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NONLOCAL CAUCHY PROBLEMS FOR FRACTIONAL EVOLUTION EQUATIONS INVOLVING VOLTERRA-FREDHOLM TYPE INTEGRAL OPERATORS

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Abstract. In this paper, nonlocal Cauchy problems for fractional evolution equations involving Volterra-Fredholm type integral operators are investigated. Some new existence theorems of mild solutions are presented by using fractional calculus, Hölder inequality, Beta function and fixed point theorems.

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Keywords: fractional evolution equations, nonlocal Cauchy problems, mild solutions, analytic compact semigroup, fixed point theorems

1. INTRODUCTION

The fractional differential equations has recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economy and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [8, 11, 12, 14, 22, 23]. There has been a significant development in fractional differential equations. For more details on fractional calculus theory, one can see the monographs of Diethelm [9], Kilbas et al. [17], Lakshmikantham et al. [19], Michalski [24], Miller and Ross [25], Podlubny [29] and Tarasov [30]. In the last years, the theory of fractional differential equations attracted the attention of many authors (see for instance [1–7, 10, 13, 15, 16, 21, 26, 27, 32, 34, 35] and references therein).

However, to our knowledge, nonlocal Cauchy problems for fractional evolution equations involving Volterra-Fredholm type integral operators has not been discussed extensively. Motivated by the above mentioned works (including our papers [6,

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20, 21, 26, 31–33, 35]), the main purpose of this paper is to consider the following nonlocal Cauchy problems for fractional evolution equations involving Volterra–Fredholm type integral operators such as

$$\begin{cases} {}^cD_t^q x(t) = -Ax(t) + t^n f(t, x(t), (Kx)(t), (Hx)(t)), \\ \qquad\qquad t \in J = [0,T],\ n \in Z^+,\ q \in (0,1), \\ x(0) = g(x) + x_0, \end{cases} \tag{1.1}$$

where the fractional derivative ${}^cD_t^q$ is understood here in the Caputo sense, $-A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{S(t), t \geq 0\}$, the Volterra type integral operator K and Fredholm type integral operator H are defined by

$$(Kx)(t) = \int_0^t k(t,s,x(s))ds, \quad (Hx)(t) = \int_0^T h(t,s,x(s))ds.$$

The function $f : J \times X_\alpha \times X_\alpha \times X_\alpha \to X($ or X_α, or $X_\mu)$ is continuous where $X_\alpha = D(A^\alpha)$, $0 \leq \mu \leq \alpha \leq 1$, is a Banach space with the norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in X_\alpha$. f, k, h and g are specified latter. It is easy to see that term t^n appears before the nonlinear term f. We remark that this term t^n will help us to overcome the essential difficult caused by the singular term $(t-s)^{q-1}$ in the formula of the solutions due to the well known Beta function.

In the present paper, we discuss the existence and uniqueness of mild solutions for system (1.1). Our results cover the cases for the nonlinear term f taking values in the spaces such as X, X_α, X_μ, where $0 \leq \mu \leq \alpha \leq 1$, the nonlocal term g is linear completely continuous or satisfies the Lipschitz continuous condition. The main techniques used here are Hölder's inequality, Beta function via Banach contraction principle, Schauder's fixed point theorem for compact maps and Sadovskii's fixed point theorem for condensing maps.

The rest of this paper is organized as follows. In Section 2, we give some known preliminary results on the fraction powers of the generator of an analytic compact semigroup and introduce the mild solution of system (1.1). In Section 3, we study the existence of mild solutions for system (1.1) by using fractional calculus, Hölder inequality via Banach contraction principle, Schauder's fixed point theorem and Sadovskii's fixed point theorem. At last, an example is given to demonstrate the applicability of our result.

2. Preliminaries

In this section, we introduce some facts about the fractional powers of the generator of a compact analytic semigroup, the Riemann-Liouville fractional integral operator that are used throughout this paper.

We denote by X a Banach space with norm $\|\cdot\|$ and $-A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{S(t), t \geq 0\}$. This means that there exists $M > 1$ such that $\|S(t)\| \leq M$.

We assume without loss of generality that $0 \in \rho(A)$. This allows us to define the fractional power A^α for $0 \le \alpha \le 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$ (see [28]).

In the sequel, we will also use $\|l\|_{L^p(J,R^+)}$ to denote the $L^p(J,R^+)$ norm of l whenever $l \in L^p(J,R^+)$ for some p with $1 \le p < \infty$. We will set $\alpha \in [0,1]$ and denote by C_α, the Banach space $C(J,X_\alpha)$ endowed with supnorm given by $\|x\|_\infty = \sup_{t\in J}\|x\|_\alpha$, for $x \in C_\alpha$.

Let us recall the following known definitions. For more details, see [17].

Definition 1. The fractional integral of order γ with lower limit zero for a function l is defined as

$$I^\gamma l(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{l(s)}{(t-s)^{1-\gamma}}ds,\ t>0,\ \gamma>0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2. The Riemann-Liouville derivative of order γ with lower limit zero for a function $l : [0,\infty) \to R$ can be written as

$${}^{L}D_t^\gamma l(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{l(s)}{(t-s)^{\gamma+1-n}}ds,\ t>0,\ n-1<\gamma<n.$$

Definition 3. The Caputo derivative of order γ for a function $l : [0,\infty) \to R$ can be written as

$${}^{c}D_t^\gamma l(t) = {}^{L}D^\gamma\left(l(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}l^{(k)}(0)\right),\ t>0,\ n-1<\gamma<n.$$

Remark 1. (i) If $l \in C^1[0,\infty)$, then

$${}^{c}D_t^\gamma l(t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{l'(s)}{(t-s)^\gamma}ds = I^{1-\gamma}l'(t),\ t>0,\ 0<\gamma<1.$$

(ii) The Caputo derivative of a constant is equal to zero.
(iii) If l is an abstract function with values in X, then integrals which appear in Definitions 1 and 2 are taken in Bochner's sense.

Motivated by Definition 3.1 of [35], we adopt the following concept of mild solution for our problem.

Definition 4. By the mild solution of system (1.1), we mean that the function $x : J \to X_\alpha$ which satisfies

$$x(t) = \mathscr{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{q-1}s^n\mathscr{S}(t-s)f(s,x(s),(Kx)(s),(Hx)(s))\,ds$$

for any $t \in J$, where

$$\mathscr{T}(t) = \int_0^\infty \xi_q(\theta) S(t^q\theta) d\theta, \quad \mathscr{S}(t) = q\int_0^\infty \theta\xi_q(\theta) S(t^q\theta) d\theta,$$

and

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}} \varpi_q(\theta^{-\frac{1}{q}}) \geq 0,$$

where

$$\varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty).$$

ξ_q is a probability density function defined on $(0,\infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0,\infty) \quad and \quad \int_0^\infty \xi_q(\theta) d\theta = 1.$$

Remark 2. It is not difficult to verify that for $v \in [0,1]$,

$$\int_0^\infty \theta^v \xi_q(\theta) d\theta = \int_0^\infty \theta^{-qv}\varpi_q(\theta) d\theta = \frac{\Gamma(1+v)}{\Gamma(1+qv)}.$$

The following results are very useful and will be used throughout this paper.

