmetric about $x=\frac{a+b}{2}$ (i.e. $w(x)=w(a+b-x)$).

In [7], Kirmaci proved the following results connected to the left part of (1.1). In [7] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

Lemma 1. *Let $f\colon I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If $f'\in L([a,b])$, then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\\ &= (b-a)\left[\int_0^{\frac{1}{2}} t f'(ta+(1-t)b)\mathrm{d}t + \int_{\frac{1}{2}}^1 (t-1)\,f'(ta+(1-t)b)\mathrm{d}t\right].\end{aligned}$$

The purpose of this paper is to establish new Hermite-Hadamard-Fejér type inequalities involving convex functions. Using functions whose first derivatives absolute values are convex, we obtained new weighted midpoint inequalities that are connected to the celebrated Hermite-Hadamard-Fejér type which cover the previously published results.

2. Weighted midpoint inequalities

Lemma 2. *Let $f\colon [a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and $g\colon [a,b]\to\mathbb{R}$ be continuous and symmetric to $\frac{a+b}{2}$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,\mathrm{d}x - \frac{1}{(b-a)}\int_a^b g(x)\,f(x)\,\mathrm{d}x \tag{2.1}$$

$$= \frac{(b-a)}{4}\left\{\int_0^1\left(\int_t^1 g\left(bs+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f'\left(at+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right.$$
$$\left.-\int_0^1\left(\int_t^1 g\left(as+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f'\left(bt+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right\}.$$

Proof. Here, we apply integration by parts in integrals of right part of (2.1), and by using the change of the variable, we have

$$\int_0^1\left(\int_t^1 g\left(bs+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f'\left(at+(1-t)\frac{a+b}{2}\right)\mathrm{d}t \tag{2.2}$$

$$= \frac{2}{a-b}\left(\int_t^1 g\left(bs+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f\left(at+(1-t)\frac{a+b}{2}\right)\Bigg|_0^1$$
$$+\frac{2}{a-b}\int_0^1 g\left(bt+(1-t)\frac{a+b}{2}\right)f\left(at+(1-t)\frac{a+b}{2}\right)\mathrm{d}t$$
$$= \frac{4}{(b-a)^2}\left(\int_{\frac{a+b}{2}}^{b} g(x)\,\mathrm{d}s\right)f\left(\frac{a+b}{2}\right)-\frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} g(a+b-x)f(x)\,\mathrm{d}x$$

and

$$\int_0^1\left(\int_t^1 g\left(as+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f'\left(bt+(1-t)\frac{a+b}{2}\right)\mathrm{d}t \tag{2.3}$$

$$= \frac{2}{b-a}\left(\int_t^1 g\left(as+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f\left(bt+(1-t)\frac{a+b}{2}\right)\Bigg|_0^1$$
$$+\frac{2}{b-a}\int_0^1 g\left(at+(1-t)\frac{a+b}{2}\right)f\left(bt+(1-t)\frac{a+b}{2}\right)\mathrm{d}t$$
$$= -\frac{4}{(b-a)^2}\left(\int_a^{\frac{a+b}{2}} g(x)\,\mathrm{d}s\right)f\left(\frac{a+b}{2}\right)+\frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^{b} g(a+b-x)f(x)\,\mathrm{d}x.$$

If we subtract (2.2) from (2.3), use the symmetry of g and multiply by $\frac{(b-a)}{4}$, we obtain the proof of (2.1). □

Remark 1. With the notations in Lemma 2, if we choose $g(x)=1$, we have

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x=\frac{1}{(b-a)}\left\{\int_a^{\frac{a+b}{2}}(x-a)\,f'(x)\,\mathrm{d}x+\int_{\frac{a+b}{2}}^b(x-b)\,f'(x)\,\mathrm{d}x\right\},$$

which is proved by Kirmaci in [7].

Now, we extend some estimates of the left hand side of a Hermite-Hadamard-Fejér type inequalities for functions whose first derivatives absolute values are convex as follows.

