



## GENERAL $((k, p), \psi)$ -HILFER FRACTIONAL INTEGRALS

BOUHARKET BENAÏSSA AND HUSEYIN BUDAK

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*Abstract.* The main motivation of this study is to establish a general version of the Riemann-Liouville fractional integrals with two exponential parameters  $k$  and  $p$  called  $((k, p), \psi)$ -Hilfer fractional integrals which is determined over the  $k$ -gamma function. We first prove that these operators are well-defined, continuous and have semi-group property. Then, particularly, we present the harmonic, geometric and arithmetic  $(k, p), \psi$ -Hilfer fractional integrals. Moreover, some special cases relating to general  $((k, p), \psi)$ -Riemann-Liouville fraction integrals are given.

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### 1. INTRODUCTION

The fractional calculus theory has been used as a mathematical tool in a variety of pure and practical fields. This approach has been used in different scientific fields. In applied mathematics, various fractional operators have been used to show a set of integral inequalities and their generalizations. One among the vital applications of fractional integrals is the  $k$ -Riemann-Liouville fractional integral operator which is an important tool and a source of many research works in field science such as the theory of inequalities, differential equations, integral inequalities. see for example [2], [9], [10], [11], [12]. The right and left-sided  $k$ -Riemann-Liouville fractional integrals of order  $\alpha > 0$ , for a continuous function  $f$  on  $[a, b]$  are defined as

$$J_{a^+, k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad a < x \leq b, \quad (1.1)$$

$$J_{b^-, k}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad a \leq x < b, \quad (1.2)$$

where the  $k$ -gamma function[3] verified

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \quad (1.3)$$

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$$

for all  $\alpha, k > 0$ .

Some basic equations satisfied by the  $k$ -gamma function are given in [3].

**Property 1.** For all  $\alpha, k > 0$  and  $n \in \mathbb{N}$ , the fundamental formulae satisfied by  $k$ -gamma function are

$$\Gamma_k(\alpha + nk) = k^n \left(\frac{\alpha}{k}\right) \left(\frac{\alpha}{k} + 1\right) \dots \left(\frac{\alpha}{k} + (n-1)\right) \Gamma_k(\alpha), \quad (1.4)$$

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha), \quad (1.5)$$

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right). \quad (1.6)$$

*Remark 1.* The above definition and properties reduce to the gamma function and its properties when  $k \rightarrow 1$ .

The  $k$ -beta function satisfies the following identities.

$$\beta_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt. \quad (1.7)$$

$$\beta_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad x > 0, \quad y > 0. \quad (1.8)$$

In 2017, Kuldeep [6] defined the two-parameter gamma function called  $(k, p)$  gamma function which is a generalization of  $k$ -gamma.

**Definition 1.** Given  $\alpha \in \mathbb{C}/k\mathbb{Z}^-; k, p \in \mathbb{R}^+ - \{0\}$  and  $Re(\alpha) > 0$ , then the integral representation of  $(k, p)$ -Gamma Function is given by

$$\Gamma_{(k,p)}(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t}{p}} dt. \quad (1.9)$$

We present certain base formulas related to the  $(k, p)$ -gamma function are mentioned in [5], [6].

**Property 2.** For all  $\alpha, k, p > 0$  and  $n \in \mathbb{N}$ , the fundamental formulae satisfied by  $(k, p)$ -gamma function are ,

$$\Gamma_{(k,p)}(\alpha + nk) = p^n \left(\frac{\alpha}{k}\right) \left(\frac{\alpha}{k} + 1\right) \dots \left(\frac{\alpha}{k} + (n-1)\right) \Gamma_{(k,p)}(\alpha), \quad (1.10)$$

$$\Gamma_{(k,p)}(\alpha + k) = \frac{p\alpha}{k} \Gamma_{(k,p)}(\alpha), \quad (1.11)$$

$$\Gamma_{(k,p)}(\alpha) = \left(\frac{p}{k}\right)^{\frac{\alpha}{k}} \Gamma_k(\alpha) = \left(\frac{p}{k}\right)^{\frac{\alpha}{k}} \Gamma\left(\frac{\alpha}{k}\right). \quad (1.12)$$

We deduce that

$$\Gamma_{(k,p)}(1) = \left(\frac{p}{k}\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k}\right), \quad \Gamma_{(k,p)}(k) = \left(\frac{p}{k}\right), \quad (1.13)$$

$$\Gamma_{(k,p)}(p) = \left(\frac{p^{\frac{p}{k}}}{k}\right) \Gamma\left(\frac{p}{k}\right).$$

*Remark 2.* The above definition and properties reduce to the  $k$ -gamma function and its properties when  $p = k$ .

