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# CONVERGENCE ANALYSIS OF SEMI-EXPONENTIAL POST-WIDDER OPERATORS

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*Abstract.* In the present article, we provide a recurrence relation for the semi-exponential Post-Widder operators (1.1) and estimate the moments for these operators. The next section discusses some convergence results in the Lipschitz-type space and estimates the rate of convergence with the help of the Ditzian-Totik modulus of smoothness and weighted modulus of continuity. At last, we estimate the rate of convergence for the functions whose derivatives are of bounded variation.

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## 1. INTRODUCTION

In the literature on approximation theory, several generalizations of exponential-type operators have been studied by many authors. The authors have focused on examining the rate of convergence of these generalizations. Recently, some researchers have introduced the concept of semi-exponential operators from the exponential-type operators.

The Post-Widder operators for $n \in \mathbb{N}$ and $x \in [0,\infty)$ considered by D.V. Widder [12] is defined as:

$$P_n(\varpi;x) = \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1}\int_0^\infty \lambda^n e^{-\frac{n\lambda}{x}}\varpi(\lambda)\mathrm{d}\lambda.$$

Following [12], For $x \in [0,\infty)$ and a parameter $p > 0$ Rathore and Singh [10] proposed an integral representation of Post-Widder operators as:

$$P_{n,p}(\varpi;x) = \frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1}\int_0^\infty \lambda^{n+p} e^{-\frac{n\lambda}{x}}\varpi(\lambda)\mathrm{d}\lambda.$$

Recently, for $x \in [0,\infty)$ and $\rho > 0$ using laplace transformation M. Herzog [5] defined the semi exponential Post-Widder operators as:

$$W_n^{\rho}(\varpi;x) = \frac{n}{\lambda^n e^{\rho\lambda}} \int_0^{\infty} \frac{\left(\frac{nu}{\rho}\right)^{\frac{n-1}{2}} I_{n-1}\left(2\sqrt{n\rho u}\right)}{e^{\frac{nu}{\lambda}}} \varpi(\lambda) d\lambda.$$

Following [5], an alternative approach of semi exponential Post-Widder operators is given by U. Abel at al. [2] which is defined as:

$$\mathcal{P}_{n,m}^{\rho}(\varpi;x) = \frac{n^n}{e^{\rho x} x^n} \sum_{m=0}^{\infty} \frac{(n\rho)^m}{m!} \frac{1}{\Gamma(n+m)} \int_0^{\infty} \lambda^{n+m-1} e^{-\frac{n\lambda}{x}} \varpi(\lambda) d\lambda \tag{1.1}$$

and an alternative form of operators (1.1) is defined as follows:

$$\mathcal{P}_{n,m}^{\rho}(\varpi;x) = \frac{n^n}{e^{\rho x} x^n} \int_0^{\infty} \left(\frac{n\lambda}{\rho}\right)^{\frac{n-1}{2}} e^{\frac{-n\lambda}{n}} I_{n-1}\left(2\sqrt{n\rho\lambda}\right) \varpi(\lambda) d\lambda. \tag{1.2}$$

Where $I_n$ represents the modified Bessel function of first kind. For more literature related to this article we may refer [1,4,6,9,11].

In this article, we study some fundamental properties of the operators (1.1), including the rate of convergence using modulus of continuity, Lipschitz-type space, and weighted space.

## 2. Basic Properties

In this section, we discuss some useful lemmas and results.

*Remark* 1. For $\rho > 0$, if we denote $H_{n,r}^{\rho} = \mathcal{P}_{n,m}^{\rho}(e_r;x)$, $e_r(\lambda) = \lambda^r$, $r = 1,2,3\cdot, x > 0$ then

$$nH_{n,r+1}^{\rho}(x) = x(\rho x + n)H_{n,r}^{\rho}(x) + x^2 H_{n,r}^{\rho}{}'(x).$$

**Lemma 1.** *Using the Remark 1, the moments of the operators $\mathcal{P}_{n,k}^{\rho}$, could be written as:*

$$\mathcal{P}_{n,m}^{\rho}(e_0;x) = 1,$$

$$\mathcal{P}_{n,m}^{\rho}(e_1;x) = \frac{x}{n}(n+\rho x),$$

$$\mathcal{P}_{n,m}^{\rho}(e_2;x) = \frac{x^2}{n^2}\left[n+n^2+2\rho x+2\rho n x+\rho^2 x^2\right],$$

$$\mathcal{P}_{n,m}^{\rho}(e_3;x) = \frac{x^3}{n^3}\left[2n+3n^2+n^3+6\rho x+9\rho n x+3\rho n^2 x+6\rho^2 x^2+3\rho^2 n x^2+\rho^3 x^3\right],$$

$$\begin{aligned}\mathcal{P}_{n,m}^{\rho}(e_4;x) = \frac{x^4}{n^4}\Big[&6n+11n^2+6n^3+n^4+24\rho x+44\rho n x+24\rho n^2 x+4\rho n^3 x+36\rho^2 x^2\\ &+30\rho^2 n x^2+6\rho^2 n^2 x^2+12\rho^3 x^3+4\rho^3 n x^3+\rho^4 x^4\Big].\end{aligned}$$

**Lemma 2.** *Central moments of the operators $\mathcal{P}^{\rho}_{n,m}$, with the help of Lemma 1, are as follows*

$$\mathcal{P}^{\rho}_{n,m}((\lambda-x);x)=\frac{\rho x^2}{n},$$
$$\mathcal{P}^{\rho}_{n,m}((\lambda-x)^2;x)=\frac{x^2}{n^2}\left[n+2\rho x+\rho^2x^2\right],$$
$$\mathcal{P}^{\rho}_{n,m}((\lambda-x)^3;x)=\frac{x^3}{n^3}\left[2n-4n^3+6\rho x+3\rho nx-6\rho n^2x+6\rho^2x^2+\rho^3x^3\right],$$
$$\mathcal{P}^{\rho}_{n,m}((\lambda-x)^4;x)=\frac{x^4}{n^4}\left[6n+3n^2+24\rho x+20\rho nx\right.$$
$$\left.+36\rho^2x^2+6\rho^2nx^2+12\rho^3x^3+\rho^4x^4\right].$$

**Lemma 3.** *Using Lemma 2, we have*

$$\mathcal{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)=\frac{x^2}{n^2}\left(n+2\rho x+\rho^2x^2\right)\le\frac{x^2}{n}\left(1+2\rho x+\rho^2x^2\right)\le\frac{\chi^2(x)}{n},$$

*and*

$$\mathcal{P}^{\rho}_{n,m}((\lambda-x)^4;x)\le C\frac{\chi^4(x)}{n^2},$$

*where, $C$ is a large positive constant and $\chi(x)=x(1+\rho x)$. Moreover*

$$\lim_{n\to\infty}n\mathcal{P}^{\rho}_{n,m}((\lambda-x);x)=\rho x^2,\quad \textit{and}\quad \lim_{n\to\infty}n\mathcal{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)=x^2.$$

