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WELL-POSEDNESS FOR FRACTIONAL HARDY-HéNON PARABOLIC EQUATIONS WITH FRACTIONAL BROWNIAN MOTION

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Abstract. In this paper, we consider fractional Hardy-Hénon parabolic equations driven by fractional Brownian motion. The local existence and uniqueness of L^q mild solutions are proved.

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1. INTRODUCTION

Stochastic partial differential equations driven by fractional Brownian motion have attracted much attention. Tindel et al. [11] studied linear stochastic evolution equations driven by infinite-dimensional fractional Brownian motion with Hurst parameter in the interval $H \in (0,1)$. A sufficient and necessary condition for the existence and uniqueness of the solution is established. Maslowski and Nualart [10] studied nonlinear stochastic evolution equations in a Hilbert space driven by a cylindrical fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ and nuclear covariance operator. Caraballo et al. [3] investigated the existence, uniqueness and exponential asymptotic behavior of mild solutions to a class of stochastic delay evolution equations perturbed by a fractional Brownian motion. Ahmed and Ragusa [1], Chadha and Bora [4], Khan et al. [7] studied some other fractional stochastic equations about the controllability, stability and fractional analysis. Clarke and Olivera [5] studied a semilinear heat equation driven by a Hilbert space-valued fractional Brownian motion, existence and uniqueness of local L^q mild solutions are shown.

In this paper, we study the Cauchy problem

$$\begin{cases} \partial_t u(t) = -(-\Delta)^{\alpha/2} u(t) + |x|^{-\gamma} |u(t)|^{p-1} u(t) + \partial_t B^H(t),\ t > 0, \\ u(0) = \varphi, \end{cases} \tag{1.1}$$

where $u \subset \mathbb{R}^d$, $p > 1$, $u_0 \in L^q(\mathbb{R}^d)$, $\gamma \geq 0$, $(-\Delta)^{\alpha/2}(0 < \alpha < 2)$ is the fractional Laplacian, $B^H(t)$ is a fractional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (\frac{1}{2}, 1)$.

When $\gamma = 0$, existence results of L^p mild solutions for the problem (1.1) without noise was studied in Weissler [12]. In the case $\gamma \in (0, \min(2,d))$, Ben Slimene, Tayachi and Weissler [2] investigated local well-posedness, global existence and large time behavior for the problem (1.1) without noise. When $\alpha = 2$ and $\gamma \in (0,2)$, Majdoub and Mliki [9] studied the problem (1.1) on $\mathbb{R}^d$ ($d = 2$ or 3), they obtained the local existence and uniqueness of mild L^q solution. The purpose of this paper is to investigate the existence and uniqueness of mild solutions to the equation (1.1). The main results are obtained by using the contraction principle.

Definition 1. A process $u\colon \Omega \times [0,T]$ is called a mild solution of equation (1.1) if

(1) $u \in C([0,T]; \mathbb{R}^d)$,

(2)

$$u(t) = e^{-t(-\Delta)^{\alpha/2}}\varphi + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}|u(s)|^{p-1}u(s))\mathrm{d}s + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\mathrm{d}B^H(s), \tag{1.2}$$

with probability one, where $e^{-t(-\Delta)^{\alpha/2}}$ is the linear fractional heat semigroup.

Theorem 1. *Let $d \geq 1$ be an integer and let*

$$0 < \gamma < \min(\alpha, d), \tag{1.3}$$

$$\max\left(\frac{1}{2}, \frac{1}{q}\right) < H < 1, \tag{1.4}$$

$$\max\left(\frac{dp}{d-\gamma}, \frac{d(p-1)}{\alpha-\gamma}\right) < q < \infty. \tag{1.5}$$

Given $\varphi \in L^q(\mathbb{R}^d)$, then there exist $T > 0$ such that (1.1) has a unique mild solution $u \in C([0,T]; L^q(\mathbb{R}^d))$.