By using the formula (1.12), we get

$$\beta_k(x, y) = \frac{\Gamma_{(k,p)}(x)\Gamma_{(k,p)}(y)}{\Gamma_{(k,p)}(x+y)} = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad x > 0, \quad y > 0. \tag{1.14}$$

## 2. $((k, p), \psi)$ -HILFER FRACTIONAL INTEGRALS

In this section, we present the general  $((k, p), \psi)$ -Hilfer fractional integrals of order  $\alpha$  with two exponential parameters  $k$  and  $p$  which generalize the  $k$ -Riemann-Liouville fractional integrals.

Let  $\varphi : (0, +\infty) \times (0, +\infty) \rightarrow (0, +\infty)$  be a map satisfying the condition  $\varphi(k, k) = k$ . For example

(1) the arithmetic mean  $\varphi_1(k, p) = \frac{k+p}{2}$ ,

(2) the geometric mean  $\varphi_2(k, p) = \sqrt{kp}$ ,

(3)  $\varphi_3(k, p) = \frac{k^2}{p}$ , called the H-case.

Now we will give definitions of generalized fractional integrals.

**Definition 2.** (See [1], [7], [13]) Let  $\psi(x)$  be an increasing positive monotone function on  $[a, b]$  such that  $\psi'(x)$  is continuous on  $(a, b)$ . The space  $X_{\psi}^r(a, b)$  ( $1 \leq r < +\infty$ ) is defined as the set of those real-valued Lebesgue measurable functions  $f$  on  $[a, b]$  for which

$$\|f\|_{X_{\psi}^r} = \left(\int_a^b |f(x)|^r \psi'(x) dx\right)^{\frac{1}{r}} < \infty. \tag{2.1}$$

In particular, when  $\psi(x) = x$  the space  $X_{\psi}^r(a, b)$  coincides with the  $L_r[a, b]$ .

**Definition 3.** Let  $[a, b] \subseteq [0, +\infty]$ ,  $f \in X_{\psi}^1(a, b)$  and  $k, p > 0$  and for an strictly increasing function  $\psi$ , where  $\psi \in C^1[a, b]$  such that  $\psi'(t) \neq 0$ , for all  $t \in [a, b]$ . The left and right-sided general  $(k, p)$ -Hilfer fractional integrals of a function  $f$  with respect to the function  $\psi$  on  $[a, b]$  are defined, respectively, as follows

$${}_{a^+}I_{\varphi(k,p)}^{\alpha, \psi} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\frac{\alpha}{\varphi(k,p)} - 1} f(t) dt, \quad a < x \leq b, \tag{2.2}$$

$${}_b^-I_{\varphi(k,p)}^{\alpha,\Psi}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_x^b \Psi'(t)(\Psi(t) - \Psi(x))^{\frac{\alpha}{\varphi(k,p)}-1} f(t)dt, \quad a \leq x < b. \quad (2.3)$$

Here,  $\Gamma_{(k,p)}$  is the  $(k,p)$ -Gamma Function defined by as in (1.9).

*Remark 3.* Set  $p = k$  in the above definition, we obtain the  $(k, \Psi)$ -Hilfer fractional integrals [8].

In the following theorem, we show that the  $((k,p), \Psi)$ -Hilfer fractional integrals are well-defined.