**Lemma 4.** *For the operators $\mathcal{P}^{\rho}_{n,m}$ and $\varpi\in[0,\infty)$, we have*

$$|\mathcal{P}^{\rho}_{n,m}(\varpi;x)|\le\|\varpi\|,$$

*where norm of the function on the positive half real line is given by $\|\varpi\|=\sup\limits_{x\in[0,\infty)}|\varpi(x)|$.*

*Proof.* From (1.2) and Lemma 1, we have

$$|\mathcal{P}^{\rho}_{n,m}(\varpi;x)|\le\frac{n^n}{e^{-\rho x}x^n}\sum_{m=0}^{\infty}\frac{(n\rho)^m}{m!}\frac{1}{\Gamma(n+m)}\int_0^{\infty}\lambda^{n+m-1}e^{-\frac{n\lambda}{x}}|\varpi(\lambda)|d\lambda\le\|\varpi\|.$$

□

**Theorem 1.** *Let $\varpi\in C_B[0,\infty)$, then $\lim\limits_{n\to\infty}\mathcal{P}^{\rho}_{n,m}(\varpi;x)=\varpi(x)$, uniformly in every closed interval in $[0,\infty)$.*

*Proof.* From Lemma 1, $\mathcal{P}^{\rho}_{n,m}(e_0;x)=1$, $\mathcal{P}^{\rho}_{n,m}(\lambda;x)=x$, $\mathcal{P}^{\rho}_{n,m}(\lambda^2;x)=x^2$, as $n\to\infty$. Therefore by the Bohman-Korovkin theorem, we get $\mathcal{P}^{\rho}_{n}(\varpi(\lambda);x)=\varpi(x)$ as $n\to\infty$, uniformly in every closed subinterval of $[0,\infty)$. □

## 3. MAIN RESULTS

Here, we assess the rate of convergence by using the Ditzian-Totik modulus of smoothness $\omega_{\chi^\gamma}(\varpi,\delta)$ and Peetre's $K-$functional $K_{\chi^\gamma}(\varpi,\delta)$, $\quad 0\leq\gamma\leq 1$. For $\varpi\in C_B[0,\infty)$ and $\chi(x)=x(1+\rho x)$, the Ditzian-Totik modulus of smoothness is explained as

$$\omega_{\chi^\gamma}(\varpi,\delta)=\sup_{0\leq i\leq\delta}\sup_{x\pm\frac{i\chi^\gamma(x)}{2}\in[0,\infty)}\left|\varpi\left(x+\frac{i\chi^\gamma(x)}{2}\right)-\varpi\left(x-\frac{i\chi^\gamma(x)}{2}\right)\right|,$$

and the Peetre's $K-$functional is defined as

$$\omega_{\chi^\gamma}(\varpi,\delta)=\inf_{\varphi\in W_\gamma}\{\|\varpi-\varphi\|-\delta\|\chi^\gamma\varphi'\|\},$$

where $W_\gamma$ is a subspace of all real valued functions defined on $[0,\infty)$, and $\varphi\in W_\gamma$ which is locally absolutely continuous with norm $\|\varpi^\gamma\varphi'\|\leq\infty$. In [3, Theorem 2.1.1], there exists a constant $\mathcal{D}\geq 0$ such that

$$\mathcal{D}^{-1}\omega_{\chi^\gamma}(\varpi,\delta)\leq K_{\chi^\gamma}(\varpi,\delta)\leq\mathcal{D}\omega_{\chi^\gamma}(\varpi,\delta). \tag{3.1}$$

**Theorem 2.** *For $\varpi\in C_B[0,\infty)$ then, we have*

$$\left|\mathcal{P}^\rho_{n,m}(\varpi;x)-\varpi(x)\right|\leq\mathcal{D}\omega_{\chi^\gamma}\left(\varpi;\frac{\chi^{2-\gamma}(x)}{\sqrt{n}}\right).$$

*Proof.* For $\varphi\in W_\gamma$, and calling the representation

$$\varphi(\lambda)=\varphi(x)+\int_x^\lambda\varphi'(s)\mathrm{d}s.$$

Applying $\mathcal{P}^\rho_{n,m}$ and using Hölder's inequality, we have

$$\begin{aligned}\left|\mathcal{P}^\rho_{n,m}(\varphi(\lambda);x)-\varphi(x)\right|&\leq\mathcal{P}^\rho_{n,m}\left(\int_x^\lambda\left|\varphi'\right|\mathrm{d}s;x\right)\\&\leq\|\varpi^\gamma\varphi'\|\mathcal{P}^\rho_{n,m}\left(\left|\int_x^\lambda\frac{ds}{\chi^\gamma(s)}\right|;x\right)\\&\leq\|\varpi^\gamma\varphi'\|\mathcal{P}^\rho_{n,m}\left(|\lambda-x|^{1-\gamma}\left|\int_x^\lambda\frac{ds}{\chi(s)}\right|^\gamma;x\right).\end{aligned} \tag{3.2}$$

Let $I=\left|\int_x^\lambda\frac{ds}{\chi(s)}\right|$, now first we simplify expression $I$

$$\begin{aligned}I&\leq\left|\int_x^\lambda\frac{ds}{\sqrt{s}}\right|\left|\left(\frac{1}{1+\rho x}+\frac{1}{1+\rho\lambda}\right)\right|\leq 2\left|\sqrt{\lambda}-\sqrt{x}\right|\left(\frac{1}{1+\rho x}+\frac{1}{1+\rho\lambda}\right)\\&\leq 2\frac{|\lambda-x|}{\sqrt{x}}\left(\frac{1}{1+\rho x}+\frac{1}{1+\rho\lambda}\right).\end{aligned} \tag{3.3}$$

Now, we use the inequality $|p+q|^{\gamma} \leq |p|^{\gamma}+|q|^{\gamma}, \quad 0 \leq \gamma \leq 1$ then from (3.3), we get

$$\left|\int_x^{\lambda} \frac{ds}{\chi(s)}\right|^{\gamma} \leq 2^{\gamma} \frac{|\lambda-x|^{\gamma}}{x^{\frac{\gamma}{2}}}\left(\frac{1}{(1+\rho x)^{\frac{\gamma}{2}}}+\frac{1}{(1+\rho\lambda)^{\frac{\gamma}{2}}}\right). \tag{3.4}$$

From (3.2) and (3.4) and using Cauchy inequality, we get

$$\left|\mathcal{P}^{\rho}_{n,m}(\varphi(\lambda);x)-\varphi(x)\right| \leq \frac{2^{\gamma}\|\chi^{\gamma}\varphi'\|}{x^{\frac{\gamma}{2}}}\mathcal{P}^{\rho}_{n,m}\left(|\lambda-x|\left(\frac{1}{(1+\rho x)^{\frac{\gamma}{2}}}+\frac{1}{(1+\rho\lambda)^{\frac{\gamma}{2}}}\right);x\right)$$

$$=\frac{2^{\gamma}\|\chi^{\gamma}\varphi'\|}{x^{\frac{\gamma}{2}}}\left(\frac{1}{(1+\rho x)^{\frac{\gamma}{2}}}\left(\mathcal{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)\right)^{\frac{1}{2}}\right.$$
$$\left.+\left(\mathcal{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)\right)^{\frac{1}{2}}\cdot\left(\mathcal{P}^{\rho}_{n,m}\left((1+\rho\lambda)^{-\gamma};x\right)\right)^{\frac{1}{2}}\right).$$

From Lemma 2, we may write

$$\left(\mathcal{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)\right)^{\frac{1}{2}} \leq \frac{\chi^2(x)}{\sqrt{n}}, \tag{3.5}$$

where $\chi(x)=x(1+\rho x)$.