Lemma 1 (Lemma 2.9, [32])**.** *The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties:*

(1) *For fixed $t \geq 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are linear and bounded operators, that is, for any $x \in X$,*

$$\|\mathscr{T}(t)x\| \leq M\|x\|, \quad \|\mathscr{S}(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|.$$

(2) *$\{\mathscr{T}(t), t \geq 0\}$ and $\{\mathscr{S}(t), t \geq 0\}$ are strongly continuous.*

(3) *For every $t > 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are also compact operators.*

(4) *For any $x \in X$, $\beta \in [0,1]$ and $\alpha \in [0,1]$, we have*

$$A\mathscr{S}(t)x = A^{1-\beta}\mathscr{S}(t)A^\beta x, \quad t \in J,$$

$$\|A^\alpha \mathscr{S}(t)\| \leq \frac{M_\alpha q\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}t^{-\alpha q}, \quad 0 < t \leq T.$$

(5) *For fixed $t \geq 0$ and any $x \in X_\alpha$, we have*

$$\|\mathscr{T}(t)x\|_\alpha \leq M\|x\|_\alpha, \quad \|\mathscr{S}(t)x\|_\alpha \leq \frac{M}{\Gamma(q)}\|x\|_\alpha.$$

(6) *For a positive number μ with $0 \leq \mu \leq \alpha \leq 1$, fixed $t \geq 0$ and any $x \in X_\mu$, we have*

$$\|\mathscr{T}(t)x\|_\alpha \leq \|A^{\alpha-\mu}\|M\|x\|_\mu, \quad \|\mathscr{S}(t)x\|_\alpha \leq \|A^{\alpha-\mu}\|\frac{M}{\Gamma(q)}\|x\|_\mu.$$

(7) *$\mathscr{T}_\alpha(t)$ and $\mathscr{S}_\alpha(t)$, $t>0$ is uniformly continuous, that is for each fixed $t>0$, and $\epsilon>0$, there exists $h>0$ such that*

$$\|\mathscr{T}_\alpha(t+\epsilon)-\mathscr{T}_\alpha(t)\|_\alpha<\varepsilon, \quad for \quad t+\epsilon\geq 0 \quad and \quad |\epsilon|<h,$$

$$\|\mathscr{S}_\alpha(t+\epsilon)-\mathscr{S}_\alpha(t)\|_\alpha<\varepsilon, \quad for \quad t+\epsilon\geq 0 \quad and \quad |\epsilon|<h.$$

where

$$\mathscr{T}_\alpha(t)=\int_0^\infty \xi_q(\theta)S_\alpha(t^q\theta)d\theta, \quad \mathscr{S}_\alpha(t)=q\int_0^\infty \theta\xi_q(\theta)S_\alpha(t^q\theta)d\theta.$$

3. Existence of mild solutions

In this section, we give theorems for the existence and uniqueness of the mild solutions of system (1.1).

We first make the following assumptions.

[Hf1]: $f: J\times X_\alpha\times X_\alpha\times X_\alpha\to X$ is continuous and there exist $m_1,m_2,m_3>0$ such that

$$\|f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)\|\leq m_1\|x_1-y_1\|_\alpha+m_2\|x_2-y_2\|_\alpha+m_3\|x_3-y_3\|_\alpha$$

for all $x_i,y_i\in X_\alpha$, $i=1,2,3$ and $t\in J$.

[Hk1]: Let $D_k=\{(t,s)\in R^2; 0\leq s\leq t\leq T\}$. The function $k: D_k\times X_\alpha\to X_\alpha$ is continuous and there exists a $m_k(t,s)\in C(D_k,R^+)$ such that

$$\|k(t,s,x)-k(t,s,y)\|_\alpha\leq m_k(t,s)\|x-y\|_\alpha$$

for each $(t,s)\in D_k$ and $x,y\in X_\alpha$. We set

$$K^*=\max_{t\in J}\int_0^t m_k(t,s)ds.$$

[Hh1]: Let $D_h=\{(t,s)\in R^2; 0\leq s,t\leq T\}$. The function $h: D_h\times X_\alpha\to X_\alpha$ is continuous and there exists a $m_h(t,s)\in C(D_h,R^+)$ such that

$$\|h(t,s,x)-h(t,s,y)\|_\alpha\leq m_h(t,s)\|x-y\|_\alpha$$

for each $(t,s)\in D_h$ and $x,y\in X_\alpha$. We set

$$H^*=\max_{t\in J}\int_0^T m_h(t,s)ds.$$

[Hg1]: $g: C_\alpha\to X_\alpha$ and there exists a constant $l_g>0$ such that

$$\|g(x)-g(y)\|_\alpha\leq l_g\|x-y\|_\infty, \text{ for arbitrary } x,y\in C_\alpha.$$

[HΩ]: A constant $\Omega_{n,\alpha,q,T}$ defined by

$$\Omega_{n,\alpha,q,T}=Ml_g+\frac{M_\alpha q\Gamma(2-\alpha)B(q,n+1)}{\Gamma(1+q(1-\alpha))}T^{n+(1-\alpha)q}(m_1+K^*m_2+H^*m_3)$$

satisfies $\Omega_{n,\alpha,q,T}<1$, where $B(\cdot,\cdot)$ denotes Beta function.

Now we are ready to give our first result which is based on the Banach contraction mapping principle.

Theorem 1. *Assume that [Hf1], [Hk1], [Hh1], [Hg1] and [HΩ] are satisfied. If* $x_0 \in X_\alpha$ *then system (1.1) has a unique mild solution* $x \in C_\alpha$.