The paper is organized as follows. In Section 2, some basic notations and preliminary facts on stochastic integrals for fractional Brownian motion and smoothing effect for the fractional heat semigroup are given. In Section 3, we give the main results of this paper.

2. Preliminaries

2.1. *Fractional Brownian motion*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 2. A centered Gaussian process $\{\beta^H(t),\ t \geq 0\}$ is called fractional Brownian motion (fBm) of Hurst index $H \in (0,1)$ if it has the covariance function

$$R_H(t,s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

In the following, we assume $\frac{1}{2} < H < 1$. Consider the square integrable kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \mathrm{d}u,$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2}))}\right]^{1/2}$ and $t > s$.

Then

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

Consider a fBm $\{\beta^H(t),\ t \in [0,T]\}$. We denote by ζ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of ζ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

Define the linear operator K_H^* from ζ to $L^2([0,T])$ by

$$(K_H^*\varphi)(s) = \int_s^b \varphi(t) \frac{\partial K_H}{\partial t}(t,s) \mathrm{d}t.$$

The operator K_H^* is an isometry between ζ and $L^2([0,T])$ that can be extended to the Hilbert space $\mathcal{H}$.

Consider the process $W = W(t), t \in [0,T]$ defined by

$$W(t) = \beta^H((K_H^*)^{-1}\mathbf{1}_{[0,t]}).$$

Then W is a Wiener process and β^H has the integral representation

$$\beta^H(t) = \int_0^t K_H(t,s) \mathrm{d}W(s). \tag{2.1}$$

We have the following relationship between the Wiener integral with respect to fBm and the Wiener integral with respect to the Wiener process W

$$\int_0^t \varphi(s) \mathrm{d}\beta^H(s) = \int_0^t (K_H^*\varphi)(s) \mathrm{d}W(s), \tag{2.2}$$

for every $t \in [0,T]$ and $\varphi \mathbf{1}_{[0,t]} \in \mathcal{H}$ if and only if $K_H^*\varphi \in L^2([0,T])$. As in [11], we define the standard cylindrical Brownian motion in X as

$$B^H(t) = \sum_{n=0}^{\infty} e_n \beta_n^H(t), \tag{2.3}$$

where $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis in X and β_n^H are real, independent fBm's.

Let f be a deterministic function with values in $\mathcal{L}_2(X,Y)$, the space of Hilbert-Schmidt operators from X to Y. We make the following assumptions on f.

(i) For each $x \in X$, $f(\cdot)x \in L^p([0,T];Y)$, for $p > \frac{1}{H}$,

(ii) $\alpha_H \int_0^T \int_0^T |f(s)|_{\mathcal{L}_2(X,Y)} |f(t)|_{\mathcal{L}_2(X,Y)} |s-t|^{2H-2} dsdt < \infty$,

where $\alpha_H = H(2H-1)$. The stochastic integral of f with respect to B^H is defined by

$$\int_0^t f(s)\mathrm{d}B^H(s) = \sum_{n=1}^{\infty} \int_0^t f(s)e_n \mathrm{d}\beta_n^H(s) = \sum_{n=1}^{\infty} (K_H^* f e_n)(s)\mathrm{d}\beta_n(s), \tag{2.4}$$

where β_n is the standard Brownian motion used to represent β_n^H as in (2.1) and the series (2.4) is finite if

$$\sum_n \|K_H^*(fe_n)\|^2_{L^2([0,b];Y)} = \sum_n \| \|fe_n\|_{\mathcal{H}} \|_Y^2 < \infty. \tag{2.5}$$

2.2. *Smoothing effect*

Let $1 \le q < \infty$ and denote $L^q(\mathbb{R}^d)$ the space of functions with the norm $\|f\|_q = (\int_{\mathbb{R}^d} |f(x)|^q \mathrm{d}x)^{1/q}$. For $t > 0$, $e^{-t(-\Delta)^{\alpha/2}}$ denotes the linear fractional heat semigroup defined by (see [8])

$$(e^{-t(-\Delta)^{\alpha/2}} f)(x) = \int_{\mathbb{R}^d} G_\alpha(t, x-y) f(y)\mathrm{d}y, \tag{2.6}$$

where G_α is the fractional heat kernel

$$G_\alpha(t,x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi - t|\xi|^\alpha} \mathrm{d}\xi,\ t > 0,\ x \in \mathbb{R}^d, \tag{2.7}$$

and $f \in L^q(\mathbb{R}^d)$, $q \in [1,\infty)$ or $f \in C_0(\mathbb{R}^d)$.