For $x\in[0,\infty)$, $\mathcal{P}^{\rho}_{n,m}((1+\rho\lambda)^{-\gamma};x)\to(1+\rho x)^{-\gamma}$ as $n\to\infty$. Thus for $\varepsilon>0$, we find a number $n_0\in\mathbb{N}$ such that

$$\mathcal{P}^{\rho}_{n,m}\left((1+\rho\lambda)^{-\gamma};x\right) \leq (1+\rho x)^{-\gamma}+\varepsilon, \qquad \text{for all } n\geq n_0.$$

By choosing $\varepsilon=(1+\rho x)^{-\gamma}$, we obtain

$$\mathcal{P}^{\rho}_{n,m}\left((1+\rho\lambda)^{-\gamma};x\right) \leq 2(1+\rho x)^{-\gamma}, \qquad \text{for all } n\geq n_0. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\left|\mathcal{P}^{\rho}_{n,m}(\varphi(\lambda);x)-\varphi(x)\right| \leq 2^{\gamma}\|\chi^{\gamma}\varphi'\|\frac{\chi^2(x)}{\sqrt{n}}\left(\chi^{-\gamma}(x)+\sqrt{2}x^{-\frac{\gamma}{2}}(1+\rho x)^{-\frac{\gamma}{2}}\right)$$
$$\leq 2^{\gamma}(1+\sqrt{2})\|\chi^{\gamma}\varphi'\|\frac{1}{\sqrt{n}}\chi^{2-\gamma}(x). \tag{3.7}$$

We may write

$$\begin{aligned}\left|\mathcal{P}^{\rho}_{n,m}(\varpi(\lambda);x)-\varpi(x)\right| &\leq \left|\mathcal{P}^{\rho}_{n,m}(\varpi(\lambda)-\varphi(\lambda);x)\right| \\ &\quad+\left|\mathcal{P}^{\rho}_{n,m}(\varphi(\lambda);x)-\varphi(x)\right|+|\varphi(x)-\varpi(x)| \\ &\leq 2\|\varpi-\varphi\|+\left|\mathcal{P}^{\rho}_{n,m}(\varphi(\lambda);x)-\varphi(x)\right|.\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) and for sufficiently large $n$, we obtain

$$\left|\mathcal{P}^{\rho}_{n,m}(\varpi(\lambda);x)-\varpi(x)\right| \leq 2\|\varpi-\varphi\|+2^{\gamma}\left(1+\sqrt{2}\right)\|\chi^{\gamma}\varphi'\|\frac{1}{\sqrt{n}}\chi^{2-\gamma}(x)$$
$$\leq C_1\{\|\varpi-\varphi\|+\chi^{2-\gamma}(x)\frac{1}{\sqrt{n}}\|\chi^{\gamma}\varphi'\|\} \leq CK_{\chi^{\lambda}}\left(\varpi;\chi^{2-\gamma}(x)\frac{1}{\sqrt{n}}\right), \tag{3.9}$$

where $C_1 = \max\{2, 2^{\lambda}(1+\sqrt{2})\}$ and $C = 2C_1$. From (3.1) and (3.9) we may conclude the required result. □

Let $C^B[0,\infty)$ be the class of all absolutely continuous and bounded functions equipped with the norm $\|\varpi\| = \sup\{|\varpi(x)| : x \in [0,\infty)\}$. Then, K-functional in terms of modulus of smoothness is given by:

$$K_2(\varpi, \zeta) = \inf_{h \in W^2}\{\|\varpi - \varphi\| + \zeta\|\varphi''\|\},$$

where $\zeta > 0$ and $W^2 = \{\varphi \in C^B[0,\infty) : \varphi', \varphi'' \in C^B[0,\infty)\}$. Now, we find a constant $C_0 > 0$ such that

$$K_2(\varpi, \zeta) \leq C_0\, \omega_2(\varpi, \sqrt{\zeta}), \tag{3.10}$$

where

$$\omega_2(\varpi, \sqrt{\zeta}) = \sup_{0<\wp\leq\sqrt{\zeta}} \sup_{x\in[0,\infty)} |\varpi(x+2\wp) - 2\varpi(x+\wp) + \varpi(x)|$$

is known as second order modulus of smoothness of $\varpi \in C^B[0,\infty)$.

**Theorem 3.** *For real valued continuous function $\varpi \in C^B[0,\infty)$, we have*

$$|\mathcal{P}^{\rho}_{n,m}(\varpi;x) - \varpi(x)| \leq C_0\omega_2(\varpi,\zeta) + \omega\left(\varpi; \left|\frac{x}{n}(n+\rho x)\right|\right).$$

*Proof.* For any real and continuous function $\xi \in C^B[0,\infty)$, by Taylor's expansion, we have

$$\xi(\lambda) = \xi(x) + (\lambda - x)\xi'(x) + \int_{\lambda}^{x} (\lambda - s)\xi''(s)\mathrm{d}s. \tag{3.11}$$

Consider an auxiliary operators associated with $\mathcal{P}^{\rho}_{n,m}$

$$\tilde{\mathcal{P}}^{\rho}_{n,m}(\varpi;x) = \mathcal{P}^{\rho}_{n,m}(\varpi;x) - \varpi\left(\frac{x}{n}(n+\rho x)\right) + \varpi(x), \tag{3.12}$$

for $\varpi(\lambda) = 1$, we have $\tilde{\mathcal{P}}^{\rho}_{n,m}(1;x) = 1$, and for $\varpi(\lambda) = \lambda$

$$\tilde{\mathcal{P}}^{\rho}_{n,m}(\lambda;x) = \mathcal{P}^{\rho}_{n,m}(\lambda;x) - \frac{x}{n}(n+\rho x) + x = x.$$

Immediately, we may write

$$\tilde{\mathcal{P}}^{\rho}_{n,m}((\lambda - x);x) = 0.$$

Now, applying the operators $\tilde{\mathcal{P}}^{\rho}_{n,m}$ on (3.11) and using (3.12), we have

$$\begin{aligned}\tilde{\mathcal{P}}^{\rho}_{n,m}(\xi;x) - \xi(x) &= \tilde{\mathcal{P}}^{\rho}_{n,m}((\lambda - x);x)\xi'(x) + \tilde{\mathcal{P}}^{\rho}_{n,m}\left(\int_x^{\infty}(\lambda - s)\xi''(s)\mathrm{d}s;x\right)\\ &= \tilde{\mathcal{P}}^{\rho}_{n,m}\left(\int_x^{\infty}(\lambda - s)\xi''(s)\mathrm{d}s;x\right)\end{aligned}$$

$$= \mathcal{P}^{\rho}_{n,m}\left(\int_x^{\infty}(\lambda - s)\xi''(s)\mathrm{d}s; x\right) - \int_x^{\frac{x}{n}(n+\rho x)}\left(\frac{x}{n}(n+\rho x)\right)\xi''(s)\mathrm{d}s.$$