Proof. Define the function $\Gamma : C_\alpha \to C_\alpha$ by

$$(\Gamma x)(t) = \mathscr{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{q-1} s^n \mathscr{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s))\, ds, t \in J. \tag{3.1}$$

Note that Γ is well defined on C_α. Now, take $t \in J$ and $x, y \in C_\alpha$. Then we have

$$\begin{aligned}
\|(\Gamma x)(t) - (\Gamma y)(t)\|_\alpha &\le \|\mathscr{T}(t)(g(x) - g(y))\|_\alpha \\
&\quad + \int_0^t (t-s)^{q-1} s^n \big\| \mathscr{S}(t-s)[f(s, x(s), (Kx)(t), (Hx)(t)) \\
&\qquad\qquad - f(s, y(s), (Ky)(s), (Hy)(s))] \big\|_\alpha ds \\
&\le M \|g(x) - g(y)\|_\alpha \\
&\quad + \int_0^t (t-s)^{q-1} s^n \|A^\alpha \mathscr{S}(t-s)\| \|f(s, x(s), (Kx)(s), (Hx)(s)) \\
&\qquad\qquad - f(s, y(s), (Ky)(s), (Hy)(s))\| ds,
\end{aligned}$$

which according to [Hf1], [Hk1], [Hh1], [Hg1], (4)-(5) of Lemma 1 and Hölder inequality,gives

$$\begin{aligned}
\|(\Gamma x)(t) - (\Gamma y)(t)\|_\alpha &\le M l_g \|x - y\|_\infty \\
&\quad + M_\alpha q t^{-\alpha q} \frac{\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} m_1 \int_0^t (t-s)^{q-1} s^n \|x(s) - y(s)\|_\alpha ds \\
&\quad + M_\alpha q t^{-\alpha q} \frac{\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} m_2 \int_0^t (t-s)^{q-1} s^n \|(Kx)(s) - (Ky)(s)\|_\alpha ds \\
&\quad + M_\alpha q t^{-\alpha q} \frac{\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} m_3 \int_0^t (t-s)^{q-1} s^n \|(Hx)(s) - (Hy)(s)\|_\alpha ds \\
&\le M l_g \|x - y\|_\infty + M_\alpha q t^{-\alpha q} \frac{\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} (m_1 + K^* m_2 + H^* m_3) \\
&\quad \times \|x - y\|_\infty \int_0^t (t-s)^{q-1} s^n ds \\
&\le \left\{ M l_g + \frac{M_\alpha q \Gamma(2-\alpha) B(q, n+1)}{\Gamma(1+q(1-\alpha))} t^{n+(1-\alpha)q} (m_1 + K^* m_2 + H^* m_3) \right\} \\
&\quad \times \|x - y\|_\infty,
\end{aligned}$$

due to

$$\int_0^t (t-s)^{q-1} s^n ds = B(q,n+1)t^{n+q},$$

$\|(Kx)(s)-(Ky)(s)\|_\alpha \le K^*\|x-y\|_\infty$ and $\|(Hx)(s)-(Hy)(s)\|_\alpha \le H^*\|x-y\|_\infty$. Therefore, we can deduce that

$$\begin{aligned}
&\|\Gamma x - \Gamma y\|_\infty \\
&\le \left\{ M l_g + \frac{M_\alpha q \Gamma(2-\alpha) B(q,n+1)}{\Gamma(1+q(1-\alpha))} t^{n+(1-\alpha)q} (m_1 + K^* m_2 + H^* m_3) \right\} \\
&\quad \times \|x-y\|_\infty \le \Omega_{n,\alpha,q,T} \|x-y\|_\infty .
\end{aligned}$$

Hence, [HΩ] allows us to conclude in view of the contraction mapping principle, that Γ has a unique fixed point $x \in C_\alpha$, and

$$x(t) = \mathscr{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{q-1} s^n \mathscr{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s))\, ds$$

which is the unique mild solution of system (1.1). □

Our second result uses Schauder's fixed point theorem.

We assume the following conditions.

[Hf2]: $f : J \times X_\alpha \times X_\alpha \times X_\alpha \to X_\alpha$ is continuous and there exists a positive function $\rho \in L^p(J, R^+)$ for some $p \in (\frac{1}{q}, \infty)$ such that

$$\|f(t,x,y,z)\|_\alpha \le \rho(t)$$

for all $x, y, z \in X_\alpha$ and $t \in J$.

[Hk2]: The function $k : D_k \times X_\alpha \to X_\alpha$ is continuous and there exists $L_1 > 0$ such that

$$\|k(t,s,x) - k(t,s,y)\|_\alpha \le L_1 \|x-y\|_\alpha$$

for each $(t,s) \in D_k$ and $x, y \in X_\alpha$.

[Hh2]: The function $h : D_h \times X_\alpha \to X_\alpha$ is continuous and there exists $L_2 > 0$ such that

$$\|h(t,s,x) - h(t,s,y)\|_\alpha \le L_2 \|x-y\|_\alpha$$

for each $(t,s) \in D_h$ and $x, y \in X_\alpha$.

[Hg2]: $g : C_\alpha \to X_\alpha$ is compact continuous and there exist $\beta_1 \ge 0$, $\beta_2 \ge 0$ such that

$$\|g(x)\|_\alpha \le \beta_1 \|x\|_\infty + \beta_2 .$$

Now we are ready to state and prove the following existence result.

Theorem 2. *Assume that the conditions [Hf2], [Hk2], [Hh2], [Hg2] are satisfied. If $x_0 \in X_\alpha$ then system (1.1) has at least one mild solution on J provided that*

$$M\beta_1 < 1.$$

Proof. Define the function $F : C_\alpha \to C_\alpha$ by

$$(Fx)(t) = \mathscr{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{q-1} s^n \mathscr{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s))\, ds,$$

and for $n \in Z^+$, we choose r such that

$$r \geq \frac{1}{1 - M\beta_1}\Bigg[M(\|x_0\|_\alpha + \beta_2) + \frac{M}{\Gamma(q)} B\left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1}\right)^{\frac{p-1}{p}} T^{\frac{pq+np-1}{p}} \|\rho\|_{L^p(J,R^+)}\Bigg].$$

Let $B_r = \{x \in C_\alpha \mid \|x\|_\infty \leq r\}$. Then we proceed in three steps.

Step 1. We show that $FB_r \subset B_r$. Let $x \in B_r$. Then for $t \in J$, using (5) of Lemma 1 and Hölder inequality, we have

$$\begin{aligned}\|(Fx)(t)\|_\alpha &\leq \|\mathscr{T}(t)(x_0 + g(x))\|_\alpha \\ &\quad + \int_0^t (t-s)^{q-1} s^n \left\|\mathscr{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s))\right\|_\alpha ds \\ &\leq M(\|x_0\|_\alpha + \|g(x)\|_\alpha) \\ &\quad + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n \|f(s, x(s), (Kx)(s), (Hx)(s))\|_\alpha ds,\end{aligned}$$

which according to [Hf2], [Hg2] and $pq > 1$ ($\Leftrightarrow \frac{(q-1)p}{p-1} > -1$), gives

$$\begin{aligned}&\|(Fx)(t)\|_\alpha \\ &\leq M(\|x_0\|_\alpha + \beta_1\|x\|_\infty + \beta_2) + \frac{M}{\Gamma(q)}\left(\int_0^t (t-s)^{q-1} s^n \rho(s) ds\right) \\ &\leq M(\|x_0\|_\alpha + \beta_1\|x\|_\infty + \beta_2) + \frac{M}{\Gamma(q)}\left(\int_0^t (t-s)^{\frac{(q-1)p}{p-1}} s^{\frac{np}{p-1}} ds\right)^{\frac{p-1}{p}} \\ &\quad \times \left(\int_0^t \rho(s)^p ds\right)^{\frac{1}{p}} \\ &\leq M(\|x_0\|_\alpha + \beta_1\|x\|_\infty + \beta_2) + \frac{M}{\Gamma(q)} B\left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1}\right)^{\frac{p-1}{p}} \\ &\quad \times t^{\frac{pq+np-1}{p}} \|\rho\|_{L^p(J,R^+)} \leq r, \text{ for } t \in J.\end{aligned}$$

Hence, we deduce $\|Fx\|_\infty \leq r$.

