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EXISTENCE OF SOLUTIONS TO NON-LOCAL UNCERTAIN DIFFERENTIAL EQUATIONS UNDER THE ψ-CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this manuscript, we look into the existence of solutions to a class of uncertain differential equations with non-local derivatives. The method relies on the Krasnosel'skii fixed point theorem as well as a lengthened Schauder fixed point theorem that is applicable on fuzzy metric spaces. This theorems implies that the topic in question has a fuzzy solution specified on a certain interval. Our strategy requires considering a linked integral problem wherein the aforementioned tools apply. We'll wrap off with a physical incentive.

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1. INTRODUCTION

Uncertain valuation is a mathematical description that aims to minimize uncertainties in computational, mathematical, and practical uses. Due to a variety of factors, uncertainty has seemed in mathematical approaches and experimental observations. The uncertainty principle, for example, is crucial in quantum physics [10].

Nonlocal derivatives, while on the other side, have grown in prominence in recent years as a result of the attention and strength of their applicability in the creation of models for complicated systems in connection to several sectors of research and engineering. Certain well potential applications include the investigation of electromagnetism or control processes, diffusion processes via diverse media, biological challenges, and electronic systems. [2, 7, 13, 14, 17].

Agarwal et al. [3] developed the notion of solutions to uncertain differential equations having non-local derivatives of fraction form in 2010 (denoted as UFDEs). The equations provided in [3] have an unusual attitude because they mix the input of uncertainty and non-local impacts. This effort has resulted in the development of a novel class of hybrid operators [2], which gather the characteristics of uncertain equations. Arshad and Lupulescu [5, 6] have extracted various conclusions about the existence and uniqueness concerns for UFDE by studying the Riemann-Liouville derivative.

The creation of novel extensions of fixed point findings is significant in the context of UFDEs; this truth is inspired by the distinctive peculiarities observed in the domains of fuzzy sets and functions. Agarwal et al. [3] for an intriguing illustration of an expansion of the Schauder fixed point theorem. The traditional fixed point finding is expanded to semi-linear spaces with the cancellation feature in their investigation. This plugin helps you to investigate the existence of UFDE solutions. The researchers in [12] used this concept to investigate UFDEs and investigate the availability of solutions. Another extended fixed point findings may be found in [15], in which the researchers established Krasnosel'skii's fixed point theorem in the setting of semi-linear Banach spaces of common class and used it to derive various existence results for UFDE's.

In this work, we look at a class of non-linear differential equations with uncertainty and non-local derivatives, which comprises a wider class of equations than [12]. The use of an extended Schauder fixed point theorem accessible in the setting of fuzzy metric spaces is the major mathematical instrument for the investigation. With this conclusion, We demonstrate that the problems of concern have at least one solution, assuming that some adequate criteria are met. The goal of this paper is to offer a finding that generalize the discoveries in [18] to the situation of uncertain differential equations. As a result, the article's reaching new is to show that there's at least one solution to a class of uncertain differential equations with non-local derivatives under the ψ-Caputo fractional derivative of order $\alpha \in (0,1]$.

This article is organized as follows: after this introduction, we have presented some concepts related to fuzzy semilinear metric spaces, the ψ-Caputo fractional derivative and we provide all of the required tools that will be utilized in this study. The main results were then discussed in Section 3. Finally, we conclude our work with an illustrative example.

2. Basic notions

To identify the topic of concern and the key outcomes, certain notions and relevant representations first be introduced.

In this paper, we indicate the space of fuzzy sets by $\mathbb{E}$. The α-cuts of an element $\phi \in \mathbb{E}$ are defined as:

$$[\phi]^\alpha := \{t \in \mathbb{R} \mid \phi(t) \geq \alpha\}, \quad \forall \alpha \in (0,1].$$

The support of ϕ is given by:

$$[\phi]^0 := \text{cl}\{t \in \mathbb{R} \mid \phi(t) > 0\}.$$

It would also be useful to denote the α-cuts of ϕ as:

$$[\phi]^\alpha = \left[\underline{\phi}^\alpha, \bar{\phi}^\alpha\right].$$

Taking two elements $u, v \in \mathbb{E}$ and a scalar $r \in \mathbb{R}$, we compute the sum $u+v$ and multiplication by scalar ru by using formulas

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \text{ and } [ru]^\alpha = r[u]^\alpha, \quad \forall \alpha \in [0,1].$$

As previously stated, these ideas are based on the classical summation of real intervals and the scalar multiplication of a real interval.

Due on the Pompeiu-Hausdorff distance between the α-cuts, the set E is simply supplied with a measure $D\colon E \times E \to \mathbb{R}^+ \cup \{0\}$, as shown in:

$$D(\phi, \varphi) := \sup_{\alpha\in[0,1]} d_H\left([\phi]^\alpha, [\varphi]^\alpha\right) = \sup_{\alpha\in[0,1]} \max\left\{ | \underline{\phi}^\alpha - \underline{\varphi}^\alpha |, | \bar{\phi}^\alpha - \bar{\varphi}^\alpha | \right\}, \quad \phi, \varphi \in \mathbb{E}.$$

The aforementioned distance possesses several notable properties, including translation invariance and positive homogeneity:

- $D(\phi+\Phi, \varphi+\Phi) = D(\phi, \varphi)$, for all $\phi, \varphi, \Phi \in E$,
- $D(r\phi, r\varphi) = |r| D(\phi, \varphi)$, for all $r \in \mathbb{R}$ and $\phi, \varphi \in E$,
- $D(\phi+\varphi, \mu+\nu) \leq D(\phi, \mu) + D(\varphi, \nu)$, for all $\phi, \varphi, \mu, \nu \in E$,
- $D(\gamma\phi, \kappa\phi) = |\gamma - \kappa| D(\phi, \tilde{0})$, for all $\gamma, \kappa \geq 0$ and $\phi \in E$, and the space (E, D) is metrically complete.

Since the space E is not a vector space, traditional fixed point concepts in Banach spaces are not applicable to solving problems in this context. Therefore, we adopt the notion of semi-linear Banach spaces, which encompasses a framework that includes the set of all fuzzy sets. The concept of semi-linear spaces has been explored in previous academic works, such as [9].

We define U as a semi-linear metric space if it is equipped with a metric $d\colon U \times U \to \mathbb{R}^+$ satisfying:

- $d(\phi, \Phi) = d(\phi+\Xi, \Phi+\Xi)$ (translation invariance),
- $\Upsilon d(\phi, \Phi) = d(\Upsilon\phi, \Upsilon\Phi)$, for $\Upsilon \geq 0$ (positive homogeneity)

for all $\phi, \Phi, \Upsilon \in U$, where $\tilde{0}$ denotes the zero element in U. The norm $\|x\| := d(x, \tilde{0})$ can be defined on U. It is feasible to establish that addition and scalar multiplication are continuous operations in semi-linear metric spaces. If U is complete with respect to the metric d, it is referred to as a semi-linear Banach space.

For instance, since $\mathbb{E}$ is not a linear space, it cannot be a Banach space. The property that $\phi + \Phi = \varphi + \Phi$ implies $\phi = \varphi$ for all $\phi, \Phi, \varphi \in U$ indicates the cancellation property in the semi-linear space U.