Now,

$$|\tilde{\mathcal{P}}^{\rho}_{n,m}(\xi;x) - \xi(x)| \leq \mathcal{P}^{\rho}_{n,m}\left(\frac{1}{2}(\lambda - x)^2\|\xi''\|; x\right) + \frac{1}{2}\left(\frac{x}{n}(n+\rho x)\right)^2\|\xi''\|$$
$$\leq \frac{1}{2}\left|\mathcal{P}^{\rho}_{n,m}\left((\lambda - x)^2; x\right) + \left(\frac{x}{n}(n+\rho x)\right)^2\right|\|\xi''\| \leq \zeta\|\xi''\|,$$

where

$$\zeta = \frac{1}{2}\left|\mathcal{P}^{\rho}_{n,m}\left((\lambda - x)^2; x\right) + \left(\frac{x}{n}(n+\rho x)\right)^2\right|.$$

Again, from equation (3.11) and (3.12), we have

$$\mathcal{P}^{\rho}_{n,m}(\varpi;x) = \tilde{\mathcal{P}}^{\rho}_{n,m}(\varpi;x) + \varpi\left(\frac{x}{n}(n+\rho x)\right) - \varpi(x)$$
$$= \tilde{\mathcal{P}}^{\rho}_{n,m}((\varpi - \xi);x) + \tilde{\mathcal{P}}^{\rho}_{n,m}(\xi;x) + \varpi\left(\frac{x}{n}(n+\rho x)\right) - \varpi(x)$$
$$= \tilde{\mathcal{P}}^{\rho}_{n,m}((\varpi - \xi);x) - (\varpi - \xi)(x) + \tilde{\mathcal{P}}^{\rho}_{n,m}(\xi;x) - \xi(x)$$
$$+ \varpi\left(\frac{x}{n}(n+\rho x)\right) - \varpi(x).$$

Now, we have

$$|\mathcal{P}^{\rho}_{n,m}(\varpi;x) - \varpi(x)| \leq \|\varpi - \xi\| + \zeta\|\xi''\| + \left|\varpi\left(\frac{x}{n}(n+\rho x)\right) - \varpi(x)\right|$$
$$\leq 2\|\varpi - \xi\| + \zeta\|\xi''\| + \omega\left(\varpi; \left|\frac{x}{n}(n+\rho x)\right|\right)$$
$$\leq 2K_2(\varpi,\zeta) + \omega\left(\varpi; \left|\frac{x}{n}(n+\rho x)\right|\right).$$

In the view of (3.10), we get the required result. □

Let $x \in (0,\infty)$ and $\lambda \in [0,\infty]$, as we can see in Özarslan and Duman [7], the Lipschitz type space is explained as:

$$Lip^*_M(\alpha) = \left\{\varpi \in C[0,\infty] : |\varpi(\lambda) - \varpi(x)| \leq M\frac{|\lambda - x|^{\alpha}}{(\lambda + x)^{\frac{\alpha}{2}}}\right\}, \quad \text{where } 0 < \alpha \leq 1.$$

In the following theorem, we obtain the rate of convergence of the operators $\mathcal{P}^{\rho}_{n,m}$ for functions in $Lip^*_M(\alpha)$.

**Theorem 4.** *Let $\varpi \in Lip^*_M(\alpha)$ and $\alpha \in (0,1]$. Then for all $x \in (0,\infty)$, we have*

$$|\mathcal{P}^{\rho}_{n,m}(\varpi(\lambda);x) - \varpi(x)| \leq M\left(\frac{x\left(n + 2\rho x + \rho^2 x^2\right)}{n^2}\right)^{\frac{\alpha}{2}}.$$

*Proof.* From the Lemma 4, we have

$$|\mathcal{P}^{\rho}_{n,m}(\varpi(\lambda);x)-\varpi(x)| \leq \mathcal{P}^{\rho}_{n,m}(|\varpi(\lambda)-\varpi(x)|;x) \leq M\mathcal{P}^{\rho}_{n,m}\left(\frac{|\lambda-x|^{\alpha}}{(\lambda+x)^{\frac{\alpha}{2}}};x\right)$$
$$\leq \frac{M}{x^{\frac{\alpha}{2}}}\mathcal{P}^{\rho}_{n,m}(|\lambda-x|^{\alpha};x). \tag{3.13}$$

Taking $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$ and applying Hölder's inequality, we obtain

$$\mathcal{P}^{\rho}_{n,m}(\varpi)(|\lambda-x|^{\alpha};x) \leq \left\{\mathcal{P}^{\rho}_{n,m}\left(|\lambda-x|^{2};x\right)\right\}^{\frac{\alpha}{2}}.\left\{\mathcal{P}^{\rho}_{n,m}\left(1^{\frac{2}{2-\alpha}};x\right)\right\}^{\frac{2-\alpha}{2}}$$
$$\leq \left\{\mathcal{P}^{\rho}_{n,m}\left(|\lambda-x|^{2};x\right)\right\}^{\frac{\alpha}{2}}. \tag{3.14}$$

On combining (3.13), (3.14) and using the Lemma 2, we get the required result. □

For $c$, d $>0$, Özarslan and Aktuğlu [8] considered the Lipschitz-type space with two parameters, as follows

$$Lip_{M}^{(c,d)}(\alpha)=\left(\varpi\in C[0,\infty):|\varpi(\lambda)-\varpi(x)|\leq M\frac{|\lambda-x|^{\alpha}}{(\lambda+cx^{2}+dx)^{\frac{\alpha}{2}}}\right),$$

where $M$ is a positive constant and $0<\alpha\leq 1$.

**Theorem 5** (Point-wise Estimate)**.** *For $f\in Lip_{M}^{(c,d)}(\alpha)$. Then, for all $x>0$, we have*

$$|\mathcal{P}^{\rho}_{n,m}(\varpi;x)-\varpi(x)|\leq M\frac{\chi(x)}{(cx^{2}+dx)^{\frac{\alpha}{2}}}.$$

*Proof.* First we prove the theorem for $\alpha=1$. Then for $\varpi\in Lip_{M}^{(c,d)}(1)$ and $x\in[0,\infty)$, we have

$$|\mathcal{P}^{\rho}_{n,m}(\varpi;x)-\varpi(x)|\leq \mathcal{P}^{\rho}_{n,m}(|\varpi(\lambda)-\varpi(x)|;x)\leq M\left\{\mathcal{P}^{\rho}_{n,m}\left(\frac{|\lambda-x|}{(\lambda+cx^{2}+dx)^{\frac{1}{2}}};x\right)\right\}$$
$$\leq \frac{M}{(cx^{2}+dx)^{\frac{1}{2}}}\mathcal{P}^{\rho}_{n,m}(|\lambda-x|;x).$$

By applying Cauchy-Schwartz inequality and using Lemma 3

$$|\mathcal{P}^{\rho}_{n,m}(\varpi;x)-\varpi(x)|\leq \frac{M}{(cx^{2}+dx)^{\frac{1}{2}}}\{\mathcal{P}^{\rho}_{n,m}\left(|\lambda-x|^{2};x\right)\}^{\frac{1}{2}}\leq M\left(\frac{\chi^{2}(x)}{cx^{2}+dx}\right)^{\frac{1}{2}}.$$