Step 2. We prove that F is continuous. Let $\{x_m\}$ be a sequence of B_r such that $x_m \to x$ in B_r. It comes from the continuity of k, h and assumptions [Hk2], [Hh2] that

$$\int_0^s k(s,\tau,x_m(\tau))d\tau \to \int_0^s k(s,\tau,x(\tau))d\tau,$$
$$\int_0^T h(s,\tau,x_m(\tau))d\tau \to \int_0^T h(s,\tau,x(\tau))d\tau$$

uniformly in $s \in J$ on C_α. Then,

$$f(s,x_m(s),(Kx_m)(s),(Hx_m)(s)) \to f(s,x(s),(Kx)(s),(Hx)(s)) \tag{3.2}$$

as $m \to \infty$, because the function f is continuous on $J \times X_\alpha \times X_\alpha$.

Further, one has

$$g(x_m) \to g(x) \text{ as } m \to \infty \tag{3.3}$$

because g is continuous on C_α.

Now for $t \in J$, according to [Hf2], [Hg2], (5) of Lemma 1 and Hölder inequality, we have

$$\begin{aligned}
\|(Fx_m)(t)-(Fx)(t)\|_\alpha &\le \|\mathscr{T}(t)(g(x_m)-g(x))\|_\alpha \\
&\quad + \int_0^t (t-s)^{q-1}s^n \big\|\mathscr{S}(t-s)[f(s,x_m(s),(Kx_m)(s),(Hx_m)(s)) \\
&\qquad\qquad - f(s,x(s),(Kx)(s),(Hx)(s))]\big\|_\alpha ds \\
&\le M\|g(x_m)-g(x)\|_\alpha \\
&\quad + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^n \|f(s,x_m(s),(Kx_m)(s),(Hx_m)(s)) \\
&\qquad\qquad - f(s,x(s),(Kx)(s),(Hx)(s))\|_\alpha ds \\
&\le M\|g(x_m)-g(x)\|_\alpha + \frac{M}{\Gamma(q)}\left(\int_0^t (t-s)^{\frac{(q-1)p}{p-1}} s^{\frac{np}{p-1}} ds\right)^{\frac{p-1}{p}} \\
&\quad \times \Bigg(\int_0^t \|f(s,x_m(s),(Kx_m)(s),(Hx_m)(s)) \\
&\qquad\qquad - f(s,x(s),(Kx)(s),(Hx)(s))\|^p ds\Bigg)^{\frac{1}{p}} \\
&\le M\|g(x_m)-g(x)\|_\alpha + \frac{M}{\Gamma(q)} B\left(\frac{pq-1}{p-1},\frac{(n+1)p-1}{p-1}\right)^{\frac{p-1}{p}} t^{\frac{pq+np-1}{p}} \\
&\quad \times \Bigg(\int_0^t \|f(s,x_m(s),(Kx_m)(s),(Hx_m)(s))
\end{aligned}$$

$$
\left. - f(s,x(s),(Kx)(s),(Hx)(s))\|^p ds \right)^{\frac{1}{p}}
$$

$$
\begin{aligned}
&\le M\|g(x_m)-g(x)\|_\alpha + \frac{M}{\Gamma(q)} B\left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1}\right)^{\frac{p-1}{p}} T^{\frac{pq+np-1}{p}} \\
&\quad \times \left(\int_0^T \|f(s,x_m(s),(Kx_m)(s),(Hx_m)(s)) \right. \\
&\qquad \left. - f(s,x(s),(Kx)(s),(Hx)(s))\|^p ds \right)^{\frac{1}{p}}.
\end{aligned}
$$

Therefore, using (3.2), (3.3), [Hf2] and the Lebesgue Dominated Convergence Theorem, it can easily been shown that

$$
\lim_{m\to\infty} \|Fx_m - Fx\|_\infty = 0, \text{ as } m \to \infty.
$$

That is, F is continuous.

Step 3. We show that F is compact. To this end, we use the famous Ascoli-Arzela's theorem. We first prove that $\{(Fx)(t) \mid x \in B_r\}$ is relatively compact in X_α, for all $t \in J$. Obviously, $\{(Fx)(0) \mid x \in B_r\}$ is compact. Let $t \in (0,T]$. For each $h \in (0,t)$, arbitrary $\delta > 0$ and $x \in B_r$, we defined the operator F_h by

$$
\begin{aligned}
(F_{h,\delta}x)(t) &= \mathscr{T}(t)[x_0 + g(x)] + S(h^q\delta)\int_0^{t-h} (t-s)^{q-1}s^n \\
&\quad \times \left(q\int_\delta^\infty \theta\xi_q(\theta)S((t-s)^q\theta - h^q\delta)d\theta \right) f(s,x(s),(Kx)(s),(Hx)(s))\,ds \\
&= \mathscr{T}(t)[x_0 + g(x)] \\
&\quad + q\int_0^{t-h}\int_\delta^\infty \theta(t-s)^{q-1}s^n\xi_q(\theta)S((t-s)^q\theta) \\
&\qquad \times f(s,x(s),(Kx)(s),(Hx)(s))\,d\theta ds.
\end{aligned}
$$