We recall the space-time estimate for the fractional semigroup on Lebesgue spaces,

$$\|e^{-t(-\Delta)^{\alpha/2}} u\|_{s_2} \le C_1 t^{-\frac{d}{\alpha}(\frac{1}{s_1} - \frac{1}{s_2})} \|u\|_{s_1}, \tag{2.8}$$

for $1 \le s_1 \le s_2 \le \infty$, $t > 0$ and $u \in L^{s_1}(\mathbb{R}^d)$.

Given $\lambda > 0$, we define the dilation operator D_λ by $D_\lambda \varphi(x) = \varphi(\lambda x)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. This operator is extended by duality to $\mathcal{S}'(\mathbb{R}^d)$.

Lemma 1. *D_λ has the following properties:*

(i) $D_\lambda(e^{-\lambda^\alpha t(-\Delta)^{\alpha/2}} \varphi) = e^{-t(-\Delta)^{\alpha/2}}(D_\lambda \varphi)$ *for all* $\varphi \in \mathcal{S}'(\mathbb{R}^d)$,

(ii) $D_\lambda(D_{\frac{1}{\lambda}} \varphi) = \varphi$ *for all* $\varphi \in \mathcal{S}'(\mathbb{R}^d)$,

(iii) $\|D_\lambda \varphi\|_r = \lambda^{-\frac{d}{r}} \|\varphi\|_r$ *for all* $\varphi \in L^r(\mathbb{R}^d)$, $r \ge 1$,

(iv) $D_\lambda(\varphi\psi) = D_\lambda \varphi D_\lambda \psi$ *for all* $\varphi, \psi, \varphi\psi \in L^1_{loc}(\mathbb{R}^d)$,

(v) $D_\lambda(|\cdot|^{-\gamma}) = \lambda^{-\gamma} |\cdot|^{-\gamma}$ *for all* $\gamma > 0$.

Proof. It is obvious that (ii)-(v) hold. We only prove (i). By (2.6) and (2.7),

$$e^{-t(-\Delta)^{\alpha/2}}(D_\lambda \varphi(x)) = \int_{\mathbb{R}^d} G_\alpha(t, x-y)(D_\lambda \varphi(y))\mathrm{d}y$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi - t|\xi|^\alpha} \varphi(\lambda y) d\xi dy. \tag{2.9}$$

On the other hand,

$$\begin{aligned} e^{-\lambda^\alpha t(-\Delta)^{\alpha/2}}\varphi(x) &= \int_{\mathbb{R}^d} G_\alpha(\lambda^\alpha t, x-y)\varphi(y)dy \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi - \lambda^\alpha t|\xi|^\alpha} \varphi(y) d\xi dy, \end{aligned} \tag{2.10}$$

then

$$\begin{aligned} D_\lambda(e^{-\lambda^\alpha t(-\Delta)^{\alpha/2}}\varphi) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(\lambda x-y)\xi - \lambda^\alpha t|\xi|^\alpha} \varphi(y) d\xi dy \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\lambda(x-z)\xi - t|\lambda\xi|^\alpha} \varphi(\lambda z)\lambda^n d\xi dz \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-z)\eta - t|\eta|^\alpha} \varphi(\lambda z) d\eta dz. \end{aligned} \tag{2.11}$$