Let $a > 0$. The space of all continuous fuzzy functions defined on $(0, a]$ is denoted by $C((0,a], \mathbb{E})$. Suppose $r \geq 0$. We define:

$$C_r([0,a], \mathbb{E}) = \{\Phi \in C((0,a], \mathbb{E}) \mid \Phi_r \in C([0,a], \mathbb{E})\},$$

where $\Phi_r(t) = \psi^r(t)\Phi(t)$ for $t \in (0,a]$. Clearly, $C_r([0,a],\mathbb{E})$ is a complete metric space with respect to the distance:

$$d_r(\Phi,\varphi) = \max_{t\in[0,a]} \psi^r(t)D(\Phi(t),\varphi(t)), \quad \Phi,\varphi \in C_r([0,a],\mathbb{E}).$$

We denote $d_r(u,\tilde{0})$ by $\|\Phi\|_r$, which is not a norm in the classical sense because $C_r([0,a],\mathbb{E})$ is not a vector space. It's important to note that $C_0([0,a],\mathbb{E}) = C([0,a],\mathbb{E})$.

We denote $\mathbb{E}^c$ as the set of fuzzy elements $\Phi \in \mathbb{E}$ where the mapping $\alpha \mapsto [\Phi]^\alpha$ is continuous with respect to the Hausdorff distance on $[0,1]$. It is well-known that the metric space $(\mathbb{E}^c, D)$ is complete [11].

If the mappings take values in $\mathbb{E}^c$, we obtain the space $C_r([0,1],\mathbb{E}^c)$ for $r \geq 0$.

Definition 1 ([11])**.** We say that M is compact-supported if there's a real compact subspace N with $[x]^0 \subseteq N, \forall x \in M$.

Definition 2 ([11])**.** Allow $M \subseteq E^c$. If, for each $\varepsilon > 0$, there's $\eta > 0$ such as

$$|\,\alpha-\beta\,| < \eta \text{ implies } D\left([x]^\alpha,[x]^\beta\right) < \varepsilon, \forall x \in M.$$

Consequently, at $\beta \in [0,1]$, we state that M is level-equicontinuous.

We say that M is level-equicontinuous on $[0,1]$ if the condition described in Definition 2 is true for each $\alpha \in [0,1]$.

Theorem 1 ([11])**.** *Assume that M is a compact supported subset of E^c. The set M is relatively compact in (E^c, D) iff it's level equicontinuous on $[0,1]$.*

Furthermore, we remember the semi-linear version of the classic Schauder fixed point theorem stated in [1].

Theorem 2. *Assume U is a semi-linear Banach space with cancellation. Assume S is a nonempty, bounded, closed, convex subset of U. Consider $P\colon S \to S$ to be a compact map. Therefore, in S, P has (at least) one fixed point.*

In [12] Khastan et al. used H-differences to demonstrate the following generalization of the Krasnosel'skii fixed point finding in the uncertain environment.

Theorem 3 (Krasnosel'skii fixed point theorem for fuzzy metric spaces)**.** *Allow $\mathcal{M}$ to be a non empty, closed and convex subset of $C(I,\mathcal{E}^c)$ and assume that $\mathcal{Q}$ and $\mathcal{H}$ map $\mathcal{M}$ into S and*

i) *$\mathcal{Q}$ is continuous and compact,*
ii) *$\mathcal{Q}t + \mathcal{H}s \in \mathcal{M}$, for every $t,s \in \mathcal{M}$,*
iii) *$\mathcal{H}$ is a contraction mapping.*

Then, there exists a fixed point for $\mathcal{Q}+\mathcal{H}$ in $\mathcal{M}$, that is, there is $s \in \mathcal{M}$ for which $\mathcal{Q}s + \mathcal{H}s = s$.

Definition 3 ([16]). The Gamma function $\Gamma(z)$ is defined by integral

$$\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}\mathrm{d}t \tag{2.1}$$

which converges on the complex half-plane $R(z)>0$.

Integrating by parts, we show that:

$$\Gamma(z+1)=z\Gamma(z),\quad \mathrm{Re}(z)>0.$$

And in particular,

$$\forall n\in\mathbb{N};\quad \Gamma(n+1)=n!,$$

$$\frac{1}{\Gamma(-m)}=0,\quad (m=0,1,2\ldots\ldots),$$

$$\Gamma(1)=1,\quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

Definition 4 ([4]). Assume $\alpha>0, g\in L^1(J,\mathbb{R})$ and $\psi\in C^n(I,\mathbb{R})$ such that $\psi'(t)>0\ \forall t\in I$. The ψ-Riemann Liouville fractional integral of level α of the function g is written as

$$I_{0^+}^{\alpha,\psi}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s)\mathrm{d}s. \tag{2.2}$$

Definition 5 ([4]). Assume $\alpha>0, g\in C^{n-1}(I,\mathbb{R})$ and $\psi\in C^n(I,\mathbb{R})$ such that $\psi'(t)>0\ \forall t\in I$. The ψ-Caputo fractional derivative of order α of the function g is expressed as

$${}^{c}D_{0^+}^{\alpha,\psi}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}g_{\psi}^{[n]}(s)\mathrm{d}s, \tag{2.3}$$

where

$$g_{\psi}^{[n]}(s)=\left(\frac{1}{\psi'(s)}\frac{d}{\mathrm{d}s}\right)^n g(s) \text{ and } n=[\alpha]+1$$

and $[\alpha]$ denotes the integer part of the real number α.

Remark 1. More specifically, if $\alpha\in]0,1[$, so there is

$${}^{c}D_{0^+}^{\alpha,\psi}g(s)=\frac{1}{\Gamma(\alpha)}\int_0^s (\psi(s)-\psi(t))^{\alpha-1}g'(t)\mathrm{d}t$$

and

$${}^{C}D_{0^+}^{\alpha,\psi}g(s)=I_{0^+}^{1-\alpha,\psi}\left(\frac{g'(s)}{\psi'(s)}\right).$$

Proposition 1 ([4]). *Allow $\alpha>0$, if $f\in C^{n-1}(I,\mathbb{R})$, so we've*

(1) ${}^{C}D_{0^+}^{\alpha,\psi}I_{0^+}^{\alpha,\psi}g(s)=g(s)$.

(2) $I_{0^+}^{\alpha,\psi C}D_{0^+}^{\alpha,\psi}g(s) = g(s) - \sum_{k=0}^{n-1}\frac{g_{\psi}^{[k]}(0)}{k!}(\psi(s)-\psi(0))^k$.

(3) $I_{0^+}^{\alpha,\psi}$ *is linear, bounded from* $C(I,\mathbb{R})$ *to* $C(I,\mathbb{R})$.

Proposition 2 ([4]). *Allow* $t>0$ *and* $\alpha,\beta>0$, *therefore we get*

a) $I_{0^+}^{\alpha,\psi}(\psi(s)-\psi(0))^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(s)-\psi(0))^{\alpha+\beta-1}$.

b) ${}^{C}D_{0^+}^{\alpha,\psi}(\psi(s)-\psi(0))^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(s)-\psi(0))^{\alpha-\beta-1}$.

c) ${}^{C}D_{0^+}^{\alpha,\psi}(\psi(s)-\psi(0))^{n} = 0, \quad$ *for all* $n\in\mathbb{N}$.