From above expression, we can see that the sets $\{(F_{h,\delta}x)(t) \mid x \in B_r\}$ are also relatively compact in X_α since the operator $S_\alpha(h^q\delta)$, $h^q\delta > 0$ are compact in X_α. Moreover, using [Hf2] and Hölder inequality, we have

$$
\begin{aligned}
&\|(Fx)(t) - (F_{h,\delta}x)(t)\|_\alpha \\
&\le q\left\| \int_0^t \int_0^\delta \theta(t-s)^{q-1}s^n\xi_q(\theta)S((t-s)^q\theta) \right. \\
&\qquad \left. \times f(s,x(s),(Kx)(s),(Hx)(s))\,d\theta ds \right\|_\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+q\Big\|\int_0^t\int_\delta^\infty \theta(t-s)^{q-1}s^n\xi_q(\theta)S((t-s)^q\theta)\\
&\qquad\qquad\qquad\times f\,(s,x(s),(Kx)(s),(Hx)(s))\,d\theta ds\\
&-\int_0^{t-h}\int_\delta^\infty \theta(t-s)^{q-1}s^n\xi_q(\theta)S((t-s)^q\theta)\\
&\qquad\qquad\qquad\times f\,(s,x(s),(Kx)(s),(Hx)(s))\,d\theta ds\Big\|_\alpha\\
\le{}& q\int_0^t\int_0^\delta \theta(t-s)^{q-1}s^n\xi_q(\theta)\big\|S((t-s)^q\theta)\\
&\qquad\qquad\qquad\times f\,(s,x(s),(Kx)(s),(Hx)(s))\,\big\|_\alpha d\theta ds\\
&+q\int_{t-h}^t\int_\delta^\infty \theta(t-s)^{q-1}s^n\xi_q(\theta)\big\|S((t-s)^q\theta)\\
&\qquad\qquad\qquad\times f\,(s,x(s),(Kx)(s),(Hx)(s))\,\big\|_\alpha d\theta ds\\
\le{}& qM\int_0^t\int_0^\delta \theta(t-s)^{q-1}s^n\xi_q(\theta)\rho(s)d\theta ds\\
&+qM\int_{t-h}^t\int_\delta^\infty \theta(t-s)^{q-1}s^n\xi_q(\theta)\rho(s)d\theta ds\\
\le{}& qM\left(\int_0^t(t-s)^{q-1}s^n\rho(s)ds\right)\int_0^\delta\theta\xi_q(\theta)d\theta\\
&+qM\left(\int_{t-h}^t(t-s)^{q-1}s^n\rho(s)ds\right)\int_0^\infty\theta\xi_q(\theta)d\theta.
\end{aligned}
$$

It comes from

$$
\int_0^\infty\theta\xi_q(\theta)d\theta=\frac{1}{\Gamma(1+q)},
$$

and

$$
\begin{aligned}
\int_0^t(t-s)^{q-1}s^n\rho(s)ds&\le\left(\int_0^t(t-s)^{\frac{(q-1)p}{p-1}}s^{\frac{np}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^t\rho(s)^p ds\right)^{\frac{1}{p}}\\
&\le B\left(\frac{pq-1}{p-1},\frac{(n+1)p-1}{p-1}\right)^{\frac{p-1}{p}}t^{\frac{pq+np-1}{p}}\|\rho\|_{L^p(J,R^+)},
\end{aligned}
$$

and for a fixed $\chi \in \left(1, \frac{p-1}{(1-q)p}\right)$,

$$\int_{t-h}^{t} (t-s)^{q-1} s^n \rho(s) ds \leq \left(\int_{t-h}^{t} (t-s)^{\frac{(q-1)p}{p-1}} s^{\frac{np}{p-1}} ds \right)^{\frac{p-1}{p}} \left(\int_{t-h}^{t} \rho(s)^p ds \right)^{\frac{1}{p}}$$

$$\leq \left(\int_{t-h}^{t} ds \right)^{\frac{\chi(p-1)}{(\chi-1)p}} \left(\int_{t-h}^{t} (t-s)^{\frac{\chi(q-1)p}{p-1}} s^{\frac{\chi np}{p-1}} ds \right)^{\frac{p-1}{\chi p}} \|\rho\|_{L^p(J,R^+)}$$

$$\leq h^{\frac{\chi(p-1)}{(\chi-1)p}} B\left(\frac{\chi(q-1)p}{p-1} + 1, \frac{\chi np}{p-1} + 1 \right)^{\frac{p-1}{\chi p}} t^{\frac{\chi p((q-1)p+n)}{p-1}+1} \|\rho\|_{L^p(J,R^+)},$$

that

$$\|(Fx)(t) - (F_{h,\delta}x)(t)\|_\alpha$$

$$\leq qMB\left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} t^{\frac{pq+np-1}{p}} \|\rho\|_{L^p(J,R^+)} \int_0^\delta \theta \xi_q(\theta) d\theta$$

$$+ \frac{M}{\Gamma(q)} B\left(\frac{\chi(q-1)p}{p-1} + 1, \frac{\chi np}{p-1} + 1 \right)^{\frac{p-1}{\chi p}} t^{\frac{\chi p((q-1)p+n)}{p-1}+1}$$

$$\times \|\rho\|_{L^p(J,R^+)} h^{\frac{\chi(p-1)}{(\chi-1)p}} \to 0$$

as $\delta \to 0^+$ and $h \to 0^+$. Therefore, $\{(Fx)(t) \mid x \in B_r\}$ is relatively compact in X_α for all $t \in (0, T]$ and since it is compact at $t = 0$ we have the relatively compactness in X_α for all $t \in J$.

Next, let us prove that $F(B_r)$ is equicontinuous. For $0 \leq t_2 < t_1 \leq T$, we have

$$\|(Fx)(t_1) - (Fx)(t_2)\|_\alpha \leq \|(\mathscr{T}(t_1) - \mathscr{T}(t_2))[x_0 - g(x)]\|_\alpha$$

$$+ \left\| \int_0^{t_2} (t_1 - s)^{q-1} s^n [\mathscr{S}(t_1 - s) - \mathscr{S}(t_2 - s)] \right.$$

$$\left. \times f(s, x(s), (Kx)(s), (Hx)(s)) ds \right\|_\alpha$$

$$+ \left\| \int_0^{t_2} [(t_1 - s)^{q-1} - t_2 - s)^{q-1}] s^n \mathscr{S}(t_2 - s) \right.$$

$$\left. \times f(s, x(s), (Kx)(s), (Hx)(s)) ds \right\|_\alpha$$

$$+ \left\| \int_{t_2}^{t_1} (t_1 - s)^{q-1} s^n \mathscr{S}(t_1 - s) f(s, x(s), (Kx)(s), (Hx)(s)) ds \right\|_\alpha.$$

Denote

$$
\begin{aligned}
I_1 &= \|(\mathscr{T}(t_1)-\mathscr{T}(t_2))[x_0-g(x)]\|_\alpha,\\
I_2 &= \left\| \int_0^{t_2} (t_1-s)^{q-1} s^n [\mathscr{S}(t_1-s)-\mathscr{S}(t_2-s)] \right.\\
&\qquad\qquad\qquad \left. \times f\,(s,x(s),(Kx)(s),(Hx)(s))\,ds \right\|_\alpha,\\
I_3 &= \left\| \int_0^{t_2} [(t_1-s)^{q-1}-(t_2-s)^{q-1}] s^n \mathscr{S}(t_2-s) \right.\\
&\qquad\qquad\qquad \left. \times f\,(s,x(s),(Kx)(s),(Hx)(s))\,ds \right\|_\alpha,\\
I_4 &= \left\| \int_{t_2}^{t_1} (t_1-s)^{q-1} s^n \mathscr{S}(t_1-s) f\,(s,x(s),(Kx)(s),(Hx)(s))\,ds \right\|_\alpha.
\end{aligned}
$$

Now, we need to check that I_1, I_2, I_3, I_4 tend to 0 independently of $x \in B_r$ when $t_1 \to t_2$.