**Theorem 1.** *The fractional integrals (2.2), (2.3) are defined for functions  $f \in L_1[a, b]$ , existing almost everywhere and*

$${}_a^+I_{\varphi(k,p)}^{\alpha,\Psi}f(x), {}_b^-I_{\varphi(k,p)}^{\alpha,\Psi}f(x) \in X_{\Psi}^1(a, b). \quad (2.4)$$

Moreover

$$\left\| I_{\varphi(k,p)}^{\alpha,\Psi}f(x) \right\|_{X_{\Psi}^1(a,b)} \leq C \|f(t)\|_{X_{\Psi}^1(a,b)}, \quad (2.5)$$

where

$$C = \max \left( \frac{(\Psi(b) - \Psi(a))^{\frac{\alpha_1}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}, \frac{(\Psi(b) - \Psi(a))^{\frac{\alpha_2}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k\left(\frac{\alpha_2}{\varphi(k,p)} + 1\right)\right)} \right), \quad 0 < \alpha_2 < \varphi(k,p) < \alpha_1.$$

*Proof.* Let  $f(x) \in L_1[a, b]$ .

- Let  $\frac{\alpha}{\varphi(k,p)} = 1$ , is clearly.
- Let  $\frac{\alpha}{\varphi(k,p)} > 1$ . Let  $\Omega = [a, b] \times [a, b]$ , we pose for all  $(x, t) \in \Omega$ , posing

$$G_1(x, t) = \begin{cases} (\Psi(x) - \Psi(t))^{\frac{\alpha}{\varphi(k,p)}-1}, & a \leq t \leq x, \\ 0, & x \leq t \leq b, \end{cases}$$

and

$$G_2(x, t) = \begin{cases} 0, & a \leq t \leq x, \\ (\Psi(t) - \Psi(x))^{\frac{\alpha}{\varphi(k,p)}-1}, & x \leq t \leq b. \end{cases}$$

Since  $\Psi$  is continuous, strictly increasing function, for  $i = 1, 2$  we have

$$\int_a^b G_i(x, t)\Psi'(x)dx \leq \int_a^b \Psi'(x)(\Psi(b) - \Psi(a))^{\frac{\alpha}{\varphi(k,p)}-1}dx = (\Psi(b) - \Psi(a))^{\frac{\alpha}{\varphi(k,p)}}.$$

By applying Fubini's Theorem, we get

$$\begin{aligned} \int_a^b \left( \int_a^b G_i(x, t)|f(t)|\Psi'(t)dt \right) \Psi'(x)dx &= \int_a^b \left( \int_a^b G_i(x, t)\Psi'(x)dx \right) |f(t)|\Psi'(t)dt \\ &\leq (\Psi(b) - \Psi(a))^{\frac{\alpha}{\varphi(k,p)}} \int_a^b |f(t)|\Psi'(t)dt \\ &= (\Psi(b) - \Psi(a))^{\frac{\alpha}{\varphi(k,p)}} \|f(t)\|_{X_{\Psi}^1(a,b)} < \infty. \end{aligned}$$

Therefore

$$\int_a^b G_i(x, t) |f(t)| \psi'(t) dt \in X_{\psi}^1(a, b).$$

This gives us (2.4) for  $i = 1, 2$ .

- Let  $0 < \frac{\alpha}{\varphi(k, p)} < 1$ , put  $\eta = 1 - \frac{\alpha}{\varphi(k, p)}$  hence  $0 < \eta < 1$ . By using Fubini's Theorem, we get

$$\begin{aligned} & \int_a^b \left| {}_{a^+}I_{\varphi(k, p)}^{\alpha, \psi} f(x) \right| \psi'(x) dx \\ &= \frac{1}{k\Gamma_{(k, p)}\left(\frac{k\alpha}{\varphi(k, p)}\right)} \int_a^b \left| \int_a^x (\psi(x) - \psi(t))^{\frac{\alpha}{\varphi(k, p)} - 1} f(t) \psi'(t) dt \right| \psi'(x) dx \\ &\leq \frac{1}{k\Gamma_{(k, p)}\left(\frac{k\alpha}{\varphi(k, p)}\right)} \int_a^b \int_a^x \frac{|f(t)|}{(\psi(x) - \psi(t))^\eta} \psi'(t) \psi'(x) dt dx \\ &= \frac{1}{k\Gamma_{(k, p)}\left(\frac{k\alpha}{\varphi(k, p)}\right)} \int_a^b |f(t)| \left( \int_t^b \frac{\psi'(x)}{(\psi(x) - \psi(t))^\eta} dx \right) \psi'(t) dt \\ &= \frac{1}{(1 - \eta)k\Gamma_{(k, p)}\left(\frac{k\alpha}{\varphi(k, p)}\right)} \int_a^b |f(t)| (\psi(b) - \psi(t))^{1-\eta} \psi'(t) dt \\ &\leq \frac{(\psi(b) - \psi(a))^{1-\eta}}{(1 - \eta)k\Gamma_{(k, p)}\left(\frac{k\alpha}{\varphi(k, p)}\right)} \int_a^b |f(t)| \psi'(t) dt. \end{aligned}$$