By (2.9) and (2.11), (i) holds. □

Lemma 2. *Let $d \geq 1$ be an integer and $0 < \gamma < d$. Let $q_1 \in (1, \infty]$ and $q_2 \in (1, \infty]$ such that*

$$0 \leq \frac{1}{q_2} < \frac{\gamma}{d} + \frac{1}{q_1} < 1.$$

Then there exists a constant $C > 0$ depending on $\alpha, d, \gamma, q_1, q_2$ such that

$$\|e^{-t(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C_2 t^{-\frac{d}{\alpha}(\frac{1}{q_1}-\frac{1}{q_2})-\frac{\gamma}{\alpha}} \|u\|_{q_1}. \tag{2.12}$$

Proof. Set $m = \frac{d}{\gamma}$ and let $\varepsilon, \delta > 0$ satisfy

$$\varepsilon < m, \ \frac{1}{q_2} \leq \frac{1}{m+\delta} + \frac{1}{q_1} \leq \frac{1}{m-\varepsilon} + \frac{1}{q_1} \leq 1.$$

In fact, from the following estimate (2.15), we can see that it is reasonable to choose such ε and δ. Consider the following decomposition

$$|\cdot|^{-\gamma} = \psi_1 + \psi_2, \ \psi_1 \in L^{m-\varepsilon}(\mathbb{R}^d), \ \psi_2 \in L^{m+\delta}(\mathbb{R}^d).$$

By the Hölder inequality, we have

$$\|\psi_1 u\|_{r_1} \leq \|\psi_1\|_{m-\varepsilon}\|u\|_{q_1}, \tag{2.13}$$

where

$$\frac{1}{r_1} = \frac{1}{m-\varepsilon} + \frac{1}{q_1}$$

and

$$\|\psi_2 u\|_{r_2} \leq \|\psi_2\|_{m+\delta}\|u\|_{q_1}, \tag{2.14}$$

where

$$\frac{1}{r_2} = \frac{1}{m+\delta} + \frac{1}{q_1}.$$

By the Minkowski inequality, (2.8), (2.13) and (2.14),

$$\begin{aligned}\|e^{-(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}u)\|_{q_2} &\le \|e^{-(-\Delta)^{\alpha/2}}(\psi_1 u)\|_{q_2} + \|e^{-(-\Delta)^{\alpha/2}}(\psi_2 u)\|_{q_2} \\ &\le C\|\psi_1 u\|_{r_1} + C\|\psi_2 u\|_{r_2} \\ &\le C(\|\psi_1\|_{m-\varepsilon} + \|\psi_2\|_{m+\delta})\|u\|_{q_1}. \end{aligned} \tag{2.15}$$

Thus

$$\|e^{-(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}u)\|_{q_2} \le C(\alpha,d,\gamma,q_1,q_2)\|u\|_{q_1}.$$

By (i) and (ii) in Lemma 1, $e^{-\lambda^\alpha t(-\Delta)^{\alpha/2}}\varphi = D_{\frac{1}{\lambda}} e^{-t(-\Delta)^{\alpha/2}} D_\lambda \varphi$, then

$$e^{-(-\Delta)^{\alpha/2}}\varphi = D_{t^{\frac{1}{\alpha}}} e^{-t(-\Delta)^{\alpha/2}} D_{t^{-\frac{1}{\alpha}}}\varphi \tag{2.16}$$

for all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$. From (2.12) and (2.16), it follows that

$$\|D_{t^{\frac{1}{\alpha}}} e^{-t(-\Delta)^{\alpha/2}} D_{t^{-\frac{1}{\alpha}}}(|\cdot|^{-\gamma}u)\|_{q_2} \le C(\alpha,d,\gamma,q_1,q_2)\|u\|_{q_1}.$$

This together with (iii) in Lemma 1 yield that

$$t^{-\frac{d}{\alpha q_2}}\|e^{-t(-\Delta)^{\alpha/2}} D_{t^{-\frac{1}{\alpha}}}(|\cdot|^{-\gamma}u)\|_{q_2} \le C(\alpha,d,\gamma,q_1,q_2)\|u\|_{q_1}.$$

