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ABOUT BOUNDED SOLUTIONS OF LINEAR STOCHASTIC ITO SYSTEMS

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Abstract. We prove that a sufficient condition for stochastic Ito systems to be exponentially dichotomous on the semiaxis is that the nonhomogeneous system has mean square bounded solutions.

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1. An introductory remark

The present paper investigates what conditions should be fulfilled for an Ito system to be exponentially dichotomous on the semiaxis. After same preparation we shall prove a theorem which is the main result of the paper.

2. Preparations and the proof of our theorem

Let us consider a system of linear stochastic Ito differential equations,

$$\mathrm{d}x = A(t)x\mathrm{d}t + \sum_{i=1}^{m} B_i(t)x\mathrm{d}W_i(t), \tag{2.1}$$

where $t \geq 0,\ x \in \mathbf{R}^n,\ A(t), B_i(t)$ are matrices determinate, continuous, and bounded on the positive semiaxis, $W_i(t), i = \overline{1, \ldots m}$, are jointly independent scalar Winner processes defined on a probability space (Ω, F, P). As implied by [1, p. 230], for $x_0 \in \mathbf{R}^n$ system (2.1) has a unique strong solution of the Cauchy problem, $x(t, x_0),\ x(0, x_0) = x_0$, defined for $t \geq 0$ and such that the second moment of the solution is finite for $t \geq 0$.

Definition 1. *System (2.1) is called exponentially dichotomous in mean square on the semiaxis $t \geq 0$, if the space $\mathbf{R}^n$ can be represented as a direct sum of two subspaces R^-, R^+ such that an arbitrary solution $x(t, x_0)$ of system (2.1), where $x_0 \in R^-$,* *satisfies the inequality*

$$\mathrm{M}\,|x(t,x_0)|^2 \le K\exp\{-\gamma(t-\tau)\}\mathrm{M}\,|x(\tau,x_0)|^2, \tag{2.2}$$

for $t \ge \tau \ge 0$, *and an arbitrary solution* $x(t,x_0)$ *of system (2.1), where* $x_0 \in R^+$, *satisfies the inequality*

$$\mathrm{M}\,|x(t,x_0)|^2 \ge K_1\exp\{\gamma_1(t-\tau)\}\mathrm{M}\,|x(\tau,x_0)|^2 \tag{2.3}$$

for $t \ge \tau \ge 0$ *with an arbitrary* $\tau > 0$. *Here* $K, K1, \gamma,\ \gamma_1$ *are positive constants independent of* τ *and* x_0.

An example of such a system is exponentially stable in a mean square system (2.1) (in this case $R^+ = \{0\}$, and $R^- = \mathbf{R}^n$).

As opposed to the ordinary differential equations, where conditions for dichotomy are well known [2, p. 230], [3], the problems in stochastic systems remain open. The author knows only the results of [1, p. 296], where the conditions for exponential dichotomy in mean square are obtained for a system of type (2.1) and for stochastic systems with delay in case where the matrices $A(t),\ B(t)$ are constant or periodic. But the results of [1, p. 296] are obtained with the help of a system of ordinary differential equations written for second moments of solutions of system (2.1) or a system of matrix equations for the correlation matrix of the solutions. It leads to an analysis of systems of a dimension much greater than the dimension of the initial system, and moreover, this system may not be exponentially dichotomous, although the initial system is.

That is why there is an interest in studying dichotomy conditions for system (2.1), when the matrices $A(t)$ and $B(t)$ are not obligatorily constant or periodic, and the conditions can be obtained in terms of the initial system, without an analysis of an auxiliary system for other moments.

From the works cited it follows that for ordinary differential equations, the question of exponential dichotomy on the semiaxis is equivalent to the question of existence of solutions, bounded on the semiaxis, of a nonhomogeneous system. This work is devoted to a study of this problem for system (2.1). Another point of view on studying dichotomy with the use of quadratic forms is published in another paper.

In the sequel, we shall assume that there is only one scalar Winner process $W(t)$ in system (2.1), and system (2.1) is of the form

$$\mathrm{d}x = \ A(t)x\,\mathrm{d}t + \ B(t)x\,\mathrm{d}W(t). \tag{2.4}$$

Let us consider a system of linear nonhomogeneous equation,

$$\mathrm{d}x = [\ A(t)x + \alpha(t)]\,\mathrm{d}t + \ B(t)x\,\mathrm{d}W(t), \tag{2.5}$$

where $\alpha(t)$ is n-dimensional and measurable, and for each $t \ge 0,\ F_t$ is a measurable stochastic process. Here, F_t is a flow of σ-algebras involved in the definition of the solution of the initial systems. We will assume that $\sup_{t\ge 0}\mathrm{M}\,|\alpha(t)|^2 < \infty$. With the norm $||\alpha||_2 = (\sup_{t\ge 0}\mathrm{M}\,|\alpha(t)|^2)^{1/2}$, the set of stochastic processes becomes a Banach space. Denote it by B.