3. Main results

Considering the uncertain non-local fuzzy fractional differential equation:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\psi}u = f(t,u)+g(t,u), & t\in I=[0,1],\\ u(0)=\tilde{0}\in\mathbb{E}, \end{cases} \tag{3.1}$$

where the derivation order is $0<\alpha<1$, the fuzzy function $f\colon [0,1]\times\mathbb{E}^c\to\mathbb{E}^c$ is continuous and compact, and $g\colon [0,1]\times\mathbb{E}^c\to\mathbb{E}^c$ fulfills

$$D(g(t,u),g(t,v)) \le LD(u,v), \quad u,v\in\mathbb{E}^c, \text{ for } L\ge 0.$$

Definition 6. A fuzzy function $u\in C((0,1],\mathbb{E})\cap L^1((0,1],\mathbb{E})$ with continuous fractional derivative ${}^{C}D_{0^+}^{\alpha,\psi}u$ on $(0,1]$ is a solution of (3.1) if

$${}^{C}D_{0^+}^{\alpha,\psi}u = f(t,u)+g(t,u) \quad \text{for all } t\in(0,1]. \tag{3.2}$$

Remark 2. Using the findings from Section 2, considering a fuzzy integral equation that permits us to solve the problem (3.1). Very much so, if $u\in C([0,1],\mathbb{E})$ is a solution to the fuzzy integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)}\odot\int_0^t \psi'(s)(\psi(t)\ominus\psi(s))^{\alpha-1}[f(s,u(s))\oplus g(s,u(s))]\mathrm{d}s,$$

and $f(t,u),\ g(t,u)\in C((0,1],\mathbb{E})\cap L^1((0,1),\mathbb{E})$. So u is a solution to equation (3.1) as well.

3.1. *Existence of solution via Schauder fixed point theorem*

Lemma 1. *If* $u\colon [0,1]\to\mathbb{E}$ *is continuous, then* u *is bounded.*

Proof. If u is continuous, the function $\underline{u}^{\alpha}$ and $\overline{u}^{\alpha}$ are continuous and bounded on $[0,1]$. Then $D(u,\tilde{0})\le\max\{|\underline{u}^{\alpha}|,|\overline{u}^{\alpha}|\}$ is bounded. □

Below we define the operator $\mathcal{M}: C([0,1],\mathbb{E}^c) \to C([0,1],\mathbb{E}^c)$ for $u \in C([0,1],\mathbb{E}^c)$ and $0 \le r < \alpha < 1$ by

$$[\mathcal{M}u](t) = \frac{1}{\Gamma(\alpha)} \odot \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} u_r(s)\mathrm{d}s$$
$$= \frac{1}{\Gamma(\alpha)} \odot \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \psi^r(s) u(s)\mathrm{d}s.$$

And

$$\mathcal{N}: \Omega \to C([0,1],\mathbb{R}^c),$$
$$\mathcal{N}u(t) = \psi^{-r}(t)\left(f(t,u(t)) + g(t,u(t))\right), \quad t \in [0,1],$$

where $\Omega = \left\{u \in C([0,1],\mathbb{E}^c) \mid d_0(u,\tilde{0}) \le R_0\right\}$. The operator $\mathcal{N}$ is well-defined if $(t,u) \to \psi^{-r}(t)\left(f(t,u(t)) + g(t,u(t))\right)$ is continuous on $[0,a] \times \mathbb{R}^c_F$ with values in $\mathbb{E}^c$.

Lemma 2. *On $C([0,1],\mathbb{E}^c)$, the operator $\mathcal{M}$ is properly delineated and continuous.*

Proof. Firstly, we demonstrate that $\mathcal{M}$ is well defined. That is, for a fixed $u \in C([0,1],\mathbb{E}^c)$, ensure that $\mathcal{M}u \in C([0,1],\mathbb{E}^c)$. In reality, we demonstrate $\mathcal{M}u$ is uniformly continuous on $[0,1]$. Allow $t_1, t_2 \in [0,1], t_1 < t_2$, and let R such as

$$D(u(s),\tilde{0}) \le R, \quad \forall s \in [0,1].$$

Then

$$D(\mathcal{M}u(t_1), \mathcal{M}u(t_2))$$
$$= \frac{1}{\Gamma(\alpha)} D\left(\int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\psi^r(s)u(s)\mathrm{d}s, \int_0^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\psi^r(s)u(s)\mathrm{d}s\right)$$
$$\le \frac{1}{\Gamma(\alpha)} D\left(\int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\psi^r(s)u(s)\mathrm{d}s, \int_0^{t_1} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\psi^r(s)u(s)\mathrm{d}s\right)$$
$$+ D\left(\int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\psi^r(s)u(s)\mathrm{d}s, \tilde{0}\right)$$
$$\le \frac{1}{\Gamma(\alpha)} \left[\int_0^{t_1} D\left(\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\psi^r(s)u(s)),\ \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\psi^r(s)u(s))\right)\mathrm{d}s \ldots\right.$$
$$\left. + \int_{t_1}^{t_2} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}\psi^r(s)D(u(s),\tilde{0})\mathrm{d}s\right]$$
$$\le \frac{R}{\Gamma(\alpha)} \left[\int_0^{t_1} \left|\psi'(s)\left((\psi(t_1)-\psi(s))^{\alpha-1}\psi^r(s) - (\psi(t_2)-\psi(s))^{\alpha-1}\psi^r(s)\right)\right|\mathrm{d}s\right.$$
$$\left. + \int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}\psi^r(s)\mathrm{d}s\right]$$
$$\le R\frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\left[\left|\left(\psi^{\alpha-r}(t_1) - \psi^{\alpha-r}(t_2)\right)\right| + 2\left|(\psi(t_1)-\psi(t_2))^{\alpha-r}\right|\right].$$

Applying the Mean Value Theorem to the functions $\psi^{\alpha-r}(t)$ and $\psi(t)$, the previous inequality becomes

$$\begin{aligned}
&D\left(\mathcal{M}u\left(t_1\right),\mathcal{M}u\left(t_2\right)\right)\\
&\leqslant R\frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\left[\left((\alpha-r)\psi'(c_1)\psi^{\alpha-r-1}(c_1)\left|t_1-t_2\right|\right)+2\left(\psi'(c_2)\left|t_1-t_2\right|\right)^{\alpha-r}\right],
\end{aligned}$$

where, $c_1,c_2\in[0,1]$, which implies that $D\left(\mathcal{M}u\left(t_1\right),\mathcal{M}u\left(t_2\right)\right)\to 0$ when $|t_1-t_2|\to 0$.