By (iv) and (v) in Lemma 1,

$$t^{-\frac{d}{\alpha q_2}} t^{\frac{\gamma}{\alpha}}\|e^{-t(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma} D_{t^{-\frac{1}{\alpha}}}u)\|_{q_2} \le C(\alpha,d,\gamma,q_1,q_2)\|u\|_{q_1}.$$

Replacing u by $D_{t^{\frac{1}{\alpha}}}u$,

$$\begin{aligned} t^{-\frac{d}{\alpha q_2}} t^{\frac{\gamma}{\alpha}}\|e^{-t(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}u)\|_{q_2} &\le C(\alpha,d,\gamma,q_1,q_2)\|D_{t^{\frac{1}{\alpha}}}u\|_{q_1} \\ &\le C(\alpha,d,\gamma,q_1,q_2)t^{-\frac{d}{\alpha q_1}}\|u\|_{q_1}. \end{aligned}$$

Therefore

$$\|e^{-t(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}u)\|_{q_2} \le C(\alpha,d,\gamma,q_1,q_2)t^{-\frac{d}{\alpha}(\frac{1}{q_1}-\frac{1}{q_2})-\frac{\gamma}{\alpha}}\|u\|_{q_1}.$$

□

3. RESULTS

Consider the linear problem

$$\begin{cases} \partial_t z(t) = -(-\Delta)^{\alpha/2} z(t) + \partial_t B^H(t),\ t > 0, \\ z(0) = 0, \end{cases} \tag{3.1}$$

the mild solution is

$$z(t) = \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} \mathrm{d}B^H(s).$$

For $H > \frac{1}{2}$ and $H \geq \frac{1}{q}$, by the result in [6], $z \in C([0,T], L^q(\mathbb{R}^d))$. For $\beta > 0$ and $1 < r < \infty$ to be fixed later, define

$$L(T) = \max\Big(\sup_{t\in[0,T]} \|z(t)\|_q,\ \sup_{t\in(0,T]} t^{\beta} \|z(t)\|_r \Big). \tag{3.2}$$

Proof of Theorem 1. Let q satisfy (1.5) and $1 < q < \infty$. Then there exists $r > q$ such that

$$\frac{1}{qp} - \frac{\gamma}{dp} < \frac{1}{r} < \frac{d-\gamma}{dp}. \tag{3.3}$$

Let

$$\beta = \frac{d}{\alpha q} - \frac{d}{\alpha r}. \tag{3.4}$$

Since $q > \frac{d(p-1)}{\alpha-\gamma}$, then

$$\frac{1}{q} - \frac{\alpha}{dp} < \frac{1}{qp} - \frac{\gamma}{dp}.$$

This together with (3.3) and (3.4) yield that

$$\beta p < 1.$$

Let $\rho > 0$, $M > 0$, $T > 0$ and $\varphi \in L^q(\mathbb{R}^d)$ such that

$$\|\varphi\|_q \leq \rho, \tag{3.5}$$

$$\max(C_1, 1)\rho + L(T) + \frac{K}{2p} M^p T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} \leq M, \tag{3.6}$$

$$K M^{p-1} T^{1-\frac{d(p-1)}{\alpha q}-\frac{\gamma}{\alpha}} < 1, \tag{3.7}$$

where K is a positive constant to be fixed later.