Theorem 1. *Let system (2.5) with an arbitrary stochastic process* $\alpha(t) \in B$ *have a positive solution* $x(t, x_0),\ x_0 \in \mathbf{R}^n$ *bounded in mean square Then system (2.4) is exponentially dichotomic in mean square on the positive semiaxis.*

Proof. Let $G_1 \subset R^n$ be the set of initial values of solutions of system (2.4), which are bounded in mean square on the semiaxis. It follows from linearity of system (2.4) that G_1 is a subspace of $\mathbf{R}^n$. Let us show that it plays the role of R^- in the definition of the exponential dichotomy. Let us prove the lemma. □

Lemma 1. *Suppose that the conditions of Theorem 1 hold. Then, for each stochastic process* $\alpha(t) \in B$*, there exists a unique solution* $x(t, x_0)$ *of system (5) bounded in mean square such that* $x(0, x_0) \in G_1^{\perp} = G_2$*. (*$G_1^{\perp}$ *is the orthogonal complement.) This solution satisfies the estimate*

$$||x||_2 \leq K||\alpha||_2, \tag{2.6}$$

where K *is some positive constant, independent of* $\alpha(t)$*.*

Proof. Let $\alpha(t)$ satisfy the condition of the Theorem. Then for this $\alpha(t)$, it follows from the conditions of the lemma that there is a solution $x(t, x_0)$ of system (2.5) bounded in mean square.

Let $P_1,\ P_2$ be a pair of complement projectors on $G_1,\ G_2$. Let $x_1(t)$ be a solution of equation (2.4), corresponding to equation (2.5), with the initial condition $x_1(0) = P_1 x_0$. From the definition of the subspace G_1, it follows that such a solution is bounded in mean square on the semiaxis $t \geq 0$. It is obvious that $x_2 = x(t, x_0) - x_1(t)$ is a solution of system (2.3). It is easy to see that it is bounded on the positive semiaxis. We have $x_2(0) = x_0 - P_1 x_0 = P - 2x_0 \in G_2$. So its initial condition belongs to G_2. The unicity of the solution follows from the fact that the difference of two such solutions is a solution of a homogeneous equation, bounded in mean square and starts in G_2, which is possible only for the zero solution.

Let us prove inequality (2.6). Let us consider the space B_1 of all solutions of the stochastic equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha(s))\mathrm{d}s + \int_0^t (B(s)x(s))\mathrm{d}W(s) \tag{2.7}$$

with the condition that $x(0) \in G_2, \alpha(t) \in B$ bounded in the norm $||\ ||_2$.

This equation defines a one-to-one linear operator $F : B_1 \to B$ which $\forall x \in B_1$ defines $\alpha \in B$ such that $x(t)$ is a solution of equation (2.5) bounded in mean square. Indeed, if $x(t) \in B_1$, then it follows from the definition of this space that there exists $\alpha(t) \in B$ such that $x(t)$ is a solution of equation (2.7) with this $\alpha(t)$. Let there exist another $\alpha_1(t) \in B$ such that $x(t)$ is a solution of the equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha_1(s))\mathrm{d}s + \int_0^t (B(s)x(s))\mathrm{d}W(s). \tag{2.8}$$

Subtracting (2.7) from (2.8) we get

$$\int_0^t (\alpha(s) - \alpha_1(s))\,\mathrm{d}s = 0. \tag{2.9}$$

Then $\alpha(t) = \alpha_1(t)$, for $t \geq 0$, with probability 1, from which it follows that $\alpha(t)$ and $\alpha_1(t)$ are equal as elements of the space B. It was shown above that for an arbitrary $\alpha(t) \in B$ there exists only one solution, $x(t)$, of equation (2.7) such that $x(0) \in G_2,\ x(t) \in B_1$. The linearity of the operator F is obvious.