In fact, by the compactness of the set $g(B_r)$ in view of (2) of Lemma 1, one can deduce that $\lim_{t_1\to t_2} I_1 = 0$ uniformly.

Next for $0 < h < t_2$, when $t_2 > 0$, we similarly derive

$$
\begin{aligned}
I_2 &\le \int_0^{t_2} (t_1-s)^{q-1} s^n \|\mathscr{S}_\alpha(t_1-s)-\mathscr{S}_\alpha(t_2-s)\|_\alpha\\
&\qquad \times \|\,f\,(s,x(s),(Kx)(s),(Hx)(s))\,\|_\alpha ds\\
&\le \int_0^{t_2-h} (t_2-s)^{q-1} s^n \rho(s) \|\mathscr{S}_\alpha(t_1-s)-\mathscr{S}_\alpha(t_2-s)\|_\alpha ds\\
&\quad + \int_{t_2-h}^{t_2} (t_2-s)^{q-1} s^n \rho(s) \|\mathscr{S}_\alpha(t_1-s)-\mathscr{S}_\alpha(t_2-s)\|_\alpha ds\\
&\le \max_{s\in[0,t_2-h]} \|\mathscr{S}_\alpha(t_1-s)-\mathscr{S}_\alpha(t_2-s)\|_\alpha \int_0^{t_2-h} (t_2-s)^{q-1} s^n \rho(s) ds\\
&\quad + \frac{2M}{\Gamma(q)} \int_{t_2-h}^{t_2} (t_2-s)^{q-1} s^n \rho(s) ds\\
&\le \max_{s\in[0,t_2-h]} \|\mathscr{S}_\alpha(t_1-s)-\mathscr{S}_\alpha(t_2-s)\|_\alpha B\left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1}\right)^{\frac{p-1}{p}}\\
&\qquad \times t_2^{\frac{pq+np-1}{p}} \|\rho\|_{L^p(J,R^+)}\\
&\quad + \frac{2M}{\Gamma(q)} h^{\frac{\chi(p-1)}{(\chi-1)p}} B\left(\frac{\chi(q-1)p}{p-1}+1, \frac{\chi np}{p-1}+1\right)^{\frac{p-1}{\chi p}} t_2^{\frac{\chi p((q-1)p+n)}{p-1}+1} \|\rho\|_{L^p(J,R^+)}
\end{aligned}
$$

from which we deduce that $\lim_{(h,t_2)\to(0,t_1)} I_2 = 0$ uniformly, since by (7) of Lemma 1.

Using the inequality $(x-y)^a \leq x^a - y^a$ for any $x, y \geq 0$ and $a > 1$, analogically we derive

$$\begin{aligned}
I_3 \leq{}& \frac{M}{\Gamma(q)} \int_0^{t_2} |(t_1-s)^{q-1} - (t_2-s)^{q-1}| s^n \\
&\qquad\qquad \times \| f(s, x(s), (Kx)(s), (Hx)(s)) \|_\alpha ds \\
\leq{}& \frac{M}{\Gamma(q)} \int_0^{t_2} |(t_1-s)^{q-1} - (t_2-s)^{q-1}| s^n \rho(s) ds \\
\leq{}& \frac{M}{\Gamma(q)} \left(\int_0^{t_2} \rho(s)^p ds \right)^{\frac{1}{p}} \left(\int_0^{t_2} s^{\frac{np}{p-1}} |(t_1-s)^{q-1} - (t_2-s)^{q-1}|^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}} \\
\leq{}& \frac{M}{\Gamma(q)} \left(\int_0^{t_2} s^{\frac{np}{p-1}} \left((t_2-s)^{\frac{(q-1)p}{p-1}} - (t_1-s)^{\frac{(q-1)p}{p-1}} \right) ds \right)^{\frac{p-1}{p}} \|\rho\|_{L^p(J,R^+)} \\
\leq{}& \frac{M}{\Gamma(q)} \left[B\left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} \left(t_2^{\frac{pq+np-1}{p}} - t_1^{\frac{pq+np-1}{p}} \right)^{\frac{p-1}{p}} \right. \\
&\left. + \left(\int_{t_2}^{t_1} s^{\frac{np}{p-1}} (t_1-s)^{\frac{(q-1)p}{p-1}} ds \right)^{\frac{p-1}{p}} \right] \|\rho\|_{L^p(J,R^+)} \\
\leq{}& \frac{M}{\Gamma(q)} \|\rho\|_{L^p(J,R^+)} \left[B\left(\frac{pq-1}{p-1}, \frac{(n+1)p-1}{p-1} \right)^{\frac{p-1}{p}} \right. \\
&\qquad\qquad \times \left(t_2^{\frac{pq+np-1}{p}} - t_1^{\frac{pq+np-1}{p}} \right)^{\frac{p-1}{p}} \\
&\left. + (t_1-t_2)^{\frac{\chi(p-1)}{(\chi-1)p}} B\left(\frac{\chi(q-1)p}{p-1} + 1, \frac{\chi np}{p-1} + 1 \right)^{\frac{p-1}{\chi p}} t_1^{\frac{\chi p((q-1)p+n)}{p-1}+1} \right].
\end{aligned}$$

Thus, $\lim_{t_1\to t_2} I_3 = 0$ uniformly.

Finally,

$$\begin{aligned}
I_4 \leq{}& \int_{t_2}^{t_1} (t_1-s)^{q-1} s^n \| \mathscr{S}(t_1-s) f(s, x(s), (Kx)(s), (Hx)(s)) \|_\alpha ds \\
\leq{}& \frac{M}{\Gamma(q)} \int_{t_2}^{t_1} (t_1-s)^{q-1} s^n \rho(s) ds \\
\leq{}& \frac{M}{\Gamma(q)} \left(\int_{t_2}^{t_1} s^{\frac{np}{p-1}} (t_1-s)^{\frac{(q-1)p}{p-1}} ds \right)^{\frac{p-1}{p}} \|\rho\|_{L^p(J,R^+)}
\end{aligned}$$

$$\leq \frac{M}{\Gamma(q)}(t_1-t_2)^{\frac{\chi(p-1)}{(\chi-1)p}} B\left(\frac{\chi(q-1)p}{p-1}+1, \frac{\chi np}{p-1}+1\right)^{\frac{p-1}{\chi p}}$$
$$\times t_1^{\frac{\chi p((q-1)p+n)}{p-1}+1} \|\rho\|_{L^p(J,R^+)},$$

from which we deduce that $\lim_{t_1\to t_2} I_4 = 0$ uniformly.