This shows that result is true for $\alpha=1$. Next we prove the stated result for $0<\alpha<1$, we have

$$|\mathcal{P}^{\rho}_{n,m}(\varpi;x)-\varpi(x)|\leq \frac{M}{(cx^{2}+dx)^{\frac{\alpha}{2}}}\mathcal{P}^{\rho}_{n,m}(|\lambda-x|^{\alpha};x).$$

Assume $p=\frac{1}{\alpha},\ q=\frac{1}{1-\alpha}$, on applying the Hölder's inequality, we have

$$|\mathcal{P}^{\rho}_{n,m}(\varpi;x)-\varpi(x)|\leq \frac{M}{(cx^2+dx)^{\frac{\alpha}{2}}}\left(\mathcal{P}^{\rho}_{n,m}(|\lambda-x|;x)\right)^{\alpha}.$$

Again, by Cauchy-Schwartz inequality and Lemma 3 required result follows. □

**Theorem 6.** *Let* $\varpi\in[0,\infty)$ *and second order derivative of* $\varpi$ *exists in* $[0,\infty)$*, then we have*

$$\lim_{n\to\infty} n\left[\mathcal{P}^{\rho}_{n,m}(\varpi;x)-\varpi(x)\right]=x^2\left(\rho\varpi'(x)+\varpi''(x)\right)$$

*Proof.* Using Taylor's series expansion, we write

$$\varpi(\lambda)=\varpi(x)+(\lambda-x)\varpi'(x)+\frac{(\lambda-x)^2}{2!}\varpi''(x)+\hbar(\lambda,x)(\lambda-x)^2, \tag{3.15}$$

where $\hbar(\lambda,x)\to 0$ as $\lambda\to x$. Applying $\mathcal{P}^{\rho}_{n,m}(.:x)$ on both the sides of (3.15), we get

$$\begin{aligned}\mathcal{P}^{\rho}_{n,m}\left(\varpi(\lambda)-\varpi(x);x\right)&=\varpi'(x)\mathcal{P}^{\rho}_{n,m}\left((\lambda-x);x\right)+\frac{\varpi''(x)}{2!}\mathcal{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)\\&\quad+\mathcal{P}^{\rho}_{n,m}\left(\hbar(\lambda,x)(\lambda-x)^2;x\right).\end{aligned}$$

From the Lemma 2 and applying the $\lim\limits_{n\to\infty}$, we get

$$\begin{aligned}\lim_{n\to\infty} n\mathcal{P}^{\rho}_{n,m}\left(\varpi(\lambda)-\varpi(x);x\right)&=\varpi'(x)\lim_{n\to\infty} n\mathcal{P}^{\rho}_{n,m}\left((\lambda-x);x\right)\\&\quad+\frac{\varpi''(x)}{2!}\lim_{n\to\infty} n\mathcal{P}^{\rho}_{n,m}\left((\lambda-x)^2;x\right)\\&\quad+\lim_{n\to\infty} n\mathcal{P}^{\rho}_{n,m}\left(\hbar(\lambda,x)(\lambda-x)^2;x\right).\end{aligned} \tag{3.16}$$

From Theorem 1, Lemma 2 and using cauchy-Schwarz inequality in the last term of (3.16), we have

$$\lim_{n\to\infty} n\mathcal{P}^{\rho}_{n,m}\left(\hbar(\lambda,x)(\lambda-x)^2;x\right)=0. \tag{3.17}$$

Using Lemma 2 and (3.16) , (3.17), required result follows. □

## 4. Bounded Variation

In the next theorem, we estimate the operators' convergence rate (1.1). Let $\varpi\in$ DBV$[0,\infty)$ be a continuous function taken from the class of absolutely continuous functions DBV$(0,\infty)$ and having a derivative $\varpi'$ on the interval $(0,\infty)$ coincides a.e. with a function which is of bounded variation on every finite partition of $[0,\infty)$. It can be observed that for all $\varpi\in$ DBV$[0,\infty)$, we have

$$\varpi(x)=\varpi(c)+\int_c^x g(s)\mathrm{d}s, \qquad x\geq c\geq 0,$$

where $g(s)$ is a function of bounded variation on every finite subinterval of $[0,\infty)$. We may write the operators (1.1) in alternate form as

$$\mathcal{P}^{\rho}_{n,m}(\varpi;x)=\int_0^{\infty}\mathcal{K}^{\rho}_{n,m}(x,\lambda)\varpi(\lambda)\mathrm{d}\lambda, \tag{4.1}$$

where

$$\mathcal{K}^{\rho}_{n,m}(x,\lambda)=\sum_{m=0}^{\infty}\frac{n^n}{e^{\rho x}x^n}\frac{(n\rho)^m}{m!}\frac{1}{\Gamma(n+m)}\lambda^{n+m-1}e^{-\frac{n\lambda}{x}}.$$

Now we define an auxiliary function $\varpi_x$ by

$$\varpi_x(\lambda)=\begin{cases}\varpi(\lambda)-\varpi(x^-) & \text{if } 0\le\lambda<x,\\ 0 & \text{if } \lambda=x,\\ \varpi(\lambda)-\varpi(x^+) & \text{if } x<\lambda<\infty.\end{cases}$$

**Lemma 5.** *For $x\in(0,\infty)$ and sufficiently large $n$, we have*

(1) *Since $0\le y<x$, therefore*

$$\eta_n(x,y)=\int_0^{y}\mathcal{K}^{\rho}_{n,m}(x,\lambda)\mathrm{d}\lambda\le\frac{\chi^2(x)}{n(x-y)^2}.$$

(2) *If $x<z<\infty$, then*

$$1-\eta_n(x,z)=\int_z^{\infty}\mathcal{K}^{\rho}_{n,m}(x,\lambda)\mathrm{d}\lambda\le\frac{\chi^2(x)}{n(z-x)^2}.$$

*Proof.* By simple computations and using Lemma 2 and Lemma 3 we get required results. □

**Theorem 7.** *Let $\varpi\in\mathrm{DBV}(0,\infty)$ then for all $x\in(0,\infty)$ and sufficiently large $n$, we have*

$$\begin{aligned}\left|\mathcal{P}^{\rho}_{n,m}(\varpi;x)-\varpi(x)\right|\le{}&\frac{1}{2}\left(\varpi'(x^+)+\varpi'(x^-)\right)\mathcal{P}^{\rho}_{n,m}((x-\lambda);x)\\&+\frac{1}{2}\frac{\chi(x)}{\sqrt{n}}\left|\varpi'(x^+)+\varpi'(x^-)\right|+\frac{\chi^2(x)}{nx}\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{m}}^{x+\frac{x}{m}}(\varpi'_x)\right)\\&+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}(\varpi'_x)\right)+\frac{\chi(x)}{\sqrt{n}}\varpi'(x^+)\\&+\frac{\chi^2(x)}{nx^2}\left|\varpi(2x)-\varpi(x)-x\varpi'(x^+)\right|+M(\gamma,r,x)\frac{\chi^2(x)|\varpi(x)|}{nx^2}.\end{aligned}$$