Hence, we obtain

$$\int_a^b \left| {}_{a^+}I_{\varphi(k, p)}^{\alpha, \psi} f(x) \right| \psi'(x) dx \leq \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{\varphi(k, p)}}}{\Gamma_{(k, p)}\left(k\left(\frac{\alpha}{\varphi(k, p)} + 1\right)\right)} \|f(t)\|_{X_{\psi}^1(a, b)} < +\infty.$$

By the similar process, one can obtain

$$\int_a^b \left| {}_{b^-}I_{\varphi(k, p)}^{\alpha, \psi} f(x) \right| \psi'(x) dx \leq \frac{(\psi(b) - \psi(a))^{\frac{\alpha}{\varphi(k, p)}}}{\Gamma_{(k, p)}\left(k\left(\frac{\alpha}{\varphi(k, p)} + 1\right)\right)} \|f(t)\|_{X_{\psi}^1(a, b)} < +\infty.$$

This gives us (2.4). □

**Theorem 2.** Let  $\frac{\alpha}{\varphi(k, p)} > 1$  and  $f \in L_1[a, b]$ , then the fractional integrals (2.2), (2.3) are

$${}_{a^+}I_{\varphi(k, p)}^{\alpha, \psi} f(x), {}_{b^-}I_{\varphi(k, p)}^{\alpha, \psi} f(x) \in C[a, b]. \tag{2.6}$$

*Proof.* Let  $x, y \in [a, b]$ ,  $y \leq x$  and  $y \rightarrow x$ . Since  $\psi$  is strictly increasing function and  $\psi \in C^1[a, b]$ , we get

$$\begin{aligned} & k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right) \left| {}_{b^-}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} f(x) - {}_{b^-}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} f(y) \right| \\ &= \left| \left( \int_x^b [(\psi(t) - \psi(x))^{\frac{\alpha}{\varphi(k,p)}-1} - (\psi(t) - \psi(y))^{\frac{\alpha}{\varphi(k,p)}-1}] f(t) \psi'(t) dt \right) \right. \\ &\quad \left. - \int_y^x (\psi(t) - \psi(y))^{\frac{\alpha}{\varphi(k,p)}-1} f(t) \psi'(t) dt \right| \\ &\leq \int_x^b |(\psi(t) - \psi(x))^{\frac{\alpha}{\varphi(k,p)}-1} - (\psi(t) - \psi(y))^{\frac{\alpha}{\varphi(k,p)}-1}| |f(t)| \psi'(t) dt \\ &\quad + (\psi(x) - \psi(y))^{\frac{\alpha}{\varphi(k,p)}-1} \|f(t)\|_{X_{\psi}^1(a,b)}. \end{aligned}$$

Thus

$$\left| {}_{b^-}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} f(x) - {}_{b^-}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} f(y) \right| \rightarrow 0 \quad \text{as } y \rightarrow x.$$

By similar process, one can prove

$$\left| {}_{a^+}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} f(x) - {}_{a^+}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} f(y) \right| \rightarrow 0 \quad \text{as } y \rightarrow x,$$

which completes the proof.  $\square$

Now, we give the commutativity and the semigroup properties of the  $((k, p), \psi)$ -Hilfer fractional integrals.