Following that, we demonstrate the continuity of $\mathcal{M}$. Allow $u_n\to u$ as $n\to\infty$ in $C\left([0,1],\mathbb{E}^c\right)$, That is, $d_0\left(u_n,u\right)\to 0$ as $n\to\infty$. Well there's

$$\begin{aligned}
d_0\left(\mathcal{M}u_n,\mathcal{M}u\right) &= \sup_{t\in[0,1]} D\left(\mathcal{M}u_n(t),\mathcal{M}u(t)\right)\\
&= \sup_{t\in[0,1]} D\left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)u_n(s)\mathrm{d}s,\right.\\
&\qquad\left.\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)u(s)\mathrm{d}s\right)\\
&\leq \frac{1}{\Gamma(\alpha)}\sup_{t\in[0,1]}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)D\left(u_n(s),u(s)\right)\mathrm{d}s\\
&\leq \frac{1}{\Gamma(\alpha)}d_0\left(u_n,u\right)\sup_{t\in[0,1]}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)\mathrm{d}s\\
&\leq d_0\left(u_n,u\right)\sup_{t\in[0,1]}\frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\psi^{\alpha-r}(t)\\
&\leq \frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}d_0\left(u_n,u\right)\sup_{t\in[0,1]}\psi^{\alpha-r}(t).
\end{aligned}$$

Therefore $\mathcal{M}u_n\to\mathcal{M}u$ as $n\to\infty$ in $C\left([0,1],\mathbb{E}^c\right)$. □

Remark 3. If $G\subseteq C\left([0,1],\mathbb{E}^c\right)$ is bounded, so $\mathcal{M}(G)$ is bounded in $C\left([0,1],\mathbb{E}^c\right)$. Obviously, for $v\in G$ we obtain

$$\begin{aligned}
D(\mathcal{M}v(t),\tilde{0}) &\leq \sup_{t\in[0,1]} D(v(t),\tilde{0})\frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)}\psi^{\alpha-r}(t)\\
&\leq \frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)}\sup_{t\in[0,1]}\psi^{\alpha-r}(t)D(v(t),\tilde{0}).
\end{aligned}$$

Lemma 3. *If $G\subseteq C\left([0,1],\mathbb{E}^c\right)$ is bounded, then $\mathcal{M}(G)$ is equicontinuous in $C\left([0,1],\mathbb{E}^c\right)$.*

Proof. Allow $d_0(u,\tilde{0}) \leq M$, for all $u \in G$ and $t_1, t_2 \in [0,1], t_1 < t_2$. According to the previous calculations of Lemma 2, we get, $\forall u \in G$,

$$D(\mathcal{M}u(t_1), \mathcal{M}u(t_2)) \leq M \left[\frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} \left(\psi^{\alpha-r}(t_1) - \psi^{\alpha-r}(t_2)\right) + \frac{2}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s) \left(\psi(t_2) - \psi(s)\right)^{\alpha-1} \psi^r(s) \mathrm{d}s \right],$$

which tends to 0 as $|t_1 - t_2| \to 0$. As a result, $\mathcal{M}(G)$ is equicontinuous in $C([0,1], \mathbb{E}^c)$. □

Proposition 3. *If $G \subseteq C([0,1], \mathbb{E}^c)$ in such a way that $\{v(s) \mid v \in G, s \in [0,1]\}$ is compact supported in $\mathbb{E}^c$, therefore G is bounded.*

Proof. Obviously, there's a compact set K in $\mathbb{R}$ such as

$$\left\{[u(s)]^0 \mid u \in G, s \in [0,1]\right\} \subseteq K.$$

In contrast, for $u \in G$,

$$\begin{aligned} d_0(u,\tilde{0}) &= \sup_{t\in[0,1]} \sup_{\theta\in[0,1]} d_H\left([u(t)]^\theta, \{0\}\right) \\ &= \sup_{t\in[0,1]} \sup_{\theta\in[0,1]} d_H\left(\left[\underline{u}^\theta(t), \overline{u}^\theta(t)\right], \{0\}\right) \\ &= \sup_{t\in[0,1]} \sup_{\theta\in[0,1]} \max\left\{\left|\underline{u}^\theta(t)\right|, \left|\overline{u}^\theta(t)\right|\right\} \\ &\leq \sup_{t\in[0,1]} \max\left\{\left|\underline{u}^0(t)\right|, \left|\overline{u}^0(t)\right|\right\} \\ &\leq \sup_{t\in[0,1]} d_H\left(\left|[u(t)]^0\right|, \{0\}\right) \\ &< +\infty. \end{aligned}$$

Well, $G \subseteq C([0,1], \mathbb{E}^c)$ is bounded. □

Lemma 4. *If $G \subseteq C([0,1], \mathbb{E}^c)$ is like*

$$\{u(s) \mid u \in G, s \in [0,1]\}$$

is compactly supported and level equicontinuous, so $\mathcal{M}(G)$ is relatively compact in $C([0,1], \mathbb{E}^c)$.

Proof. We demonstrate that $\mathcal{M}(G)$ is a uniformly equicontinuous subset of $C([0,1], \mathbb{E}^c)$ and that $\mathcal{M}(G)$ is relatively compact in $\mathbb{E}^c$ for all $t \in [0,1]$, using the Arzelà-Ascoli theorem. Since $\{u(s) \mid u \in G, s \in [0,1]\}$ has compact support, Proposition 3 implies that G is bounded. Additionally, by applying Lemma 3 and given that G is bounded, we conclude that $\mathcal{M}(G)$ is uniformly equicontinuous. Therefore, it suffices to show that $\mathcal{M}(G)(t)$ is relatively compact in $\mathbb{E}^c$ for all $t \in [0,1]$. According to Theorem 1, this is equivalent to proving that $\mathcal{M}(G)(t)$ is a subset of $\mathbb{E}^c$ with compact support and uniformly equicontinuous over $[0,1]$ for all $t \in [0,1]$. Given that $\{u(s) \mid u \in G, s \in [0,1]\}$ has compact support, there exists a compact set $K \subseteq \mathbb{R}$ such that $[u(s)]^0 \subseteq K$ for each $s \in [0,1]$ and $u \in G$. Therefore, for every $u \in G$ and $t \in [0,1]$,

$$\begin{aligned}
[\mathcal{M}(u)(t)]^0 &= \left[\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)u(s)\mathrm{d}s\right]^0 \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)[u(s)]^0\mathrm{d}s \\
&\subseteq \frac{K}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)\mathrm{d}s \\
&\subseteq \frac{\Gamma(-r+1)\sup_{t\in[0,1]}\psi^{\alpha-r}(t)}{\Gamma(-r+\alpha+1)} K.
\end{aligned}$$

Hence, $\mathcal{M}(G)(t)$ is compactly supported for any $t \in [0,1]$.

Furthermore, to demonstrate level equicontinuity, we fix $t \in [0,1]$ and $\varepsilon > 0$. If $w \in \mathcal{M}(G)$, we get for $u \in G$ such as $w = \mathcal{M}(u)(t)$,

$$[w]^\theta = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)[u(s)]^\theta\mathrm{d}s.$$

Then

$$d_H\left([w]^\theta, [w]^\beta\right) \le \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s) d_H\left([u(s)]^\theta, [u(s)]^\beta\right)\mathrm{d}s.$$

Since $\{u(s) \mid u \in G, s \in [0,1]\}$ is level equicontinuous, as well as for provided

$$\varepsilon\frac{\Gamma(-r+\alpha+1)}{2\lambda\Gamma(-r+1)} > 0$$

with $\lambda = \sup_{t\in[0,1]}\psi^{\alpha-r}(t)$, there's $\delta > 0$ satisfies $|\theta-\beta| < \delta$, well

$$d_H\left([u(s)]^\theta, [u(s)]^\beta\right) < \frac{\varepsilon\Gamma(-r+\alpha+1)}{2\lambda\Gamma(-r+1)} \quad u \in G, \quad s \in [0,1].$$