Define

$$X = \{u \in C([0,T]; L^q(\mathbb{R}^d)) \cap C((0,T]; L^r(\mathbb{R}^d));$$
$$\max_{t\in[0,T]} \|u(t)\|_q \leq M,\ \max_{t\in(0,T]} t^{\beta} \|u(t)\|_r \leq M\}.$$

Endowed with the metric

$$d(u,v)=\max\Big(\sup_{t\in[0,T]}\|u(t)-v(t)\|_q,\ \sup_{t\in(0,T]}t^{\beta}\|u(t)-v(t)\|_r\Big),$$

(X,d) is a complete metric space. Given $u\in X$, set

$$\begin{aligned}F_{\varphi}(u)(t)&=e^{-t(-\Delta)^{\alpha/2}}\varphi+\int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}|u(s)|^{p-1}u(s))\mathrm{d}s\\&\quad+\int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\mathrm{d}B^H(s),\end{aligned}\tag{3.8}$$

Let $\varphi,\psi\in L^q(\mathbb{R}^d)$, $u,v\in X$. For $q_1=\frac{r}{p}$, $q_2=q$, by Lemma 2 and the Hölder inequality,

$$\begin{aligned}&\|F_{\varphi}(u)(t)-F_{\psi}(v)(t)\|_q\\&\le\|e^{-t(-\Delta)^{\alpha/2}}(\varphi-\psi)\|_q\\&\quad+\int_0^t\|e^{-(t-s)(-\Delta)^{\alpha/2}}[|\cdot|^{-\gamma}(|u(s)|^{p-1}u(s)-|v(s)|^{p-1}v(s))]\|_q\mathrm{d}s\\&\le\|\varphi-\psi\|_q+C_2\int_0^t(t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}}\||u(s)|^{p-1}u(s)-|v(s)|^{p-1}v(s)\|_{\frac{r}{p}}\mathrm{d}s\\&\le\|\varphi-\psi\|_q+C_2p\int_0^t(t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}}\|u-v\|_r(\|u\|_r^{p-1}+\|v\|_r^{p-1})\mathrm{d}s\\&\le\|\varphi-\psi\|_q+\left(2C_2pM^{p-1}\int_0^t(t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}}s^{-\beta p}\mathrm{d}s\right)d(u,v).\end{aligned}$$

By (3.4),

$$\begin{aligned}&\|F_{\varphi}(u)(t)-F_{\psi}(v)(t)\|_q\\&\le\|\varphi-\psi\|_q+\left(2C_2pM^{p-1}t^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}}\int_0^1(1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}}\tau^{-\beta p}\mathrm{d}\tau\right)d(u,v).\end{aligned}$$

Since $r>q>\frac{d(p-1)}{\alpha-\gamma}$, then

$$1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}>0,\ \ \frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})+\frac{\gamma}{\alpha}<\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})+\frac{\gamma}{\alpha}<1.$$

We note $\beta p<1$ and recall the Beta function

$$B(x,y)=\int_0^1\tau^{x-1}(1-\tau)^{y-1}\mathrm{d}\tau,\ x,y>0,$$

therefore

$$\|F_{\varphi}(u)(t)-F_{\psi}(v)(t)\|_q\le\|\varphi-\psi\|_q+C_3M^{p-1}T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}}d(u,v),\tag{3.9}$$

where

$$C_3 = 2C_2 p \int_0^1 (1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} \tau^{-\beta p} \mathrm{d}s \tag{3.10}$$

is a finite positive constant.

For $s_1 = q < s_2 = r$, by (2.8),

$$\|e^{-t(-\Delta)^{\alpha/2}}(\varphi-\psi)\|_r \leq C_1 t^{-\frac{d}{\alpha}(\frac{1}{q}-\frac{1}{r})}\|\varphi-\psi\|_q = C_1 t^{-\beta}\|\varphi-\psi\|_q. \tag{3.11}$$