*Where $\bigvee_a^b(x)$ denotes the total variation of $\varpi_x$ on $[a,b]$ and*

$$M(\gamma,r,x)=M2^{\gamma}\left(\int_0^{\infty}(\lambda-x)^{2r}\mathcal{K}^{\rho}_{n,m}(x,\lambda)\mathrm{d}\lambda\right)^{\frac{\gamma}{2r}}.$$

*Proof.* Using the operator (4.1) for all $x \in [0,\infty)$, we obtain

$$\begin{aligned} \mathcal{P}^{\rho}_{n,m}(\varpi;x) - \varpi(x) &= \int_0^{\infty} \mathcal{K}^{\rho}_{n,m}(x,\lambda)\left(\varpi(\lambda) - \varpi(x)\right) \mathrm{d}\lambda \\ &= \int_0^{\infty} \left( \mathcal{K}^{\rho}_{n,m}(x,\lambda) \int_x^{\lambda} \varpi(s)\mathrm{d}s \right) \mathrm{d}\lambda. \end{aligned} \tag{4.2}$$

For $\varpi \in \mathrm{DBV}[0,\infty)$, we can write

$$\begin{aligned} \varpi'(s) &= \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-)) + \varpi'_x(x) + \frac{1}{2}(\varpi'(x^+) - \varpi'(x^-))sgn(x) \\ &\quad + \delta_x(s)(\varpi' - \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-)). \end{aligned} \tag{4.3}$$

Where

$$\delta_x(s) = \begin{cases} 1, & s = x; \\ 0, & s \neq x. \end{cases}$$

It can be easily seen that

$$\int_0^{\infty} \left( \int_x^{\lambda} \left( \varpi'(s) - \frac{1}{2}\left(\varpi'(x^+) + \varpi'(x^-)\right) \right) \delta_x(s)\mathrm{d}s \right) \mathcal{K}^{\rho}_{n,m}(x,\lambda)\mathrm{d}\lambda = 0.$$

Using the equation (4.1),we obtain

$$\begin{aligned} &\int_0^{\infty} \left( \int_x^{\lambda} \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-))\mathrm{d}s \right) \mathcal{K}^{\rho}_{n,m}(x,\lambda)\mathrm{d}\lambda \\ &= \frac{1}{2}\left(\varpi'(x^+) + \varpi'(x^-)\right) \mathcal{P}^{\rho}_{n,m}\left((x-\lambda);x\right). \end{aligned}$$

Again using (4.1), we have

$$\begin{aligned} &\int_0^{\infty} \left( \int_x^{\lambda} \frac{1}{2}(\varpi'(x^+) - \varpi'(x^-))sgn(s-x)\mathrm{d}s \right) \mathcal{K}^{\rho}_{n,m}(x,\lambda)\mathrm{d}\lambda \\ &= \int_0^{\infty} \frac{1}{2}(\varpi'(x^+) - \varpi'(x^-))(\lambda - x)\mathcal{K}^{\rho}_{n,m}(x,\lambda)\mathrm{d}\lambda \\ &\leq \frac{1}{2}\left|\varpi'(x^+) - \varpi'(x^-)\right| \left(\mathcal{P}^{\rho}_{n,m}\left((x-\lambda)^2;x\right)\right)^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

From (4.2), (4.4) and Lemma 3, we have

$$\begin{aligned} \mathcal{P}^{\rho}_{n,m}(\varpi;x) - \varpi(x) &\leq \frac{1}{2}\left(\varpi'(x^+) + \varpi'(x^-)\right) \mathcal{P}^{\rho}_{n,m}\left((x-\lambda);x\right) \\ &\leq \frac{1}{2}\left(\varpi'(x^+) + \varpi'(x^-)\right) \mathcal{P}^{\rho}_{n,m}\left((x-\lambda);x\right) \\ &\quad + \frac{1}{2}\frac{\chi(x)}{\sqrt{n}}\left|\varpi'(x^+) + \varpi'(x^-)\right| + \int_0^{\infty} \left( \int_x^{\lambda} \varpi'_x(s)\mathrm{d}s \right) \mathcal{K}^{\rho}_{n,m}(x,\lambda)\mathrm{d}\lambda. \end{aligned}$$

We obtain

$$\left|\mathcal{P}_{n,m}^{\rho}(\varpi;x)-\varpi(x)\right| \leq \frac{1}{2}\left(\varpi'(x^{+})+\varpi'(x^{-})\right)\mathcal{P}_{n,m}^{\rho}\left((x-\lambda);x\right) + \frac{1}{2}\frac{\chi(x)}{\sqrt{n}}\left|\varpi'(x^{+})+\varpi'(x^{-})\right|+A_{n1}(x)+A_{n2}(x). \tag{4.5}$$

Where

$$A_{n1}(x)=\left|\int_0^x\left(\int_x^\lambda \varpi_x'(s)\mathrm{d}s\right)\mathcal{K}_{n,m}^{\rho}(x,\lambda)\mathrm{d}\lambda\right|,$$

and

$$A_{n2}(x)=\left|\int_x^\infty\left(\int_x^\lambda \varpi_x'(s)\mathrm{d}s\right)\mathcal{K}_{n,m}^{\rho}(x,\lambda)\mathrm{d}\lambda\right|.$$

Now applying Lemma 5, integrating by parts and taking $y=x-\frac{x}{\sqrt{n}}$, we get

$$\begin{aligned}A_{n1}(x)&=\left|\int_0^x\left(\int_x^\lambda \varpi_x'(s)\mathrm{d}s\right)\mathrm{d}_\lambda\eta_n(x,\lambda)\mathrm{d}\lambda\right|=\left|\int_0^x\eta_n(x,\lambda)\varpi_x'(\lambda)\mathrm{d}\lambda\right|\\&\leq\int_0^y|\eta_n(x,\lambda)|\left|\varpi_x'(\lambda)\right|\mathrm{d}\lambda+\int_y^x|\eta_n(x,\lambda)|\left|\varpi_x'(\lambda)\right|\mathrm{d}\lambda\\&=\int_0^{x-\frac{x}{\sqrt{n}}}\eta_n(x,\lambda)\left|\varpi_x'(\lambda)\right|\mathrm{d}\lambda+\int_{x-\frac{x}{\sqrt{n}}}^x\eta_n(x,\lambda)\left|\varpi_x'(\lambda)\right|\mathrm{d}\lambda.\end{aligned}$$

Since $|\eta_n(x,\lambda)|\leq 1$ and $\varpi_x'(x)=0$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^x\eta_n(x,\lambda)\left|\varpi_x'(\lambda)\right|\mathrm{d}\lambda&=\int_{x-\frac{x}{\sqrt{n}}}^x\eta_n(x,\lambda)\left|\varpi_x'(\lambda)-\varpi_x'(x)\right|\mathrm{d}\lambda\\&\leq\int_{x-\frac{x}{\sqrt{n}}}^x\bigvee_{\lambda}^{x}(\varpi_x')\mathrm{d}\lambda\leq\frac{x}{\sqrt{n}}\bigvee_{\lambda-\frac{x}{\sqrt{n}}}^{x}(\varpi_x').\end{aligned}$$

Again using Lemma 5 and substituting $\lambda=x-\frac{x}{s}$.