Let us introduce the norm

$$|||x||| = ||x|| + ||Fx||_2. \tag{2.10}$$

This at once implies continuity of the operator F. Let us prove completeness of the space B_1. Let $\{x_n(t)\}$ be a fundamental sequence. There is fundamentality in B, as follows from (10). Hence, there exists a limit $x(t) \in B$. So, $\forall t \geq 0,\ M\,|x_n(t) - x(t)|^2 \to 0,\ n \to \infty$. And thus $|x_n(0) - x(0)| \to 0, n \to \infty$. Since $x_n(0) \in G_2$ and G_2 is a subspace of in $\mathbf{R}^n$, $x(0) \in G_2$.

It follows from the inequality $||F(x_n - x_m)||_2 \leq ||F||\,|||x_n - x_m|||$ that the sequence $Fx_n = \alpha_n$ is fundamental in B, and so there is a limit $\alpha(t)$ such that $\sup_{t\geq 0} M\,|\alpha_n(t) - \alpha(t)|^2 \to 0$, as $n \to \infty$, and $\alpha(t) \in B$.

Let us show that $x(t)$ satisfies the equation

$$x(t) = x(0) + \int_0^t (A(s)x(s) + \alpha(s))\mathrm{d}s + \int_0^t (B(s)x(s))\mathrm{d}W(s). \tag{2.11}$$

Since $A(t)$ and $B(t)$ are continuous and bounded, $x(t) \in B$, we have that $x(t)$ is F_t-measurable and both integrals in (2.11) exist. Let us estimate, for each $t \geq 0$, the expression

$$\begin{aligned} &M\left|x(t) - x(0) - \int_0^t (A(s)x(s) + \alpha(s))\mathrm{d}s - \int_0^t B(s)x(s)\mathrm{d}W(s)\right|^2 \\ &\qquad \leq M\left(|x(t) - x_n(t)| + \left|x_n(t) - x(0) - \int_0^t (A(s)x(s) + \alpha(s))\,\mathrm{d}s\right.\right. \\ &\qquad \left.\left. - \int_0^t (B(s)x(s))\mathrm{d}W(s)\right|\right)^2 \leq 2\left[M\,|x(t) - x_n(t)|^2 + M\left|x_n(t) - x(0)\right.\right. \\ &\qquad \left.\left. - \int_0^t (A(s)x(s) + \alpha(s))\mathrm{d}s - \int_0^t (B(s)x(s))\mathrm{d}W(s)\right|^2\right]. \end{aligned} \tag{2.12}$$

The first summand in the last expression tends to zero when $n \to \infty$. Let us estimate the second summand. Since $x_n(t)$, for each n, belongs to B_1, it satisfies the equation

$$x_n(t) = x_n(0) + \int_0^t (A(s)x_n(s) + \alpha_n(s))\mathrm{d}s + \int_0^t B(s)x_n(s)\mathrm{d}W(s). \tag{2.13}$$

Let us substitute (2.13) into (2.12). We get that the second summand in (2.10) does not exceed the following expression

$$3\Bigg[M\,|x_n(0) - x(0)|^2 + M\left(\int_0^t (||A(s)||\,|x_n(s) - x(s)| + |\alpha_n(s) - \alpha(s)|)\mathrm{d}s\right)^2$$

$$+ M\left|\int_0^t B(s)(x_n(s) - x(s))\mathrm{d}W(s)\right|^2\Bigg]$$

$$\leq 3\Bigg[M\,|x_n(t) - x(0)|^2 + 2t\int_0^t ||A(s)||^2 M\,|x_n(s) - x(s)|^2\mathrm{d}s$$

$$+ 2t\int_0^t M\,|\alpha_n - \alpha(s)|^2\mathrm{d}s + \int_0^t ||B(s)||^2 M\,|x_n(s) - x(s)|^2\mathrm{d}s\Bigg].$$

Each of the summands in the last expression tends to zero as $n \to \infty$. >From (2.12) it follows that $x(t)$ satisfies (2.11) with probability of 1 for each $t \geq 0$. So, the space B_1 is complete. That is why the linear continuous operator F defines a one-to-one mapping of the Banach space B_1 onto the Banach space B. By the Banach theorem, the inverse operator B^{-1} is also continuous. Then, for the solution of equation (2.3), we have the estimate

$$||x||_2 \leq |||x||| \leq ||F^{-1}||\,||\alpha||_2,$$

which is necessary estimate (2.4). The Lemma is proved. □

Let $x(t)$ be a nonzero solution of system (2.4), so that $x(0) \in G_1$. Let

$$y(t) = x(t)\int_0^t \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s, \tag{2.14}$$

where

$$\beta(t) = \begin{cases} 1, & 0 \leq t \leq t_0 + \tau, \\ 1 - (t - t_0 - \tau), & t_0 + \tau \leq t \leq t_0 + \tau + 1, \\ 0, & t \geq t_0 + \tau + 1. \end{cases}$$