So

$$d_H\left([w]^\theta, [w]^\beta\right) \le \frac{\varepsilon\Gamma(-r+\alpha+1)}{2\lambda\Gamma(-r+1)} \frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\psi^{\alpha-r}(t) \le \frac{\varepsilon}{2} < \varepsilon.$$

Thus, $\mathcal{M}(G)(t)$ is level equicontinuous in $\mathbb{E}^c$ on $[0,1]$, $\forall t \in [0,1]$. □

Define $f_{-r}(t,u) = \psi^{-r}(t)f(t,u), t \in [0,a]$. Let we now consider

$$S = \left\{x \in \mathbb{E}^c \mid D(x,\tilde{0}) \le R\right\}.$$

As a consequence, we get the next result:

Lemma 5. *Allow* $f_{-r}+g_{-r}\colon [0,1]\times\mathbb{E}^c\to\mathbb{E}^c$ *to be uniformly continuous and bounded in* $[0,1]\times S$*. Hence, the operator* $\mathcal{N}$ *is continuous and bounded in* $C([0,a],\mathbb{E}^c)$.

Proof. Take $u_n,u\in\Omega$, if $u_n\to u$, as $n\to\infty$ in $C([0,1],\mathbb{E}^c)$. Then, for $\varepsilon>0$ by the uniform continuity of $f_{-r}+g_{-r}$ in $[0,1]\times S$, there's $\delta>0$ such as for $(t,u),(s,v)\in[0,1]\times S$ we obtain

$$H((t,u),(s,v))=|t-s|+D(u,v)<\delta,$$

which means that

$$D\left(\psi^{-r}(t)(f(t,u)+g(t,u)),\psi^{-r}(s)(f(s,v)+g(s,v))\right)<\varepsilon.$$

However, provided $\delta>0$, since $u_n\to u$ as $n\to\infty$, in terms of the limit there's $n_0\in\mathbb{N}$ in such a way that, for $n\ge n_0$, we gain

$$d_0(u_n,u)<\delta\quad i.e.,\quad \sup_{t\in[0,1]}D(u_n(t),u(t))<\delta.$$

In this case

$$H((t,u_n(t)),(t,u(t)))=D(u_n(t),u(t))<\delta,\quad \forall t\in[0,1],\forall n\ge n_0,$$

so that

$$D(f_{-r}(t,u_n(t))+g_{-r}(t,u_n(t)),f_{-r}(t,u(t))+g_{-r}(t,u(t)))<\varepsilon$$

and

$$\begin{aligned}d_0(\mathcal{N}u_n,\mathcal{N}u)&=\sup_{t\in[0,a]}D(\mathcal{N}u_n(t),\mathcal{N}u(t))\\&=\sup_{t\in[0,1]}D(f_{-r}(t,u_n(t))+g_{-r}(t,u_n(t)),f_{-r}(t,u(t))+g_{-r}(t,u(t)))\\&\le\varepsilon.\end{aligned}$$

This demonstrates that $\mathcal{N}u_n\to\mathcal{N}u$ in $C([0,a],\mathbb{E}^c)$.

In contrast, let set B is bounded in Ω, so $\forall u\in B$ we have

$$d_0(u,\tilde{0})\le M,\quad \text{i.e.,}\quad \sup_{t\in[0,1]}D(u(t),\tilde{0})\le M.$$

Then,

$$d_0(\mathcal{N}u,\tilde{0})=\sup_{t\in[0,1]}D(\mathcal{N}u(t),\tilde{0})=\sup_{t\in[0,1]}D\left(f_{-r}(t,u(t))+g_{-r}(t,u(t)),\tilde{0}\right),\forall u\in B.$$

Since f_{-r} is bounded in $[0,a]\times S$ and g_{-r} is continuous, then there's a $K>0$ such that $d_0(\mathcal{N}u,\tilde{0})\le K,\forall u\in B$.

Hence, $\mathcal{N}(B)$ is bounded. □

Lemma 6. *If $\{(f_{-r}+g_{-r})(s,x) \mid (s,x) \in [0,a] \times S\}$ is compactly supported and level equicontinuous, then*

$$\{u(s) \mid u \in N(\Omega), s \in [0,a]\}$$

is relatively compact.

Proof. Because

$$\begin{aligned}\{(f_{-r}+g_{-r})(s,u(s)) \mid u \in \Omega, s \in [0,a]\} &= \{(\mathcal{N}u)(s) \mid u \in \Omega, s \in [0,a]\} \\ &= \{u(s) \mid u \in \mathcal{N}(\Omega), s \in [0,a]\}\end{aligned}$$

is compact supported and level equicontinuous, it is relatively compact. □

Lemma 7. *Allow $f_{-r}(t,u)$ and $g_{-r}(t,u)$ to be a continuous functions on $[0,1] \times S$. Thus, they are compact on $[0,1] \times S$ iff the set*

$$\left\{\psi^{-r}(t)\,(f(t,u)+g(t,u)) \mid t \in [0,1], u \in S\right\}$$

is compact supported and level equicontinuous.

Proof. Firstly, consider $f_{-r}(t,x)$ and $g_{-r}(t,u)$ as two functions continuous and compact on $[0,1] \times S$. Then by Theorem 1, we have

$$\{f_{-r}(t,u)+g_{-r}(t,u) \mid t \in [0,1], u \in S\}$$

is compact supported and level equicontinuous.

Now let $\{f_{-r}(t,u)+g_{-r}(t,u) \mid t \in [0,1], u \in S\}$ is compact supported and level equicontinuous. By Theorem 1 once again, it is relatively compact. Hence f_{-r} an g_{-r} are compact on $[0,1] \times S$. because for any bounded set $B \subseteq [0,1] \times S$, the set

$$\{(f_{-r}+g_{-r})(t,u) \mid (t,u) \in B\}$$

is relatively compact. □

Definition 7. $\mathcal{A}\colon C([0,1],\mathbb{E}) \to C([0,1],\mathbb{E})$ is a bounded operator if for every bounded B in $C([0,1],\mathbb{E})$, $\mathcal{A}(B)$ is bounded in $C([0,1],\mathbb{E})$.

The local existence theorem for fuzzy fractional differential equations (3.1) is shown here.