**Theorem 3.** *Let  $\alpha, \beta > 0$ . Then we have the following equalities for  $((k, p), \psi)$ -Hilfer fractional integrals.*

$${}_{a^+}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} \left( {}_{a^+}\mathbf{I}_{\varphi(k,p)}^{\beta,\psi} f(x) \right) = {}_{a^+}\mathbf{I}_{\varphi(k,p)}^{\alpha+\beta,\psi} f(x) = {}_{a^+}\mathbf{I}_{\varphi(k,p)}^{\beta,\psi} \left( {}_{a^+}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} f(x) \right). \quad (2.7)$$

$${}_{b^-}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} \left( {}_{b^-}\mathbf{I}_{\varphi(k,p)}^{\beta,\psi} f(x) \right) = {}_{b^-}\mathbf{I}_{\varphi(k,p)}^{\alpha+\beta,\psi} f(x) = {}_{b^-}\mathbf{I}_{\varphi(k,p)}^{\beta,\psi} \left( {}_{b^-}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} f(x) \right). \quad (2.8)$$

Equalities (2.7) and (2.8) are satisfied in any point for  $f(t) \in C([a, b])$  and in almost every point for  $f(t) \in L_1[a, b]$ .

*Proof.* Using Fubini's Theorem, we get

$$\begin{aligned} & \left[ k^2 \Gamma_{(k,p)} \left( \frac{k\alpha}{\varphi(k,p)} \right) \Gamma_k \left( \frac{k\beta}{\varphi(k,p)} \right) \right] {}_{a^+}\mathbf{I}_{\varphi(k,p)}^{\alpha,\psi} \left( {}_{a^+}\mathbf{I}_{\varphi(k,p)}^{\beta,\psi} f(x) \right) \\ &= \int_a^x (\psi(x) - \psi(t))^{\frac{\alpha}{\varphi(k,p)}-1} \left( \int_a^t (\psi(t) - \psi(s))^{\frac{\beta}{\varphi(k,p)}-1} f(s) \psi'(s) ds \right) \psi'(t) dt \\ &= \int_a^x f(s) \left( \int_s^x (\psi(x) - \psi(t))^{\frac{\alpha}{\varphi(k,p)}-1} (\psi(t) - \psi(s))^{\frac{\beta}{\varphi(k,p)}-1} \psi'(t) dt \right) \psi'(s) ds. \end{aligned} \quad (2.9)$$

If we use the change of variable  $y = \frac{\psi(t) - \psi(s)}{\psi(x) - \psi(s)}$  in the inner integral in (2.9), we obtain

$$\begin{aligned} & \int_s^x (\psi(x) - \psi(t))^{\frac{\alpha}{\varphi(k,p)} - 1} (\psi(t) - \psi(s))^{\frac{\beta}{\varphi(k,p)} - 1} dt \\ &= (\psi(x) - \psi(s))^{\frac{\alpha+\beta}{\varphi(k,p)} - 1} \int_0^1 (1-y)^{\frac{\alpha}{\varphi(k,p)} - 1} (y)^{\frac{\beta}{\varphi(k,p)} - 1} dy \\ &= k (\psi(x) - \psi(s))^{\frac{\alpha+\beta}{\varphi(k,p)} - 1} \beta_k \left( \frac{k\alpha}{\varphi(k,p)}, \frac{k\beta}{\varphi(k,p)} \right). \end{aligned} \tag{2.10}$$

By using  $k$ -beta propriety (1.14) and (2.9) in (2.10), we deduce that

$${}_{a^+}I_{\varphi(k,p)}^{\alpha, \psi} \left( {}_{a^+}I_{\varphi(k,p)}^{\beta, \psi} f(x) \right) = \frac{1}{k\Gamma_{(k,p)} \left( \frac{k(\alpha+\beta)}{\varphi(k,p)} \right)} \int_a^x (\psi(x) - \psi(s))^{\frac{\alpha+\beta}{\varphi(k,p)} - 1} f(s) \psi'(s) ds.$$

This gives the equality (2.7). By similar way, the equality (2.8) can be proved easily. □

### 3. SPECIAL CASES OF GENERAL $((k, p), \psi)$ -HILFER FRACTIONAL INTEGRALS

In this section, we present three interesting cases of the general integrals  $(k, p), \psi$ -Hilfer depending on the choice of the function  $\psi$ .

#### 3.1. $(k, p)$ -Riemann-Liouville fractional operators

By setting  $\psi(t) = t$ , general fractional operators  $(k, p), \psi$ -Hilfer reduce to fractional operators  $(k, p)$ -Riemann-Liouville as defined below.