Theorem 2. *Let $f\colon [a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b), $g\colon [a,b]\to\mathbb{R}$ be continuous and symmetric to $\frac{a+b}{2}$ with $a<b$ and $f'\in L[a,b]$. If $|f'|$ is convex on $[a,b]$, then the following inequalities hold:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,\mathrm{d}x-\frac{1}{(b-a)}\int_a^b g(x)\,f(x)\,\mathrm{d}x\right| \qquad (2.4)\\
&\leq\frac{1}{(b-a)^2}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_a^{\frac{a+b}{2}}|g(x)|\left(\frac{a+b}{2}-x\right)^2\mathrm{d}x\\
&\quad+\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|\int_a^{\frac{a+b}{2}}|g(x)|\left(1-4\left(\frac{x-a}{b-a}\right)^2\right)\mathrm{d}x\\
&\leq\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\left\{\frac{1}{(b-a)^2}\int_a^{\frac{a+b}{2}}|g(x)|\left(\frac{a+b}{2}-x\right)^2\mathrm{d}x\right.\\
&\qquad\left.+\frac{1}{4}\int_a^{\frac{a+b}{2}}|g(x)|\left(1-4\left(\frac{x-a}{b-a}\right)^2\right)\mathrm{d}x\right\},
\end{aligned}$$

where

$$\left|f'\left(\frac{a+b}{2}\right)\right|\leq\frac{|f'(a)|+|f'(b)|}{2}.$$

Proof. Using Lemma 2, by change of the order of integration and the convexity of $|f'|$, we find that

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,\mathrm{d}x - \frac{1}{(b-a)}\int_a^b g(x)f(x)\,\mathrm{d}x\right| \\
&\leq \frac{(b-a)}{4}\left\{\int_0^1\int_t^1 \left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|\left|f'\left(at+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}s\mathrm{d}t\right. \\
&\qquad\left. + \int_0^1\int_t^1 \left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|\left|f'\left(bt+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}s\mathrm{d}t\right\} \\
&\leq \frac{(b-a)}{4}\int_0^1\int_0^s \left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|\left[(1-t)\left|f'\left(\frac{a+b}{2}\right)\right| + t\left|f'(a)\right|\right]\mathrm{d}t\mathrm{d}s \\
&\quad + \frac{(b-a)}{4}\int_0^1\int_0^s \left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|\left[(1-t)\left|f'\left(\frac{a+b}{2}\right)\right| + t\left|f'(b)\right|\right]\mathrm{d}t\mathrm{d}s \\
&= \frac{(b-a)}{8}\int_0^1 \left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|\left[\left(1-(1-s)^2\right)\left|f'\left(\frac{a+b}{2}\right)\right| + s^2\left|f'(a)\right|\right]\mathrm{d}s \\
&\quad + \frac{(b-a)}{8}\int_0^1 \left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|\left[\left(1-(1-s)^2\right)\left|f'\left(\frac{a+b}{2}\right)\right| + s^2\left|f'(b)\right|\right]\mathrm{d}s.
\end{aligned}$$

By using the change of the variable it follows that

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,\mathrm{d}x - \frac{1}{(b-a)}\int_a^b g(x)f(x)\,\mathrm{d}x\right| \\
&\leq \frac{1}{(b-a)^2}\left|f'(a)\right|\int_{\frac{a+b}{2}}^b |g(x)|\left(x-\frac{a+b}{2}\right)^2\mathrm{d}x \\
&\quad + \frac{1}{(b-a)^2}\left|f'(b)\right|\int_a^{\frac{a+b}{2}} |g(x)|\left(x-\frac{a+b}{2}\right)^2\mathrm{d}x \\
&\quad + \frac{1}{4}\left|f'\left(\frac{a+b}{2}\right)\right|\int_{\frac{a+b}{2}}^b \left[|g(x)|+|g(a+b-x)|\right]\left(1-\frac{4}{(b-a)^2}(b-x)^2\right)\mathrm{d}x.
\end{aligned}$$

Since $g(x)$ is symmetric to $x=\frac{a+b}{2}$, for $g(x)=g(a+b-x)$, we write

$$\int_{\frac{a+b}{2}}^{b}|g(x)|\left(x-\frac{a+b}{2}\right)^{2}\mathrm{d}x=\int_{a}^{\frac{a+b}{2}}|g(x)|\left(\frac{a+b}{2}-x\right)^{2}\mathrm{d}x$$

and

$$\int_{\frac{a+b}{2}}^{b}|g(x)|\left(1-4\left(\frac{b-x}{b-a}\right)^{2}\right)\mathrm{d}x=\int_{a}^{\frac{a+b}{2}}|g(x)|\left(1-4\left(\frac{x-a}{b-a}\right)^{2}\right)\mathrm{d}x,$$

which completes the proof of (2.4). □

Remark 2. If in Theorem 2, we get $g(x)=1$, then inequality (2.4) becomes

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_{a}^{b}f(x)\,\mathrm{d}x\right|\leq\frac{(b-a)^{2}}{4}\left[\frac{|f'(a)|+|f'(b)|}{2}\right],$$

which is proved by Kirmaci in [7].