It is obvious that $y(t)$ is F_t-dimensional and has a stochastic differential. Let us evaluate it,

$$\mathrm{d}y = \int_0^t \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s\mathrm{d}x + x(t)\frac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}}\mathrm{d}t$$

$$= \int_0^t \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s(A(t)x\mathrm{d}t + B(t)x\mathrm{d}W(t)) + x(t)\frac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}}\mathrm{d}t$$

$$= A(t)y\mathrm{d}t + x(t)\frac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}}\mathrm{d}t + B(t)y\mathrm{d}W(t).$$

So $y(t)$ is a solution of equation (2.5) with $\alpha(t) = x(t)\dfrac{\beta(t)}{(\mathrm{M}\,|x(t)|^2)^{1/2}}$. Obviously, $||y||_2 < \infty$ and $\alpha(t) \in B$. And since $y(0) = 0 \in G_2$, the previous Lemma gives that

$$||y||_2 \le K||\alpha||_2.$$

Whence,

$$(M|y(t)|^2)^{\frac{1}{2}} \le K(\sup_{t\ge 0}\mathrm{M}\,|\alpha(t)|^2)^{\frac{1}{2}} \le K$$

for $t \ge 0$. In particular, if $t = t_0 + \tau$, then

$$(\mathrm{M}\,|y(t)|^2)^{\frac{1}{2}} = (\mathrm{M}\,|x(t_0+\tau)|^2)^{\frac{1}{2}} \le K. \tag{2.15}$$

Let us consider the function

$$\psi(t) = \int_{t_0}^t \frac{1}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s.$$

Since the second moments of system (2.4) satisfy a system of ordinary linear differential equations, see [1, p. 236], this function is continuously differentiable. Then (2.15) gives that

$$\frac{\psi'(t_0+\tau)}{\psi^(t_0+\tau)} \ge \frac{1}{K}.$$

If we integrate the last inequality from 1 to τ , we get

$$\psi(t_0+\tau) \ge \psi(t_0+1)\exp\left\{\frac{\tau-1}{K}\right\} \tag{2.16}$$

for $\tau \ge 1$. Since $x(t)$ is a solution of system (2.4),

$$x(t) = x(t_0) + \int_{t_0}^t A(s)x(s)\,\mathrm{d}s + \int_{t_0}^t B(s)x(s)\,\mathrm{d}W(s). \tag{2.17}$$

And, hence, if $t \in [t_0\ t_0+1]$, we have

$$\mathrm{M}\,|x(t)|^2 \le 3\left(\mathrm{M}\,|x(t_0)|^2 + \int\limits_{t_0}^{t_0+1} ||A(s)||^2 \mathrm{M}\,|x(s)|^2\,\mathrm{d}s + \int\limits_{t_0}^{t_0+1} ||B(s)||^2 \mathrm{M}\,|x(s)|^2\,\mathrm{d}s\right).$$

This and the Gronwall–Bellman inequality give

$$\mathrm{M}\,|x(t)|^2 \le 3\mathrm{M}\,|x(t_0)|^2 \exp\{C\}, \tag{2.18}$$

where $C > 0$ is a constant independent of t_0. Therefore,

$$\psi(t_0+1) = \int\limits_{t_0}^{t_0+1} \frac{1}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s \ge \frac{1}{3^{\frac{1}{2}}}(\mathrm{M}\,|x(t_0)|^2)^{-\frac{1}{2}} \exp\left\{-\frac{C}{2}\right\}.$$

From this inequality using (2.15) and (2.16), we get for $\tau \ge 1$ that

$$(\mathrm{M}\,|x(t_0+\tau)|^2)^{\frac{1}{2}} \le \frac{K}{\psi(t_0+\tau)} \le N(\mathrm{M}\,|x(t_0)|^2)^{\frac{1}{2}} \exp\left\{-\frac{\tau}{K}\right\}, \tag{2.19}$$

where $N > 0$ is a constant independent of τ and t_0. If $\tau \le 1$, it follows from inequality (2.18) that

$$\mathrm{M}\,|x(t_0+\tau)|^2 \le 3\mathrm{M}\,|x(t_0)|^2 \exp\left\{\frac{2}{K} + C - \frac{2\tau}{K}\right\}. \tag{2.20}$$