For $q_1 = \frac{r}{q}$, $q_2 = r$, by (3.11), Lemma 2 and the Hölder inequality,

$$\begin{aligned}
&\|F_\varphi(u)(t) - F_\psi(v)(t)\|_r \\
&\leq \|e^{-t(-\Delta)^{\alpha/2}}(\varphi-\psi)\|_r \\
&\quad + \int_0^t \|e^{-(t-s)(-\Delta)^{\alpha/2}}[|\cdot|^{-\gamma}(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s))]\|_r \mathrm{d}s \\
&\leq C_1 t^{-\beta}\|\varphi-\psi\|_q + C_2 \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \||u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)\|_{\frac{r}{p}} \mathrm{d}s \\
&\leq C_1 t^{-\beta}\|\varphi-\psi\|_q + C_2 p \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \|u-v\|_r (\|u\|_r^{p-1} + \|v\|_r^{p-1}) \mathrm{d}s \\
&\leq C_1 t^{-\beta}\|\varphi-\psi\|_q + \left(2C_2 p M^{p-1} \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} s^{-\beta p} \mathrm{d}s\right) d(u,v).
\end{aligned}$$

Then

$$\begin{aligned}
&t^\beta |F_\varphi(u)(t) - F_\psi(v)(t)\|_r \\
&\leq C_1\|\varphi-\psi\|_q + \left(2C_2 p M^{p-1} t^\beta \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} s^{-\beta p} \mathrm{d}s\right) d(u,v) \\
&= C_1\|\varphi-\psi\|_q + \left(2C_2 p M^{p-1} t^{\beta+1-\frac{d(p-1)}{\alpha r}-\frac{\gamma}{\alpha}-\beta p} \int_0^1 (1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \tau^{-\beta p} \mathrm{d}\tau\right) d(u,v) \\
&= C_1\|\varphi-\psi\|_q + \left(2C_2 p M^{p-1} t^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} \int_0^1 (1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \tau^{-\beta p} \mathrm{d}\tau\right) d(u,v) \\
&= C_1\|\varphi-\psi\|_q + C_4 M^{p-1} T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} d(u,v),
\end{aligned} \tag{3.12}$$

where

$$C_4 = 2C_2 p \int_0^1 (1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \tau^{-\beta p} \mathrm{d}\tau \tag{3.13}$$

is a finite positive constant. By (3.9) and (3.17),

$$d(F_\varphi(u), F_\psi(v)) \leq \max(C_1, 1)\|\varphi-\psi\|_q + K M^{p-1} T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} d(u,v), \tag{3.14}$$

where $K = \max(C_3, C_4)$.

By (3.14), we see that for $u \in X$, $F_\varphi(u) \in C([0,T]; L^q(\mathbb{R}^d)) \cap C((0,T]; L^r(\mathbb{R}^d))$.

Given $u \in X$, by (3.8), (3.2), we have

$$\begin{aligned}\|F_\varphi(u)(t)\|_q &\le \|e^{-t(-\Delta)^{\alpha/2}}\varphi\|_q + \int_0^t \|e^{-(t-s)(-\Delta)^{\alpha/2}}[|\cdot|^{-\gamma}(|u(s)|^{p-1}u(s)]\|_q \mathrm{d}s + \|z(t)\|_q \\ &\le \|\varphi\|_q + C_2\int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} \||u(s)|^{p-1}u(s)\|_{\frac{r}{p}} \mathrm{d}s + L(T) \\ &\le \rho + L(T) + C_2 M^p \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} s^{-\beta p}\mathrm{d}s \\ &= \rho + L(T) + \frac{C_3}{2p} M^p T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}}. \end{aligned} \tag{3.15}$$

Similarly, by (3.11) and Lemma 2, we have

$$\begin{aligned}\|F_\varphi(u)(t)\|_r &\le \|e^{-t(-\Delta)^{\alpha/2}}\varphi\|_r + \int_0^t \|e^{-(t-s)(-\Delta)^{\alpha/2}}[|\cdot|^{-\gamma}(|u(s)|^{p-1}u(s)]\|_r \mathrm{d}s + \|z(t)\|_r \\ &\le C_1\rho t^{-\beta} + C_2 M^p \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} s^{-\beta p}\mathrm{d}s + \|z(t)\|_r. \end{aligned} \tag{3.16}$$

Then

$$\begin{aligned} t^\beta |F_\varphi(u)(t)\|_r &\le C_1\rho + \left(C_2 M^p t^\beta \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} s^{-\beta p}\mathrm{d}s\right) + t^\beta\|z(t)\|_r \\ &= C_1\rho + L(T) + \frac{C_4}{2p} M^p T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}}, \end{aligned} \tag{3.17}$$

where C_4 is the same as in (3.13).