In summary, we have proven that $F(B_r)$ is relatively compact, for $t \in J$, $\{Fx \mid x \in B_r\}$ is a family of equicontinuous functions. Hence by the Arzela-Aascoli Theorem, F is compact. By Schauder fixed point theorem F has a fixed point $x \in B_r$. Consequently, system (1.1) has at least one mild solution on J. □

Our next result is based on the following well-known fixed point theorem.

Lemma 2 ([18]). *Let Γ be a condensing operator on a Banach space X. If $\Gamma(\mathcal{B}) \subset \mathcal{B}$ for a convex, closed and bounded set $\mathcal{B}$ of X, then Γ has a fixed point in $\mathcal{B}$.*

Now, we assume the following conditions and apply the above fixed point theorem.

[Hf3]: (1) There exists μ with $0 \leq \mu \leq \alpha \leq 1$ such that $f : J \times X_\alpha \times X_\alpha \times X_\alpha \to X_\mu$ is continuous and there exist $L_f^{(1)}, L_f^{(2)}, L_f^{(3)} > 0$ such that

$$\|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\|_\mu \leq L_f^{(1)}\|x_1-y_1\|_\alpha + L_f^{(2)}\|x_2-y_2\|_\alpha + L_f^{(3)}\|x_3-y_3\|_\alpha$$

for all $x_i, y_i \in X_\alpha$, $i = 1,2,3$ and $t \in J$.

(2) There exist two positive constants $c^{(1)}$, $d^{(1)}$ such that for each $(t,x,y,z) \in J \times X_\alpha \times X_\alpha \times X_\alpha$

$$\|f(t,x,y,z)\|_\mu \leq c^{(1)}(\|x\|_\alpha + \|y\|_\alpha + \|z\|_\alpha) + d^{(1)}.$$

[Hk3]: (1) The function $k : D_k \times X_\alpha \to X_\alpha$ is continuous and there exists a $L_k^{(1)} > 0$ such that for $(t,s) \in D_k$ and $x, y \in X_\alpha$,

$$\left\| \int_0^t [k(t,s,x) - k(t,s,y)]ds \right\|_\alpha \leq L_k^{(1)}\|x-y\|_\alpha.$$

(2) There exists a constant $L_k^{(2)} > 0$ such that for $(t,s) \in D_k$ and $x, y \in X_\alpha$,

$$\left\| \int_0^t k(t,s,x)ds \right\|_\alpha \leq L_k^{(2)}(1 + \|x\|_\alpha).$$

[Hh3]: (1) The function $h : D_h \times X_\alpha \to X_\alpha$ is continuous and there exists a $L_h^{(1)} > 0$ such that for $(t,s) \in D_h$ and $x, y \in X_\alpha$,

$$\left\| \int_0^T [h(t,s,x) - h(t,s,y)]ds \right\|_\alpha \leq L_h^{(1)}\|x-y\|_\alpha.$$

(2) There exists a constant $L_h^{(2)} > 0$ such that for $(t,s) \in D_h$ and $x, y \in X_\alpha$,

$$\left\| \int_0^T h(t,s,x)ds \right\|_\alpha \le L_h^{(2)}(1 + \|x\|_\alpha).$$

[Hg3]: $g : C_\alpha \to X_\alpha$ is compact continuous and there exists a nondecreasing function $\phi : R^+ \to R^+$ such that for all $x \in C_\alpha$,

$$\|g(x)\|_\alpha \le \phi(\|x\|_\infty), \text{ and } \lim_{l\to+\infty} \inf \frac{\phi(l)}{l} = \delta < \infty.$$

Now we are ready to state and prove the following existence result.

Theorem 3. *Assume that the conditions [Hf3], [Hk3], [Hh3], [Hg3] are satisfied. If $x_0 \in X_\alpha$ then system (1.1) admits at least one mild solution on J provided that*

$$M\left\{\delta + \frac{\|A^{\alpha-\mu}\|\Gamma(n)}{\Gamma(n+q+1)}\left[c^{(1)}(1 + L_k^{(2)} + L_h^{(2)})\right]T^{n+q}\right\} < 1 \tag{3.4}$$

and

$$\frac{M\|A^{\alpha-\mu}\|\Gamma(n)}{\Gamma(n+q+1)}T^{n+q}(L_f^{(1)} + L_f^{(2)}L_k^{(1)} + L_f^{(3)}L_h^{(1)}) < 1. \tag{3.5}$$

Proof. Define the operator $\Gamma : C_\alpha \to C_\alpha$ given by (3.1). For each positive number l, let $B_l = \{x \in C_\alpha \mid \|x\|_\infty \le l\}$ then, for each l, B_l is obviously a bounded closed convex set in C_α.

First, we claim that $\Gamma(B_l) \subset B_l$ for some $l > 0$. If it is not true, then for each $l > 0$, there would exist $x_l \in B_l$ and $t_l \in J$ such that $\|(\Gamma x_l)(t_l)\|_\alpha > l$. However, on the other hand, by [Hf3], [Hk3] and [Hh3]

$$\begin{aligned}
l &\le \|(\Gamma x_l)(t_l)\|_\alpha \le \|\mathscr{T}(t)(x_0 + g(x_l))\|_\alpha \\
&\quad + \int_0^{t_l}(t_l - s)^{q-1}s^n\left\|\mathscr{S}(t_l - s)f(s, x_l(s), (Kx_l)(t), (Hx_l)(t))\right\|_\alpha ds \\
&\le M(\|x_0\|_\alpha + \|g(x_l)\|_\alpha) \\
&\quad + \frac{M\|A^{\alpha-\mu}\|}{\Gamma(q)}\left[c^{(1)}(l + L_k^{(2)}(1+l) + L_h^{(2)}(1+l) + d^{(1)}\right]\int_0^{t_l}(t_l - s)^{q-1}s^n ds \\
&\le M(\|x_0\|_\alpha + \phi(l)) \\
&\quad + \frac{\|A^{\alpha-\mu}\|M}{\Gamma(q)}\left[c^{(1)}(l + L_k^{(2)}(1+l) + L_h^{(2)}(1+l) + d^{(1)}\right]B(q, n+1)t_l^{n+q} \\
&\le M(\|x_0\|_\alpha + \phi(l)) \\
&\quad + \frac{M\|A^{\alpha-\mu}\|\Gamma(n)}{\Gamma(n+q+1)}\left[c^{(1)}(l + L_k^{(2)}(1+l) + L_h^{(2)}(1+l) + d^{(1)}\right]T^{n+q}.
\end{aligned}$$

Dividing both sides by l and taking the lower limit as $l \to +\infty$, we obtain

$$1 \leq M \left\{ \delta + \frac{\|A^{\alpha-\mu}\| \Gamma(n)}{\Gamma(n+q+1)} \left[c^{(1)}(1 + L_k^{(2)} + L_h^{(2)}) \right] T^{n+q} \right\},$$

which contradicts the expression (3.4). Thus, for some positive number l, $\Gamma(B_l) \subset B_l$.