Theorem 3. *Let $f\colon [a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b), $g\colon [a,b]\to\mathbb{R}$ be continuous and symmetric to $\frac{a+b}{2}$ with $a<b$ and $f'\in L[a,b]$. If $|f'|^{q}$ is convex on $[a,b]$ for some $q>1$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_{a}^{b}g(x)\,\mathrm{d}x-\frac{1}{(b-a)}\int_{a}^{b}g(x)f(x)\,\mathrm{d}x\right| \tag{2.5}$$

$$\leq\frac{(b-a)^{1-\frac{2}{p}}}{2^{2-\frac{2}{p}}}\left(\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)|g(x)|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2, Hölder's inequality and the convexity of $|f'|^{q}$, we find that

$$\left|f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_{a}^{b}g(x)\,\mathrm{d}x-\frac{1}{(b-a)}\int_{a}^{b}g(x)f(x)\,\mathrm{d}x\right|$$

$$\leq\frac{(b-a)}{4}\left\{\int_{0}^{1}\int_{t}^{1}\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|\left|f'\left(at+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}s\mathrm{d}t\right.$$

$$+\int_0^1\int_t^1\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|\left|f'\left(bt+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}s\mathrm{d}t\Bigg\}$$

$$\le\frac{(b-a)}{4}\Bigg\{\left(\int_0^1\int_t^1\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\int_t^1\left|f'\left(at+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_t^1\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\int_t^1\left|f'\left(bt+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}\Bigg\}$$

$$\le\frac{(b-a)}{4}\left(\int_0^1\int_0^s\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left[(1-t)^2\left|f'\left(\frac{a+b}{2}\right)\right|^q+t(1-t)\left|f'(a)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)}{4}\left(\int_0^1\int_0^s\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left[(1-t)^2\left|f'\left(\frac{a+b}{2}\right)\right|^q+t(1-t)\left|f'(b)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)}{4}\left(\int_0^1 s\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{6}\left|f'(a)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)}{4}\left(\int_0^1 s\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{6}\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

$$\le\frac{(b-a)}{4}\left(\int_0^1 s\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)}{4}\left(\int_0^1 s\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}.$$

By using the change of the variable

$$\left|f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,\mathrm{d}x-\frac{1}{(b-a)}\int_a^b g(x)f(x)\,\mathrm{d}x\right|$$

$$\leq \frac{(b-a)}{4}\frac{2^{\frac{2}{p}}}{(b-a)^{\frac{2}{p}}}\left(\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)|g(x)|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)}{4}\frac{2^{\frac{2}{p}}}{(b-a)^{\frac{2}{p}}}\left(\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)|g(x)|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}.$$

Since $g(x)$ is symmetric to $x=\frac{a+b}{2}$, for $g(x)=g(a+b-x)$, we write

$$\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)|g(x)|^{p}\,\mathrm{d}x=\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)|g(x)|^{p}\,\mathrm{d}x,$$

which completes the proof of (2.5). □

Corollary 1. *With the notations in Theorem 3, we get $g(x)=1$, then (2.5) becomes the following inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)\,\mathrm{d}x\right|$$
$$\leq\frac{(b-a)}{2^{2+\frac{1}{p}}}\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}\right].$$

3. Applications to special means

For arbitrary positive real numbers a,b $(a\neq b)$, we shall consider the following special means:

(M1) The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\geq 0.$$

(M2) The geometric mean:

$$G=G(a,b):=\sqrt{ab},\quad a,b\geq 0.$$

(M3) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a, & \text{if } a=b,\\ \frac{b-a}{\ln b-\ln a}, & \text{if } a\neq b,\end{cases}\qquad a,b>0.$$

(M4) The $p-$logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if} \quad a \neq b, \\ a, & \text{if} \quad a = b, \end{cases} \qquad p \in \mathbb{R} \setminus \{-1,0\};\, a,b>0.$$

It is well known that L_p is monotonic non-decreasing over $p \in \mathbb{R}$ with $L_{-1} := L$. In particular, we have the following inequality

$$L \leq A.$$

Now, using the results of Section 2, some new inequalities is derived for the above means.