Since $t_0 \ge 0$ is arbitrary, (2.19), (2.20) and the first inequality in Definition 1 hold with

$$\gamma = \frac{2}{K},\ K_1 = \max\left\{N^2; 3\exp\left\{\frac{2}{K} + C\right\}\right\}.$$

Let us prove the second inequality in Definition 1. Let $x(t)$ be a nonzero solution of equation (2.4) with $x(0) \in G_2$. It is easy to see that

$$y(t) = x(t)\int\limits_{t}^{\infty} \frac{\beta(s)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\,\mathrm{d}s$$

is a solution of equation (2.5) with

$$\alpha(t) = -\frac{x(t)}{(\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}}}\beta(t)$$

and, since $t \ge t_0+\tau$, $\sup_{t\ge 0} \mathrm{M}\,|y(t)|^2 < \infty$. It is obvious that $y(0) \in G_2$. Therefore, because of the Lemma,

$$(\mathrm{M}\,|y(t)|^2)^{\frac{1}{2}} = (\mathrm{M}\,|x(s)|^2)^{\frac{1}{2}} \int\limits_{t}^{\infty} \frac{\beta(s)}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,\mathrm{d}s \le K.$$

So $\forall\tau \ge 0$ and $\forall t \ge 0$,

$$\int\limits_{t}^{\infty} \frac{\beta(s)}{\sqrt{\mathrm{M}\,|x(s)|^2}}\mathrm{d}s \le \frac{K}{\sqrt{\mathrm{M}\,|x(t)|^2}}. \tag{2.21}$$

The left-hand side of this inequality is monotone increasing for $\tau \geq 0$ and is bounded, so it has a limit for $\tau \to \infty$. But

$$\int_t^\infty \frac{\beta(s)}{\sqrt{(\mathrm{M}\,|x(s)|^2)}}\,\mathrm{d}s = \int_t^{t_0+\tau} \frac{1}{\sqrt{(\mathrm{M}\,|x(s)|^2)}}\,\mathrm{d}s + \int_{t_+\tau}^{t_0+\tau+1} \frac{\beta(s)}{\sqrt{(\mathrm{M}\,|x(s)|^2)}}\,\mathrm{d}s,$$

where the second summand tends to zero as $\tau \to \infty$ (since the integral is convergent), and so if $\tau \to \infty$, we get the inequality

$$\int_t^\infty \frac{1}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,\mathrm{d}s \leq \frac{K}{\sqrt{\mathrm{M}\,|x(t)|^2}}. \tag{2.22}$$

Let

$$\psi(t) = \int_t^\infty \frac{1}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,\mathrm{d}s.$$

Then it follows from (2.22) that

$$\psi'(t) \leq -\frac{1}{K}\psi(t).$$

Whence we get

$$\psi(t) \leq \psi(t_0)\exp\left\{-\frac{1}{K}(t-t_0)\right\}. \tag{2.23}$$

Since $x(t)$ is a solution of system (2.4), we have for $\tau \geq t$ that

$$\mathrm{M}\,|x(\tau)|^2 \leq C_1 \mathrm{M}|x(t)|^2 \exp\{L(\tau-t)\},$$

where L, C_1 are positive constants, independent of τ and t. From the last inequality, it follows that

$$\mathrm{M}\,|x(\tau)|^2 \leq 3\mathrm{M}\,|x(t)|^2 \exp\{3L(\tau-t+1)(\tau-t)\}.$$

Therefore,

$$(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}}\psi(t) = (\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}} \int_t^\infty \frac{1}{\sqrt{\mathrm{M}\,|x(s)|^2}}\,\mathrm{d}s$$

$$\geq \int_t^\infty \frac{1}{\sqrt{3}} \exp\left\{-\frac{3}{2}L(s-t+1)(s-t)\right\}\mathrm{d}s = l.$$

Here L is a positive constant. Then (2.22) and (2.23) give

$$(\mathrm{M}\,|x(t)|^2)^{\frac{1}{2}} \geq \frac{l}{\psi(t)} \geq \frac{l}{\psi(t_0)}\exp\left\{\frac{1}{K}(t-t_0)\right\} \geq \frac{l}{K}\exp\left\{\frac{1}{K}(t-t_0)\right\}(\mathrm{M}\,|x(t_0)|^2)^{\frac{1}{2}}.$$

This estimate is the second inequality, which figures in the definition of the exponential dichotomy. The theorem is proved.