By (3.16), (3.17) and (3.6), we get $F_\varphi(u) \in X$. That is F_φ maps X into X.

Letting $\varphi = \psi$ in (3.14), we have

$$d(F_\varphi(u), F_\varphi(v)) \le K M^{p-1} T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} d(u,v). \tag{3.18}$$

By (3.7), we conclude that F_φ is a contraction mapping from X into X. Therefore F_φ has a unique fixed point in X, which is the mild solution of (1.1). □

REFERENCES

[1] H. M. Ahmed and M. A. Ragusa, "Nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential." *Bull. of the Malaysian Mathematical Sciences Society*, vol. 45, pp. 3239–3253, 2022, doi: 10.1007/s40840-022-01377-y.

[2] B. Ben Slimene, S. Tayachi, and F. B. Weissler, "Well-posedness, global existence and large time behavior for Hardy–Hénon parabolic equations," *Nonlinear Anal.*, vol. 152, pp. 116–148, 2017, doi: 10.1016/j.na.2016.12.008.

[3] T. Caraballo, M. J. Garrido-Atienza, and T. Taniguchi, "The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion." *Nonlinear Anal.*, vol. 64, pp. 3671–3684, 2011, doi: 10.1016/j.na.2011.02.047.

[4] A. Chadha and S. N. Bora, "Stability results on mild solution of impulsive neutral fractional stochastic integro-differential equations involving Poisson jumps." *Filomat*, vol. 35, pp. 3383–3406, 2021, doi: 10.2298/FIL2110383C.

[5] J. Clarke and C. Olivera, "Local L^p-solution for semilinear heat equation with fractional noise." *Ann. Acad. Sci. Fenn. Math.*, vol. 45, pp. 305–312, 2020, doi: 10.5186/aasfm.2020.4505.

[6] P. Coupek, B. Maslowski, and M. Ondrejat, "L^p-valued stochastic convolution integral driven by Volterra noise." *Stoch. Dyn*, vol. 18, no. 06, 2018, 1850048, doi: 10.1142/S021949371850048X.

[7] I. Khan, H. Ullah, H. AlSalman, M. Fiza, S. Islam, M. Shoaib, M. A. Z. Raja, A. Gumaei, and F. Ikhlaq, "Fractional analysis of MHD boundary layer flow over a stretching sheet in porous medium: a new stochastic method." *Journal of Function Spaces*, vol. 2021, 2021, 5844741, doi: 10.1155/2021/5844741.

[8] B. Lai, C. Miao, and X. Zheng, "Forward self-similar solutions of the fractional Navier-Stokes equations." *Adv. Math.*, vol. 352, pp. 981–1043, 2019, doi: 10.1016/j.aim.2019.06.021.

[9] M. Majdoub and E. Mliki, "Well-posedness for Hardy-Hénon parabolic equations with fractional Brownian noise." *Anal. Math. Phys.*, vol. 11, 2021, 20, doi: 10.1007/s13324-020-00442-8.

[10] B. Maslowski and D. Nualart, "Evolution equations driven by a fractional Brownian motion," *J. Funct. Anal.*, vol. 202, pp. 277–305, 2003, doi: 10.1016/S0022-1236(02)00065-4.

[11] S. Tindel, C. A. Tudor, and F. Viens, "Stochastic evolution equations with fractional Brownian motion," *Probab. Theory Relat. Fields*, vol. 127, pp. 186–204, 2003, doi: 10.1007/s00440-003-0282-2.

[12] F. B. Weissler, "Semilinear evolution equations in Banach spaces." *J. Funct. Anal.*, vol. 32, pp. 277–296, 1979, doi: 10.1016/0022-1236(79)90040-5.

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