**Definition 4.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$ , for all  $t \in [a, b]$  and  $k, p > 0$ . The left and right-sided  $(k, p)$ -Riemann-Liouville fractional integrals of a function  $f$  are defined, respectively, as follows

$${}_{a^+}I_{\varphi(k,p)}^{\alpha} f(x) = \frac{1}{k\Gamma_{(k,p)} \left( \frac{k\alpha}{\varphi(k,p)} \right)} \int_a^x (x-t)^{\frac{\alpha}{\varphi(k,p)} - 1} f(t) dt, \quad a < x \leq b, \tag{3.1}$$

$${}_{b^-}I_{\varphi(k,p)}^{\alpha} f(x) = \frac{1}{k\Gamma_{(k,p)} \left( \frac{k\alpha}{\varphi(k,p)} \right)} \int_x^b (t-x)^{\frac{\alpha}{\varphi(k,p)} - 1} f(t) dt, \quad a \leq x < b, \tag{3.2}$$

where  $\Gamma_{(k,p)}$  is the  $(k, p)$ -Gamma Function defined by (1.9).

We present some specially cases to  $(k, p)$ -Riemann-Liouville fractional operators

#### 3.1.1. $H$ - $(k, p)$ -Riemann-Liouville fractional integrals

By choosing  $\varphi(k, p) = \frac{k^2}{p}$  in Definition 4, we have:

**Definition 5.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$  and  $k, p > 0$ . The right and the left-sided harmonic  $(k, p)$ -Riemann-Liouville fractional integrals of order  $\alpha > 0$  are defined as

$${}_{a^+}H_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_a^x (x-t)^{\frac{p\alpha}{k^2}-1} f(t) dt, \quad a < x \leq b. \quad (3.3)$$

$${}_{b^-}H_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_x^b (t-x)^{\frac{p\alpha}{k^2}-1} f(t) dt, \quad a \leq x < b. \quad (3.4)$$

### 3.1.2. Geometric $(k, p)$ -Riemann-Liouville fractional integrals

By taking  $\varphi(k, p) = \sqrt{kp}$  in Definition 4, we have:

**Definition 6.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$  and  $k, p > 0$ . The right and the left-sided geometric  $(k, p)$ -Riemann-Liouville fractional integrals of order  $\alpha > 0$  are defined as

$${}_{a^+}G_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_a^x (x-t)^{\frac{\alpha}{\sqrt{kp}}-1} f(t) dt, \quad a < x \leq b. \quad (3.5)$$

$${}_{b^-}G_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_x^b (t-x)^{\frac{\alpha}{\sqrt{kp}}-1} f(t) dt, \quad a \leq x < b. \quad (3.6)$$

### 3.1.3. Arithmetic $(k, p)$ -Riemann-Liouville fractional integrals

By putting  $\varphi(k, p) = \frac{k+p}{2}$  in Definition 4, we have:

**Definition 7.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$  and  $k, p > 0$ . The right and the left-sided arithmetic  $(k, p)$ -Riemann-Liouville fractional integrals of order  $\alpha > 0$  are defined as

$${}_{a^+}A_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_a^x (x-t)^{\frac{2\alpha}{k+p}-1} f(t) dt, \quad a < x \leq b. \quad (3.7)$$

$${}_{b^-}A_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b (t-x)^{\frac{2\alpha}{k+p}-1} f(t) dt, \quad a \leq x < b. \quad (3.8)$$

*Remark 4.* Choosing  $p = k$  in Definitions 5, 6 and 7 give us the  $k$ -Riemann-Liouville fraction operators given in [9].

## 3.2. $(k, p)$ -Hadamard fractional operators

Taking  $\psi(t) = \ln(t)$  on the definition of general fractional operators  $((k, p), \psi)$ -Hilfer, we get  $(k, p)$ -Hadamard fractional operators defined out below.