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}\eta_n(x,\lambda)\left|\varpi_x'(\lambda)\right|\mathrm{d}\lambda&\leq\frac{\chi^2(x)}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\left|\varpi_x'(\lambda)\right|}{(x-\lambda)^2}\mathrm{d}\lambda\leq\frac{\chi^2(x)}{nx}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{s}}^{x}(\varpi_x')\mathrm{d}s\\&\leq\frac{\chi^2(x)}{nx}\sum_{m=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{s}}^{x}(\varpi_x').\end{aligned}$$

Therefore,

$$A_{n1}(x)\leq\frac{\chi^2(x)}{nx}\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{m}}^{x}(\varpi_x')\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(\varpi_x')\right). \tag{4.6}$$

Now we observe, integration by parts and applying Lemma 5, we have

$$\begin{aligned}
A_{n2}(x) &= \int_x^\infty \mathcal{K}^{\rho}_{n,m}(x,\lambda)\left(\int_x^\lambda \varpi'_x(s)\mathrm{d}s\right)\mathrm{d}\lambda \\
&\le \left|\int_x^{2x} \mathcal{K}^{\rho}_{n,m}(x,\lambda)\left(\int_x^\lambda \varpi'_x(s)\mathrm{d}s\right)\mathrm{d}\lambda\right| + \left|\int_{2x}^{\infty} \mathcal{K}^{\rho}_{n,m}(x,\lambda)\left(\int_x^\lambda \varpi'_x(s)\mathrm{d}s\right)\mathrm{d}\lambda\right| \\
&\le B_{n1}(x)+B_{n2}(x), \qquad (4.7)
\end{aligned}$$

where,

$$\begin{aligned}
B_{n1}(x) &= \left|\int_x^{2x} \mathcal{K}^{\rho}_{n,m}(x,\lambda)\left(\int_x^\lambda \varpi'_x(s)\mathrm{d}s\right)\mathrm{d}\lambda\right|, \\
B_{n2}(x) &= \left|\int_{2x}^{\infty} \mathcal{K}^{\rho}_{n,m}(x,\lambda)\left(\int_x^\lambda \varpi'_x(s)\mathrm{d}s\right)\mathrm{d}\lambda\right|.
\end{aligned}$$

Applying integration by parts, using (4.3) and Lemma 5. Since $1-\eta_n(x,\lambda)\le 1$, substituting $\lambda = x+\frac{x}{s}$, we have

$$\begin{aligned}
B_{n1}(x) &= \left|\int_x^{2x} \varpi'_x(s)\mathrm{d}s\,(1-\eta_n(x,2x)) - \int_x^{2x}(1-\eta_n(x,\lambda))\,\varpi'_x(\lambda)\mathrm{d}\lambda\right| \\
&\le \left|\int_x^{2x}\left(\varpi'(s)-\varpi'(x^+)\right)\mathrm{d}s\right| |1-\eta_n(x,2x)| + \int_x^{2x} |\varpi'_x(\lambda)|\,|1-\eta_n(x,\lambda)|\,v\lambda \\
&\le \frac{\chi^2(x)}{nx^2}\left|\varpi(2x)-\varpi(x)-x\varpi'(x^+)\right| + \int_x^{x+\frac{x}{\sqrt{n}}} |\varpi'_x(\lambda)|\,|1-\eta_n(x,\lambda)|\,\mathrm{d}\lambda \\
&\quad + \int_{x+\frac{x}{\sqrt{n}}}^{2x} |\varpi'_x(\lambda)|\,|1-\eta_n(x,\lambda)|\,\mathrm{d}\lambda \\
&\le \frac{\chi^2(x)}{nx^2}\left|\varpi(2x)-\varpi(x)-x\varpi'(x^+)\right| + \int_x^{x+\frac{x}{\sqrt{n}}} \bigvee_\lambda^x(\varpi'_x)\mathrm{d}\lambda \\
&\quad + \frac{\chi^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{\bigvee_x^\lambda(\varpi'_x)}{(\lambda-x)^2}\mathrm{d}\lambda \\
&\le \frac{\chi^2(x)}{nx^2}\left|\varpi(2x)-\varpi(x)-x\varpi'(x^+)\right| + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}(\varpi'_x) \\
&\quad + \frac{\chi^2(x)}{n}\int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{1}{(\lambda-x)^2}\bigvee_x^\lambda(\varpi'_x)\mathrm{d}\lambda \qquad (4.8) \\
&\le \frac{\chi^2(x)}{nx^2}\left|\varpi(2x)-\varpi(x)-x\varpi'(x^+)\right| + \frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{n}}(\varpi'_x) + \frac{\chi^2(x)}{nx}\sum_{m=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{k}}(\varpi'_x).
\end{aligned}$$

And

$$B_{n2}(x) = \left| \int_{2x}^{\infty} \left( \int_{x}^{\lambda} \varpi'(s) - \varpi'(x+) \mathrm{d}s \right) \mathcal{K}_{n,m}^{\rho}(x,\lambda)\mathrm{d}\lambda \right|$$

$$\leq \int_{0}^{\infty} |\varpi(\lambda) - \varpi(x)| \mathcal{K}_{n,m}^{\rho}(x,\lambda)\mathrm{d}\lambda + \int_{2x}^{\infty} |\lambda - x| \varpi'(x^{+}) \mathcal{K}_{n,m}^{\rho}(x,\lambda)\mathrm{d}\lambda$$

$$\leq M \int_{2x}^{\infty} \lambda^{\gamma} \mathcal{K}_{n,m}^{\rho}(x,\lambda)\mathrm{d}\lambda + |\varpi(x)| \int_{2x}^{\infty} \mathcal{K}_{n,m}^{\rho}(x,\lambda)\mathrm{d}\lambda + \frac{\chi(x)}{\sqrt{n}} \varpi'(x^{+}).$$

It is obvious that

$$\lambda \leq 2(\lambda - x) \quad \text{and} \quad x \leq \lambda - x, \quad \text{when} \quad \lambda \geq 2x.$$

Applying Holder's inequality, we get

$$B_{n2}(x) \leq M 2^{\gamma} \left( \int_{0}^{\infty} (\lambda - x)^{2r} \mathcal{K}_{n,m}^{\rho}(x,\lambda)\mathrm{d}\lambda \right)^{\frac{\gamma}{2r}} + \frac{\chi^{2}(x)}{nx^{2}} |\varpi(x)| + \sqrt{\frac{1}{n}} \chi(x) \varpi'(x^{+})$$

$$\leq M(\gamma, r, x) + \frac{\chi^{2}(x)}{nx^{2}} |\varpi(x)| + \frac{\chi(x)}{\sqrt{n}} \varpi'(x^{+}). \tag{4.9}$$