Theorem 4. *Let $0 \le r < \alpha < 1$ and let f and g be two continuous applications from $(0,1] \times \mathbb{E}^c$ to $\mathbb{E}^c$. If $f_{-r}+g_{-r}$ is compact and uniformly continuous on $[0,1] \times \mathbb{E}^c$. As a result, for a sufficient $0 < \xi \le 1$, the fuzzy integral equation has at least one continuous solution specified on $[0,\xi]$.*

Proof. We simply need to examine the fuzzy integral equation, according to Remark 2:

$$u(t) = \frac{1}{\Gamma(\alpha)} \odot \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}[f(s,u(s)) \oplus g(s,u(s))]\mathrm{d}s.$$

Let's take the set

$$\Omega=\left\{u\in C\left([0,1],\mathbb{E}^{c}\right)\mid d_0(u,\tilde{0})\leq R_0\right\}.$$

It is straightforward to demonstrate that Ω is a closed, bounded, convex subset of the semilinear Banach space $C\left([0,1],\mathbb{E}^{c}\right)$. We define the operator $\mathcal{A}\colon\Omega\to C\left([0,1],\mathbb{E}^{c}\right)$ as

$$(\mathcal{A}u)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(f(s,u(s))+g(t,u(s)))\mathrm{d}s.$$

The operator $\mathcal{A}$ is declared to be continuous and compact. Clearly, the operator is the combination of the continuous and bounded operators $\mathcal{A}=\mathcal{M}\circ\mathcal{N}$, with

$$\mathcal{N}u(t)=\psi^{-r}(t)(f(t,u(t))+g(t,u(t))),\quad t\in[0,1]$$

and

$$\mathcal{M}u(t)=\frac{1}{\Gamma(\alpha)}\odot\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^{r}(s)u(s)\mathrm{d}s,\quad u\in C\left([0,1],\mathbb{E}^{c}\right).$$

The operator $\mathcal{A}$ is properly defined since it is the combination of $\mathcal{M}$ and $\mathcal{N}$. Because f_r and g_r are continuous and compact, $\{(f_{-r}+g_{-r})(t,u)\mid t\in[0,1],u\in S\}$ is compact-supported and level-equicontinuous according to Lemma 8. Thus, according to Lemma 6, $\{u(s)\mid u\in N(\Omega),s\in[0,1]\}$ is compact supported and level equicontinuous. As a result, according to Lemma 4, $\mathcal{M}(\mathcal{N}(\Omega))$ is relatively compact in $C\left([0,1],\mathbb{E}^{c}\right)$. The operator $\mathcal{A}$ is then compact on Ω. Besides that, given $0\leq t\leq\delta\leq1$ such as $\psi(t)\leq\delta$, we obtain

$$\begin{aligned}D\left(\mathcal{M}u(t),\tilde{0}\right)&\leq\sup_{t\in[0,\delta]}D\left(u(t),\tilde{0}\right)\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{q-1}\psi^{r}(s)\mathrm{d}s\\&\leq\frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)}\psi^{\alpha-r}(t)\|u\|_0\\&\leq\frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)}\delta^{\alpha-r}\|u\|_0.\end{aligned}$$

Hence, we have

$$\|\mathcal{M}u\|_0\leq\varepsilon\|u\|_0,$$

where we are able to reduce $\delta>0$ to make $\varepsilon>0$ as small as we wish. Furthermore define $B\rho$ as the field of the operator $\mathcal{A}$, where $B\rho=\{u\in C\left([0,\delta],\mathbb{E}^{c}\right):\|u\|0\leq\rho\}$, a convex, limited, and close subset of the full metric space $C\left([0,\delta],\mathbb{E}^{c}\right)$. For $\delta>0$ small enough, we get

$$\mathcal{A}\left(B_\rho\right)\subseteq B_\rho.$$

Theorem 2 guarantees that the operator $\mathcal{A}$ has at least one fixed point. As a result, equation (3.1) has at least one solution u given on $[0,\delta]$, where $\delta>0$ and $\delta\leq1$. □

Corollary 1. *Under the circumstances of Theorem 4 and with the assumption that* $f(\cdot,u(\cdot)) \in L^1((0,1),\mathbb{E}^c)$ *and* $g(\cdot,u(\cdot)) \in L^1((0,1),\mathbb{E}^c)$, $\forall u \in C([0,1],\mathbb{E}^c)$, *the fuzzy fractional differential equation (3.1) must thus have at least one solution specified on* $[0,\delta]$, *seeking an appropriate* $0 < \delta \leq 1$.

Proof. Suppose $u \in C([0,\delta],\mathbb{E}^c)$ is a solution to the fuzzy equation

$$u(t) = I^{\alpha}(f(t,u(t)) + g(t,u(t)))$$

utilizing that $f : (0,\delta] \times \mathbb{E}^c \longrightarrow \mathbb{E}^c$ and $g : (0,\delta] \times \mathbb{E}^c \longrightarrow \mathbb{E}^c$ are continuous and $f(\cdot,u(\cdot)) \in L^1((0,\delta),\mathbb{E}^c)$ and $g(\cdot,u(\cdot)) \in L^1((0,\delta),\mathbb{E}^c)$, for each $u \in C([0,\delta],\mathbb{E}^c)$, hence it is obvious that $f(t,u(t)) + g(t,u(t)) \in C((0,\delta],\mathbb{E}) \cap L^1((0,\delta),\mathbb{E})$ and u is a solution to equation (3.1) in $[0,\delta]$. □

3.2. *Existence of solution via Krasnosel'skii fixed point theorem*

Let $I_\delta := [0,\delta]$, where $\delta \leq 1$, and $r_0 > 0$ such as

$$R := \sup\left\{D(\tilde{0}, f(t,x)) \mid t \in I_{\delta_0}, D(x,\tilde{0}) \leq r_0\right\} < +\infty.$$

It is also feasible to pick δ to be minimal enough so that $\dfrac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\psi^{\alpha-r}(\delta) \leq r_0$.
Define the set

$$\Phi := \left\{x \in C(I_\delta,\mathbb{E}^c) \mid D(\tilde{0},x(t)) \leq r_0, \text{ for all } t \in I_\delta, \text{ and } x(0) = \tilde{0}\right\}.$$

We can easily confirm that the set $\Phi \subseteq C(I_\delta,\mathbb{E}^c)$ is bounded, closed, and convex. Additionally, $C(I_\delta,\mathbb{E}^c)$ is a semi-linear Banach space. Within the set Φ, we examine two mappings corresponding to the functions f and g. Specifically, consider the mapping $\mathcal{F} : \Phi \to C(I_\delta,\mathbb{E}^c)$, defined as follows:

$$(\mathcal{F}u)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)f(s,u(s))\mathrm{d}s, \text{ for } t \in I_\delta. \tag{3.3}$$

To simplify the method, we refine several notations related to the various constraints on the function f that arise due to the non-linearity of the equation.

We introduce the criterion (H) for the function $f \in C([0,1]\times\mathbb{E}^c,\mathbb{E}^c)$, which involves the following condition: the inequality

$$D(f(t,u(t)), f(t,v(t))) \leq p(t)\, w\,(d_0(u,v))$$

holds for any pair of points $(t,u),(t,v) \in [0,a]\times C([0,1],\mathbb{E}^c)$, where w is a real continuous function on $[0,\infty)$ with $w(0) = 0$, and $p\colon [0,1] \to \mathbb{R}^+$ satisfies $I^\alpha p(t) < N$ for every $t \in [0,1]$.