**Definition 8.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$ , for all  $t \in [a, b]$  and  $k, p > 0$ . The left and right-sided  $(k, p)$ -Hadamard fractional integrals are defined,



respectively, as follows

$${}_a^+ I_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^x \left(\ln \frac{x}{t}\right)^{\frac{\alpha}{\varphi(k,p)}-1} \frac{f(t)}{t} dt, \quad a < x \leq b, \quad (3.9)$$

$${}_b^- I_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_x^b \left(\ln \frac{t}{x}\right)^{\frac{\alpha}{\varphi(k,p)}-1} \frac{f(t)}{t} dt, \quad a \leq x < b, \quad (3.10)$$

where  $\Gamma_{(k,p)}$  is the  $(k, p)$ -Gamma Function defined by (1.9).

Now, we give some particular cases to  $(k, p)$ -Hadamard fractional operators.

### 3.2.1. $H$ - $(k, p)$ -Hadamard fractional integrals

By choosing  $\varphi(k, p) = \frac{k^2}{p}$  in Definition 8, we get:

**Definition 9.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$  and  $k, p > 0$ . The right and the left-sided harmonic  $(k, p)$ -Hadamard fractional integrals of order  $\alpha > 0$  are defined as

$${}_a^+ H_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_a^x \left(\ln \frac{x}{t}\right)^{\frac{p\alpha}{k^2}-1} \frac{f(t)}{t} dt, \quad a < x \leq b. \quad (3.11)$$

$${}_b^- H_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_x^b \left(\ln \frac{t}{x}\right)^{\frac{p\alpha}{k^2}-1} \frac{f(t)}{t} dt, \quad a \leq x < b. \quad (3.12)$$

### 3.2.2. Geometric $(k, p)$ -Hadamard fractional integrals

By taking  $\varphi(k, p) = \sqrt{k p}$  in Definition 8, we get:

**Definition 10.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$  and  $k, p > 0$ . The right and the left-sided geometric  $(k, p)$ -Hadamard fractional integrals of order  $\alpha > 0$  are defined as

$${}_a^+ G_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_a^x \left(\ln \frac{x}{t}\right)^{\frac{\alpha}{\sqrt{k p}}-1} \frac{f(t)}{t} dt, \quad a < x \leq b. \quad (3.13)$$

$${}_b^- G_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_x^b \left(\ln \frac{t}{x}\right)^{\frac{\alpha}{\sqrt{k p}}-1} \frac{f(t)}{t} dt, \quad a \leq x < b. \quad (3.14)$$

### 3.2.3. Arithmetic $(k, p)$ -Hadamard fractional integrals

By putting  $\varphi(k, p) = \frac{k+p}{2}$  in Definition 8, we get:

**Definition 11.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$  and  $k, p > 0$ . The right and the left-sided arithmetic  $(k, p)$ -Hadamard fractional integrals of order  $\alpha > 0$  are defined as

$${}_a^+ A_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_a^x \left(\ln \frac{x}{t}\right)^{\frac{2\alpha}{k+p}-1} \frac{f(t)}{t} dt, \quad a < x \leq b. \quad (3.15)$$

$${}_b^- \mathbf{A}_{\varphi(k,p)}^\alpha f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b \left(\ln \frac{t}{x}\right)^{\frac{2\alpha}{k+p}-1} \frac{f(t)}{t} dt, \quad a \leq x < b. \quad (3.16)$$

*Remark 5.* Choosing  $p = k$  in Definitions 9, 10 and 11 give us the  $k$ -Hadamard fractional operators defined in [4].

### 3.3. $(k, p)$ -Katugompola fractional operators

Putting  $\psi(t) = \frac{t^{\rho+1}}{\rho+1}$  where  $\rho > -1$ , then the general  $((k, p), \psi)$ -Hilfer fractional operators reduce to the  $(k, p)$ -Katugompola operators define as follows.

**Definition 12.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$ , for all  $t \in [a, b]$ ,  $\rho > -1$  and  $k, p > 0$ . The left and right-sided  $(k, p)$ -Katugompola fractional integrals of a function  $f$  are defined, respectively, as follows

$${}_a^+ \mathbf{I}_{\varphi(k,p)}^{\alpha, \rho} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{\alpha}{\varphi(k,p)}-1} t^\rho f(t) dt, \quad a < x \leq b, \quad (3.17)$$

$${}_b^- \mathbf{I}_{\varphi(k,p)}^{\alpha, \rho} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_x^b (t^{\rho+1} - x^{\rho+1})^{\frac{\alpha}{\varphi(k,p)}-1} t^\rho f(t) dt, \quad a \leq x < b, \quad (3.18)$$

where  $\Gamma_{(k,p)}$  is the  $(k, p)$ -Gamma Function defined by (1.9).