From (4.8) and (4.9), we get

$$A_{n2}(x) \leq \frac{\chi^{2}(x)}{nx^{2}} \left| \varpi(2x) - \varpi(x) - x\varpi'(x^{+}) \right| + \frac{x}{\sqrt{n}} \bigvee_{x}^{x+\frac{x}{n}} (\varpi'_{x})$$

$$+ \frac{\chi^{2}(x)}{nx} \sum_{m=1}^{[\sqrt{n}]} \bigvee_{x}^{x+\frac{x}{m}} (\varpi'_{x}) + M(\gamma, r, x) \frac{\chi^{2}(x)|\varpi(x)|}{nx^{2}} + \sqrt{\frac{1}{n}} \chi(x) \varpi'(x^{+}). \tag{4.10}$$

On combining (4.5)-(4.7) and (4.10), we get required result. □

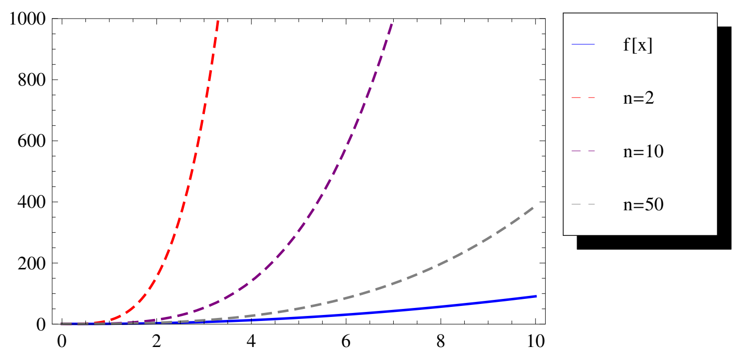


FIGURE 1.

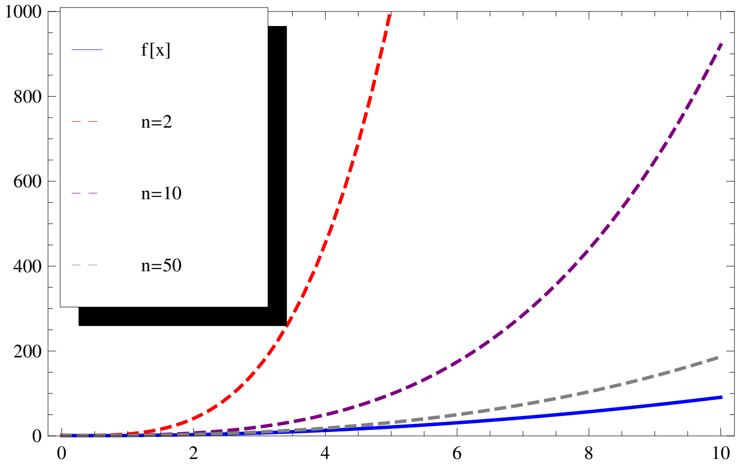


FIGURE 2.

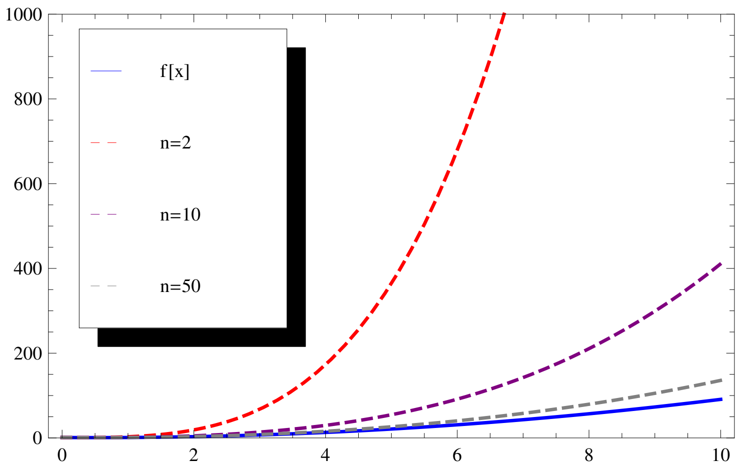


FIGURE 3.

*Example* 1. The Graphical representation of the convergence of the operators $\mathcal{P}_{n,k}^{\beta}(f(t);x)$ to the test function $f(x) = x^2 - x + 1$ are given in the Figure-1 for $\beta = 5,$ and $n = \{2, 10, 50\}$. Figure-2 shows the convergence of the operators $\mathcal{P}_{n,k}^{\beta}(f(t);x)$ for $\beta = 2,$ and $n = \{2, 10, 50\}$. And Figure-3 represents the convergence of the operators $\mathcal{P}_{n,k}^{\beta}(f(t);x)$ for $\beta = 1,$ and $n = \{2, 10, 50\}$. From the graphical representation we easily seen that the operators converges fast when $\beta$ decreases.

## REFERENCES

[1] U. Abel and V. Gupta, "Rate of convergence of exponential type operators related to $p(x) = 2x^{3/2}$ for functions of bounded variation," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 114, no. 4, pp. Paper No. 188, 8, 2020, doi: 10.1007/s13398-020-00919-y.

[2] U. Abel, V. Gupta, and M. Sisodia, "Some new semi-exponential operators," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 116, no. 2, pp. Paper No. 87, 12, 2022, doi: 10.1007/s13398-022-01228-2.

[3] Z. Ditzian and V. Totik, *Moduli of smoothness*, ser. Springer Series in Computational Mathematics. Springer-Verlag, New York, 1987, vol. 9, doi: 10.1007/978-1-4612-4778-4.

[4] V. Gupta and M. T. Rassias, *Moments of linear positive operators and approximation*, ser. SpringerBriefs in Mathematics. Springer, Cham, 2019. doi: 10.1007/978-3-030-19455-0.

[5] M. Herzog, "Semi-exponential operators," *Symmetry*, vol. 13, no. 4, 2021, doi: 10.3390/sym13040637.

[6] K. Lipi and N. Deo, "General family of exponential operators," *Filomat*, vol. 34, no. 12, pp. 4043–4060, 2020, doi: 10.2298/fil2012043l.

[7] M. A. Özarslan and O. Duman, "Local approximation behavior of modified SMK operators," *Miskolc Math. Notes*, vol. 11, no. 1, pp. 87–99, 2010, doi: 10.18514/mmn.2010.228.

[8] M. A. Özarslan and H. Aktuğlu, "Local approximation properties for certain King type operators," *Filomat*, vol. 27, no. 1, pp. 173–181, 2013, doi: 10.2298/FIL1301173O.

[9] R. S. Phillips, "An inversion formula for Laplace transforms and semi-groups of linear operators," *Ann. of Math. (2)*, vol. 59, pp. 325–356, 1954, doi: 10.2307/1969697.

[10] R. K. S. Rathore and O. P. Singh, "On convergence of derivatives of Post-Widder operators," *Indian J. Pure Appl. Math.*, vol. 11, no. 5, pp. 547–561, 1980.

[11] A. Tyliba and E. Wachnicki, "On some class of exponential type operators," *Comment. Math. (Prace Mat.)*, vol. 45, no. 1, pp. 59–73, 2005.

[12] D. V. Widder, *The Laplace Transform*, ser. Princeton Mathematical Series, vol. 6. Princeton University Press, Princeton, N. J., 1941.

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