Lemma 8. *Consider f fulfill (H) on $[0,a]\times\mathbb{E}^c$. The following assumptions are thus meet: Let f fulfill (H) on $[0,a]\times\mathbb{E}^c$; the following assumptions are thus verified:*

i) *$\mathcal{F}$ is well-defined on $(I_\delta,\mathbb{E}^c)$.*
ii) *$\mathcal{F}$ is a continuous mapping on $C(I_\delta,\mathbb{E}^c)$.*

Proof. To begin, we demonstrate that the map $\mathcal{F}$ is quite defined. For any $uin\Phi$, $(\mathcal{F}u)(0)=\tilde{0}$ is evident by construction. With a given $uin\Phi$, we prove that $\mathcal{F}u\in C(I_\delta,\mathbb{E}^c)$. Furthermore, we demonstrate that $\mathcal{F}$ is a uniformly continuous function on the interval I_δ. We choose a determined $t,t'\in I_\delta$, with $t<t'$. We derive these components

$$\begin{aligned}
D\left((\mathcal{F}u)(t),(\mathcal{F}u)\left(t'\right)\right)=&\frac{1}{\Gamma(\alpha)}D\left(\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)f(s,u(s))\mathrm{d}s,\right.\\
&\left.\int_0^{t'}\psi'(s)(\psi(t')-\psi(s))^{\alpha-1}\psi^r(s)f(s,u(s))\mathrm{d}s\right)\\
=&\frac{1}{\Gamma(\alpha)}D\left(\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)f(s,u(s))\mathrm{d}s,\right.\\
&\int_0^t\psi'(s)(\psi(t')-\psi(s))^{\alpha-1}\psi^r(s)f(s,u(s))\mathrm{d}s\ldots\\
&\left.+\int_t^{t'}\psi'(s)(\psi(t')-\psi(s))^{\alpha-1}\psi^r(s)f(s,u(s))\mathrm{d}s\right)\\
\leq&\frac{1}{\Gamma(\alpha)}\left[D\left(\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)f(s,u(s))\mathrm{d}s,\right.\right.\\
&\left.\int_0^t\psi'(s)(\psi(t')-\psi(s))^{\alpha-1}\psi^r(s)f(s,u(s))\mathrm{d}s\right)\ldots\\
&\left.+D\left(\int_t^{t'}\psi'(s)(\psi(t')-\psi(s))^{\alpha-1}\psi^r(s)f(s,u(s))\mathrm{d}s,\tilde{0}\right)\right]\\
\leq&\frac{R}{\Gamma(\alpha)}\left[\int_0^t\psi'(s)\left|(\psi(t)-\psi(s))^{\alpha-1}-(\psi(t')-\psi(s))^{\alpha-1}\right|\psi^r(s)\mathrm{d}s\right.\\
&\left.+\int_t^{t'}\psi'(s)\left|\psi(t')-\psi(s)\right|^{\alpha-1}\psi^r(s)\mathrm{d}s\right]\\
\leq&\frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\left(2\left|\psi(t')-\psi(t)\right|^{\alpha-r}+\left|\psi^{\alpha-r}(t)-\psi^{\alpha-r}(t')\right|\right).
\end{aligned}$$

Therefore, $D\left((\mathcal{F}u)(t),(\mathcal{F}u)\left(t'\right)\right)\to 0$, when $|t-t'|$ tends to 0, so $\mathcal{F}u$ is a continuous mapping on I_δ, for $u\in\Phi$. In addition, we have

$$\begin{aligned}
D((\mathcal{F}u)(t),\tilde{0})\leq&\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)D\left(f(s,u(s)),\tilde{0}\right)\mathrm{d}s\\
\leq&\frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\psi^{\alpha-r}(\delta)\\
\leq&r_0
\end{aligned}$$

$\forall t\in I_\delta$. As a result, $\mathcal{F}$ is a self-mapping $\mathcal{F}:\Phi\to\Phi$.

We next demonstrate that $\mathcal{T}$ is a continuous map. Consider $y_n,y\in\Phi,n=1,2,\ldots$ such as $y_n\underset{n\to\infty}{\longrightarrow}y$, with convergence in the set $C([0,1],\mathbb{E}^c)$, i.e, fulfill $d_0(y_n,y)\underset{n\to\infty}{\longrightarrow}0$.

As a result, we may derive that for any $t \in J := [0,1]$,

$$
\begin{aligned}
d_0(\mathcal{F}y_n, \mathcal{F}y) &= \sup_{t\in J} D\left((\mathcal{F}y_n)(t), (\mathcal{F}y)(t)\right) \\
&= \sup_{t\in I_{\delta_0}} D\left(\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s) f(s, y_n(s))\,\mathrm{d}s,\right. \\
&\qquad \left.\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s) f(s, y(s))\,\mathrm{d}s\right) \\
&\leq \frac{1}{\Gamma(\alpha)} \sup_{t\in I_\delta}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s) D\left(f(s, y_n(s)), f(s, y(s))\right)\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha)} \sup_{t\in I_\delta}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s) p(s) w\left(d_0(y_n, y)\right)\mathrm{d}s \\
&\leq \sup_{t\in I_{\delta_0}} I^\alpha p(t) w\left(\rho_0(y_n, y)\right).
\end{aligned}
$$

Because, according to the criteria in (H), $w(0) = 0$ and w is continuous on its domain $[0,\infty)$, it is obvious that $w(r) \underset{r\to 0^+}{\longrightarrow} 0$. Due to $\rho_0(y_n, y) \underset{n\to\infty}{\longrightarrow} 0$ and the estimation $I^\nu p(t) < N$, we get to the conclusion that $\mathcal{F}y_n \to \mathcal{F}y$ as $n \to \infty$, i.e., $\mathcal{F}$ is continuous. □

Theorem 5. *Take a given $u \in (0,1)$ and assume that the fuzzy function $f\colon [0,1]\times\mathbb{E}^c \to \mathbb{E}^c$ is continuous and compact, fulfilling (H), and suppose that $g\colon [0,1]\times\mathbb{E}^c \to \mathbb{E}^c$ fulfills (3.2). Given these constraints, the non-local UDE (3.1) has (at least) one solution specified on $[0,\delta]$. This is a continuous function, with δ being an acceptable positive integer $\delta < 1$.*

Proof. Using Remark 2, we are restricted to investigate the uncertain integral equation

$$
\begin{aligned}
u(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s) f(s,u(s))\mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s) g(s,u(s))\mathrm{d}s.
\end{aligned}
$$

The mapping $\mathcal{F}$ is precisely considered by Lemma 8, equation (3.3), as well as its continuity on the set $C(I_\delta, \mathbb{E}^c)$. The next section argues that $\mathcal{F}$ is a compact mapping. If we choose an arbitrary determined $u \in \Phi$ and let $t, t' \in I\delta$ with $t \leq t'$, so

$$
\begin{aligned}
D\left((\mathcal{F}u)(t), (\mathcal{F}u)(t')\right) = \frac{1}{\Gamma(\alpha)} D\Bigg(&\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s) f(s,u(s))\mathrm{d}s\,, \\
&\int_0^{t'} \psi'(s)(\psi(t')-\psi(s))^{\alpha-1}\psi^r(s) f(s,u(s))\mathrm{d}s\Bigg)
\end{aligned}
$$

$$\leq \frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\left(2\left(\psi(t')-\psi(t)\right)^{\alpha-r}+\psi^{\alpha-r}(t)-\psi^{\alpha-r}(t')\right).$$

Thus means that $\mathcal{F}(\Phi)$ is equicontinuous in $C(I_\delta,\mathbb{E}^c)$.

Furthermore, we employ Theorem 1 to demonstrate the relative compactness of $\mathcal{F}(\Phi)(t)$ in $\mathbb{E}^c$, implying that $\mathcal{F}(\Phi)(t)$ is compact supported and level equicontinuous in $\mathbb{E}^c$.