In the theory of ordinary differential equation one proves a converse result that exponential dichotomy of a homogeneous system implies the existence of a bounded solution of the nonhomogeneous system and that

$$y(t) = \int_0^\infty G(t,\tau) f(\tau)\,\mathrm{d}\tau, \tag{2.24}$$

where $G(t,\tau)$ is Green's function

$$G(t,\tau) = \begin{cases} \Phi(0,t)P_1(\Phi(0,\tau))^{-1}, & t \geq \tau, \\ -\Phi(0,t)P_2(\Phi(0,\tau))^{-1}, & t < \tau, \end{cases} \tag{2.25}$$

$\Phi(0,t)$ is the matriciant of the homogeneous system. For stochastic nonhomogeneous systems,

$$\mathrm{d}x = (A(t)x + \alpha(t))dt + \ (B(t)x + \beta(t))]\mathrm{d}W(t), \tag{2.26}$$

one can also write the representation

$$y(t) = \int_0^\infty G(t,\tau)\alpha(\tau)\mathrm{d}\tau \ + \int_0^\infty G(t,\tau)\beta(\tau)\mathrm{d}W(\tau), \tag{2.27}$$

however, in such a case, $y(t)$ will not be F_t-measurable any more. Thus, by using Green's function, one succeeds in getting a similar result only in the case where the homogeneous system is exponentially stable and the nonhomogeneous system has the form

$$\mathrm{d}x = [\ A(t)x + \alpha(t)]\mathrm{d}t + \ \beta(t)\mathrm{d}W(t). \tag{2.28}$$

Theorem 2. *Let the homogeneous system*

$$\mathrm{d}x = \ A(t)x\mathrm{d}t \tag{2.29}$$

be exponentially stable on the positive semiaxis. Then, for arbitrary $\alpha(t), \beta(t) \in B$, system (2.28) has a solution bounded in mean square on the positive semiaxis. In addition, all bounded solutions of system (2.28) are given by

$$x = \psi(t) + \int_0^t \Phi(t,\tau)\alpha(\tau)\,\mathrm{d}\tau + \int_0^t \Phi(t,\tau)\beta(\tau)\,\mathrm{d}W(\tau), \tag{2.30}$$

where $\psi(t)$ is an arbitrary solution of system (2.29), and $\Phi(t,\tau)$ is the matriciant of system (2.29), $\Phi(\tau,\tau) = E$.

Proof. Since system (2.29) is exponentially stable, its matriciant satisfies the estimate

$$||\Phi(t,\tau)|| \leq K \exp\{-\gamma(t-\tau)\}, \tag{2.31}$$

for $t \geq \tau \geq 0$, with some positive K and γ. Let us show that $x(t)$, defined by (2.30), is bounded in mean square for $t \geq 0$. To do that, it is sufficient to prove the boundedness of each of its summands. Indeed, $\psi(t)$ is a function bounded on the semiaxis. Let us estimate the second summand. From the Cauchy–Bunyakovskii inequality, we have

$$M\,|\int_0^t \Phi(t,\tau)\alpha(\tau)\,\mathrm{d}\tau|^2 \le M\left(\int_0^t ||\Phi(t,\tau)|||\alpha(\tau)|\,\mathrm{d}\tau|\right)^2$$

$$\le K^2 M\left(\int_0^t \exp\left\{\frac{-\gamma(t-\tau)}{2}\right\}\exp\left\{\frac{-\gamma(t-\tau)}{2}\right\}|\alpha(\tau)\,\mathrm{d}\tau\right)^2$$

$$\le K^2\int_0^t \exp\{-\gamma(t-\tau)\}\,\mathrm{d}\tau\int_0^t \exp\{-\gamma(t-\tau)\}M\,|\alpha(\tau)|^2\,\mathrm{d}\tau) < C,$$

where $C>0$ is a constant, since $\alpha(t)\in B$. Let us use the properties of the stochastic integral

$$M\,|\int_0^t \Phi(t,\tau)\beta(\tau)\,\mathrm{d}W(\tau)|^2 \le \int_0^t ||\Phi(t,\tau)||^2 M\,|\beta(\tau)|^2\,\mathrm{d}\tau$$

$$\le K^2\int_0^t \exp\{-2\gamma(t-\tau)\}\,\mathrm{d}\tau \sup_{t\ge 0} M\,|\beta(t)|^2 < C_1, \qquad C_1>0.$$

So the function $x(t)$ in (2.30) is bounded in mean square. It is obvious that it is F_t-dimensional, which follows from [1, p. 234], and gives a solution of system (2.28). The theorem is proved. □

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