(1) Let $f\colon [a,b] \to \mathbb{R}$, $(0<a<b)$, $f(x) = x^s$, $s \geq 1$ and $g(x) = \left(x - \frac{a+b}{2}\right)^2$. Then,

$$\frac{1}{b-a}\int_a^b g(x)f(x)\mathrm{d}x = L_{s+2}^{s+2}(a,b) - 2A(a,b)L_{s+1}^{s+1}(a,b) + A^2(a,b)L_s^s(a,b),$$

$$\int_a^b \left(x - \frac{a+b}{2}\right)^2 \mathrm{d}x = \frac{(b-a)^3}{12},$$

$$f\left(\frac{a+b}{2}\right) = A^s(a,b).$$

It is easy to verify that the function $|f'(x)|^q = s^q x^{(s-1)q}$ is convex on $[a,b]$.

(a) From Theorem 2, we obtain

$$\left|A^s(a,b)\frac{(b-a)^2}{12} - L_{s+2}^{s+2}(a,b) + 2A(a,b)L_{s+1}^{s+1}(a,b) - A^2(a,b)L_s^s(a,b)\right|$$

$$\leq sA(a^{s-1},b^{s-1})\frac{(b-a)^3}{5.2^5} + sA^{s-1}(a,b)\frac{3(b-a)^3}{5.2^4} \leq sA(a^{s-1},b^{s-1})\frac{(b-a)^3}{5.2^3}.$$

(b) From Theorem 3, we have

$$\left|A^s(a,b)\frac{(b-a)^2}{12} - L_{s+2}^{s+2}(a,b) + 2A(a,b)L_{s+1}^{s+1}(a,b) - A^2(a,b)L_s^s(a,b)\right|$$

$$\leq s\frac{(b-a)^{2+\frac{2}{p}}}{8(p+2)^{\frac{1}{p}}}\left[\left(\frac{2a^{(s-1)q}+b^{(s-1)q}}{6}\right)^{\frac{1}{q}} + \left(\frac{a^{(s-1)q}+2b^{(s-1)q}}{6}\right)^{\frac{1}{q}}\right]$$

$$= s\frac{(b-a)^{2+\frac{2}{p}}}{8(p+2)^{\frac{1}{p}}}$$

$$\times \left[\left(\frac{1}{3}A\left(a^{(s-1)q},b^{(s-1)q}\right) + \frac{a^{(s-1)q}}{6}\right)^{\frac{1}{q}} + \left(\frac{1}{3}A\left(a^{(s-1)q},b^{(s-1)q}\right) + \frac{b^{(s-1)q}}{6}\right)^{\frac{1}{q}}\right],$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

(2) Let $f\colon [a,b]\subseteq[0,\infty)\to\mathbb{R}$, $(0<a<b)$, $f(x)=\dfrac{1}{x^s}$, $s\in(0,1]$. Then,

$$\frac{1}{b-a}\int_a^b g(x)f(x)\mathrm{d}x=L_{2-s}^{2-s}(a,b)-2A(a,b)L_{1-s}^{1-s}(a,b)+A^2(a,b)L_{-s}^{-s}(a,b),$$

$$f\left(\frac{a+b}{2}\right)=A^{-s}(a,b).$$

(a) From Theorem 2, we obtain

$$\left|A^{-s}(a,b)\frac{(b-a)^2}{12}-L_{2-s}^{2-s}(a,b)+2A(a,b)L_{1-s}^{1-s}(a,b)-A^2(a,b)L_{-s}^{-s}(a,b)\right|$$
$$\leq s(s+1)\frac{L_s^s(a,b)}{G^{2(s+1)}(a,b)}\frac{(b-a)^4}{5.2^5}+s\frac{3(b-a)^3}{A^{s+1}(a,b)\cdot 5.2^4}$$
$$\leq s(s+1)\frac{L_s^s(a,b)}{G^{2(s+1)}(a,b)}\frac{(b-a)^4}{2^6}.$$

(b) From Theorem 3, we have

$$\left|A^{-s}(a,b)\frac{(b-a)^2}{12}-L_{2-s}^{2-s}(a,b)+2A(a,b)L_{1-s}^{1-s}(a,b)-A^2(a,b)L_{-s}^{-s}(a,b)\right|$$
$$\leq\frac{s(b-a)}{(p+1)^{1/p}2^{1+\frac{1}{p}}(s+1)^{1/q}}\left[m\left\{2^{-(s+1)}a^{-q(s+1)}+\left(1-2^{-(s+1)}\right)b^{-q(s+1)}\right\}^{(p-1)/p}\right.$$
$$\left.+n\left\{\left(1-2^{-(s+1)}\right)a^{-q(s+1)}+2^{-(s+1)}b^{-q(s+1)}\right\}^{(p-1)/p}\right],$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

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