Consider $t\in[0,\delta]$, we check that $\mathcal{F}(\Phi)(t)\subseteq\mathbb{E}^c$. For every $u\in\mathbb{T}(\Phi)(t)$, it is possible to write

$$u=\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)f(s,y(s))\mathrm{d}s,\ \text{for some } y\in\Phi.$$

Given that f is a compact mapping, $f(I\delta\times\Phi)$ is a relatively compact in $\mathbb{E}^c$. According to Theorem 1, $f(I\delta\times\Phi)$ is a level equicontinuous. As a result, we can assert the existence of $\delta'>0$ for any $\varepsilon>0$.

$$D\left([f(s,y(s))]^{\alpha_1},[f(s,y(s))]^{\alpha_2}\right)<\frac{\Gamma(-r+\alpha+1)}{\psi^{\alpha-r}(\delta)\Gamma(-r+1)}\varepsilon,\forall(s,y)\in I_{\delta_0}\times\Phi,$$

provided that $|\alpha_1-\alpha_2|<\delta$. Therefore, similarly to [1], for $|\alpha_1-\alpha_2|<\delta$, we deduce

$$\begin{aligned}
D\left([u]^{\alpha_1},[u]^{\alpha_2}\right)&=D\left([\mathcal{F}(y)(t)]^{\alpha_1},[\mathcal{F}(y)(t)]^{\alpha_2}\right)\\
&\leq\frac{1}{\Gamma(\alpha)}D\left(\left[\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)f(s,y(s))\mathrm{d}s\right]^{\alpha_1},\right.\\
&\qquad\left.\left[\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)f(s,y(s))\mathrm{d}s\right]^{\alpha_2}\right)\\
&\leq\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)D\left([f(s,y(s))]^{\alpha_1},[f(s,y(s))]^{\alpha_2}\right)\mathrm{d}s\\
&\leq\frac{\psi^{\alpha-r}(\delta)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}D\left([f(s,y(s))]^{\alpha_1},[f(s,y(s))]^{\alpha_2}\right)\leq\varepsilon.
\end{aligned}$$

Thus, $\mathcal{F}(\Phi)(t)$ is level equicontinuous in $\mathbb{E}^c$. Consequently, given the relative compactness of $f(I_\delta\times\Phi)$, for any $(s,y)\in I_\delta\times\Phi$, there exists a compact set $M\subset\mathbb{R}^n$ such that $[f(s,y(s))]_0\subseteq M$. Therefore, we obtain

$$\begin{aligned}
&\left[\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)f(s,y(s))\mathrm{d}s\right]^0\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)[f(s,y(s))]^0\mathrm{d}s\\
&\subseteq\frac{M}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)\mathrm{d}s=\frac{\psi^{\alpha-r}(t)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}M.
\end{aligned}$$

As a consequence, as a subset of $\mathbb{E}^c,\mathcal{F}(\Phi)(t)$ is compact supported. Hence, the relative compactness of $\mathcal{F}(\Phi)$ in $C([0,\delta],\mathbb{E}^c)$ is inferred using the Arzelà-Ascoli principle.

On the other hand, we take into account the mapping $\widetilde{\mathcal{F}}: \Phi \to C(I\delta, \mathbb{E}^c)$ is described as

$$(\tilde{\mathcal{F}}u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)g(s,u(s))\mathrm{d}s, \quad t \in I_{\delta_0}.$$

Therefore,

$$\begin{aligned} D((\widetilde{\mathcal{F}}x)(t),(\widetilde{\mathcal{F}}y)(t)) &\le \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)D(g(s,x(s)),g(s,y(s)))\mathrm{d}s \\ &\le \frac{L}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\psi^r(s)D(x(s),y(s))\mathrm{d}s \\ &\le \frac{L\psi^{\alpha-r}(t)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} d_0(x,y) \\ &\le \frac{L\psi^{\alpha-r}(\delta)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} d_0(x,y). \end{aligned}$$

As a result, if δ is short enough that $L\psi^{\alpha-r}(\delta)\Gamma(-r+1) < \Gamma(-r+\alpha+1)$, therefore the mapping $\widetilde{\mathcal{F}}$ is contraction. As a result, invoking Theorem 3, the mapping $\mathcal{F} + \widetilde{\mathcal{F}}$ has at least one fixed point in Φ. As a consequence, for a given $0 < \delta \le a$, there's (at least) one fuzzy solution u for (3.1) on I_δ. □

4. AN ILLUSTRATIVE EXAMPLE

The research has highlighted the practical use of arbitrary-order mathematics to physical issues. We concentrate on a particular model presented in [8] as an illustration of how our approach might be used.

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\psi}u(t) = \eta v(t) \oplus \rho w(t)(x(t)\sin(\xi) \ominus v(t)\cos(\xi)) \ominus \frac{1}{T_2}u(t), \\ {}^{C}D_{0^+}^{\alpha,\psi}v(t) = (-\eta)u(t) \ominus w(t) \oplus \rho w(t)(v(t)\sin(\xi) \oplus u(t)\cos(\xi)) \ominus \frac{1}{T_2}v(t), \\ {}^{C}D_{0^+}^{\alpha,\psi}w(t) = v(t) \ominus \rho\sin(\xi)\left(u^2(t) \oplus v^2(t)\right) \ominus \frac{1}{T_1}(w(t)-1), \end{cases} \tag{4.1}$$

where $\alpha \in (0,1)$, $\eta = -0.4\pi$, $\rho = 30$, $\xi = 0.173$, $T_1 = 5$ and $T_2 = 2.5$.

The ψ-Caputo fractional derivative is used. In f nuclear magnetic resonance (NMR) difficulties, the spinning mechanism is disrupted by a sequence of short-duration radiofrequency (RF) pulses (generally under a millisecond), which is lengthy in relation to the frequency of resonance period (about 10 ns for 3 T magnetic resonance imaging (MRI) at 127 MHz), yet small in comparison to the T_1 and T_2 relaxation times (300 and 100 ms, respectively). Thus we has to be capable of expressing the Bloch equation with a derivative that we may switch on and off in a pulse- or step-like way. Under such conditions, the Caputo is clearly defined, but the Riemann-Liouville is not, and it also requires fractional order beginning conditions, which we cannot provide.

This system's initial equation relates to the framework under consideration. Alternatively, comparable results may be achieved for the space $C([0,1],\mathbb{E}\times\mathbb{E}\times\mathbb{E})$ with noticeable modifications; thus, it might be feasible to recast issue (4.1) as a higher dimension extension of problem (3.1), by selecting

$$f(t,u,v,w) = (\eta v \oplus \rho w(x\sin(\xi)\ominus v\cos(\xi)), (-\eta)u\ominus w\oplus \rho w(v\sin(\xi)\oplus u\cos(\xi)), \\ v\ominus \rho\sin(\xi)\left(u^2\oplus v^2\right))$$

and

$$g(t,u,v,w) = \left(\ominus\frac{1}{T_2}u, \ominus\frac{1}{T_2}v, \ominus\frac{1}{T_1}(w\ominus 1)\right)$$

which meet the relevant requirements if g is well-defined.

Conclusion

In this study, we investigated the existence of solutions to non-local uncertain differential equations including Caputo type fractional derivatives with regard to another function ψ. The formulation of mild solutions is provided. Our demonstrations of existence are based on the fixed point theorems of Schauder and Krasnosel'skii. An example is provided as an application to demonstrate the acquired results.

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