The specially cases to  $(k, p)$ -Katugompola fractional operators are as follows.

#### 3.3.1. $H$ - $(k, p)$ -Katugompola fractional integrals

By choosing  $\varphi(k, p) = \frac{k^2}{p}$  in Definition 12, we have:

**Definition 13.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$ ,  $\rho > -1$  and  $k, p > 0$ . The right and the left-sided harmonic  $(k, p)$ -Katugompola fractional integrals of order  $\alpha > 0$  are defined as

$${}_a^+ \mathbf{H}_{\varphi(k,p)}^{\alpha, \rho} f(x) = \frac{(\rho+1)^{\frac{p\alpha}{k^2}-1}}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{p\alpha}{k^2}-1} t^\rho f(t) dt, \quad a < x \leq b. \quad (3.19)$$

$${}_b^- \mathbf{H}_{\varphi(k,p)}^{\alpha, \rho} f(x) = \frac{(\rho+1)^{\frac{p\alpha}{k^2}-1}}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_x^b (t^{\rho+1} - x^{\rho+1})^{\frac{p\alpha}{k^2}-1} t^\rho f(t) dt, \quad a \leq x < b. \quad (3.20)$$

#### 3.3.2. Geometric $(k, p)$ -Katugompola fractional integrals

By taking  $\varphi(k, p) = \sqrt{k}p$  in Definition 12, we have:

**Definition 14.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$ ,  $\rho > -1$  and  $k, p > 0$ . The right and the left-sided geometric  $(k, p)$ -Katugompola fractional integrals of order  $\alpha > 0$  are defined as

$${}_a^+ G_{\varphi(k,p)}^{\alpha,\rho} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\sqrt{k}p}}}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{\alpha}{\sqrt{k}p}-1} t^\rho f(t) dt, \quad a < x \leq b. \quad (3.21)$$

$${}_b^- G_{\varphi(k,p)}^{\alpha,\rho} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\sqrt{k}p}}}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_x^b (t^{\rho+1} - x^{\rho+1})^{\frac{\alpha}{\sqrt{k}p}-1} t^\rho f(t) dt, \quad a \leq x < b. \quad (3.22)$$

### 3.3.3. Arithmetic $(k, p)$ -Katugompola fractional integrals

By putting  $\varphi(k, p) = \frac{k+p}{2}$  in Definition 12, we have:

**Definition 15.** Let  $[a, b] \subseteq [0, +\infty]$ , where  $a < b$ ,  $f \in X_t^1(a, b)$ ,  $\rho > -1$  and  $k, p > 0$ . The right and the left-sided arithmetic  $(k, p)$ -Katugompola fractional integrals of order  $\alpha > 0$  are defined as

$${}_a^+ A_{\varphi(k,p)}^{\alpha,\rho} f(x) = \frac{(\rho+1)^{1-\frac{2\alpha}{k+p}}}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{2\alpha}{k+p}-1} t^\rho f(t) dt, \quad a < x \leq b. \quad (3.23)$$

$${}_b^- A_{\varphi(k,p)}^{\alpha,\rho} f(x) = \frac{(\rho+1)^{1-\frac{2\alpha}{k+p}}}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b (t^{\rho+1} - x^{\rho+1})^{\frac{2\alpha}{k+p}-1} t^\rho f(t) dt, \quad a \leq x < b. \quad (3.24)$$

*Remark 6.* Putting  $p = k$  in Definitions 13, 14 and 15 give us the  $k$ -Katugompola fractional operators.

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*Authors' addresses*

**Bouharket Benaissa**

Faculty of Material Sciences, Laboratory of Informatics and Mathematics, University of Tiaret, Tiaret, Algeria

*E-mail address:* bouharket.benaissa@univ-tiaret.dz

**Huseyin Budak**

(**Corresponding author**) Department of Mathematics, Faculty of Science and Arts, Düzce University, 81620, Düzce, Türkiye

*E-mail address:* hsyn.budak@gmail.com