We decompose $\Gamma = \Gamma_1 + \Gamma_2$ as

$$(\Gamma_1 x)(t) = \mathscr{T}(t)[x_0 + g(x)],$$

$$(\Gamma_2 x)(t) = \int_0^t (t-s)^{q-1} s^n \mathscr{S}(t-s) f(s, x(s), (Kx)(s), (Hx)(s))\, ds.$$

Second, we show that Γ_1 is compact continuous and Γ_2 is a contraction. By [Hg3], we can infer that Γ_1 is compact continuous on X_α. Next, we prove that Γ_2 is a contraction on B_l. In fact, for each $t \in J$, $x, y \in B_l$, by [Hg3] we have

$$\begin{aligned}
&\|(\Gamma_2 x)(t) - (\Gamma_2 y)(t)\|_\alpha \\
&\leq \|A^{\alpha-\mu}\| \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n \big\| f(s, x(s), (Kx)(s), (Hx)(s)) \\
&\qquad\qquad - f(s, y(s), (Ky)(s), (Hy)(s)) \big\|_\mu ds \\
&\leq \|A^{\alpha-\mu}\| \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n \big[L_f^{(1)} \|x(s) - y(s)\|_\alpha \\
&\quad + L_f^{(2)} \|(Kx)(s) - (Ky)(s)\|_\alpha + L_f^{(3)} \|(Hx)(s) - (Hy)(s)\|_\alpha \big] ds \\
&\leq \|A^{\alpha-\mu}\| \frac{M}{\Gamma(q)} (L_f^{(1)} + L_f^{(2)} L_k^{(1)} + L_f^{(3)} L_h^{(1)}) \int_0^t (t-s)^{q-1} s^n ds \|x - y\|_\infty \\
&\leq \|A^{\alpha-\mu}\| \frac{M\Gamma(n)}{\Gamma(n+q+1)} T^{n+q} (L_f^{(1)} + L_f^{(2)} L_k^{(1)} + L_f^{(3)} L_h^{(1)}) \|x - y\|_\infty.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\Gamma_2 x - \Gamma_2 y\|_\infty \leq \frac{M\|A^{\alpha-\mu}\|\Gamma(n)}{\Gamma(n+q+1)} T^{n+q} (L_f^{(1)} + L_f^{(2)} L_k^{(1)} + L_f^{(3)} L_h^{(1)}) \\
\times \|x - y\|_\infty,
\end{aligned}$$

which implies that Γ_2 is a contraction by (3.5).

At last, we can conclude that $\Gamma = \Gamma_1 + \Gamma_2$ is a condensing map on B_l. By Lemma 2, system (1.1) admits at least one mild solution on J. □

4. AN EXAMPLE

In this section, we present an example, which indicate how our theorems can be applied to concrete problems.

Consider the following problem:

$$\begin{cases} {}^cD_t^q x(t,y) - \Delta x(t,y) \\ = \left(\frac{e^{-t}}{e^t+e^{-t}} + e^{-t}\right)\cos\Big[x(t,y) + \int_0^t \sin(t+s)x(s,y)ds \\ \qquad\qquad\qquad\qquad + \int_0^T \cos(ts)x(s,y)ds\Big], \\ \qquad\qquad y \in \Omega,\ t,s \in (0,T],\ q = \frac{19}{20}, \\ x(t,y)\mid_{y\in\partial\Omega} = 0, \quad t > 0, \\ x(0,y) = \int_\Omega \int_0^T h(t,y)\log(1+|x(t,\xi)|^{\frac{1}{2}})dtd\xi, \end{cases} \tag{4.1}$$

where Δ is the Laplace operator in $\mathbb{R}^3$, $\Omega \subset \mathbb{R}^3$ is a bounded domain, $\partial\Omega \in C^3$, and $h(t,y) \in C(J \times \bar{\Omega})$.

We apply Theorem 2 by taking $X = L^2(\Omega)$, $D(A) = H^2(\Omega) \bigcap H_0^1(\Omega)$, and $Ax = -\Delta x$ for $x \in D(A)$ and set $\alpha = 0$.

Define $x(t)(y) = x(t,y)$, $(Kx)(t)(y) = \int_0^t \sin(t+s)x(s,y)ds$, $(Hx)(t)(y) = \int_0^T \cos(ts)x(s,y)ds$, and

$$f\,(t,x(t),(Kx)(t),(Hx)(t))\,(y)$$
$$= \left(\frac{e^{-t}}{e^t+e^{-t}} + e^{-t}\right)\cos\left[x(t) + \int_0^t \sin(t+s)x(s)ds + \int_0^T \cos(ts)x(s)ds\right](y),$$

$$g(x)(y) = \int_\Omega \int_0^T h(t,y)\log(1+|x(t,\xi)|^{\frac{1}{2}})dtd\xi,\ y \in \overline{\Omega},\ x \in C(J,X).$$

Then, A generates a compact analytic semigroup in X with $M = 1$, and

$$\|f\,(t,x(t),(Kx)(t),(Hx)(t))\| \le \rho(t) = \left(\frac{e^{-t}}{e^t+e^{-t}} + e^{-t}\right)(\text{mes}(\Omega))^{\frac{1}{2}}$$

with $\rho \in L^p(J,R^+)$, $p = 10$. Moreover, g is compact (see [20]). Next, using $\log(1+a) \le a$ for any $a \ge 0$, we derive

$$\int_\Omega \log(1+|x(\xi)|^{\frac{1}{2}})d\xi \le \int_\Omega |x(\xi)|^{\frac{1}{2}}d\xi \le (\text{mes}(\Omega))^{\frac{3}{4}}\|x\|^{\frac{1}{2}} \le \frac{(\text{mes}(\Omega))^{\frac{3}{4}}}{2}(1+\|x\|)$$

for any $x \in X$. Hence using $\|x(t,\cdot)\| \le \|x\|_\infty$ for any $x \in C(J,X)$, we obtain

$$\|g(x)\| \le \frac{T}{2}(\text{mes}(\Omega))^{\frac{7}{4}} \max_{t\in J, y\in\bar{\Omega}} |h(t,y)|\,(1+\|x\|_\infty),\ x \in C(J,X).$$

Thus problem (4.1) can be rewritten as

$$\begin{cases} {}^cD_t^q x(t) = -Ax(t) + t^n f\,(t,x(t),(Kx)(t),(Hx)(t))\,, \\ \qquad\qquad\qquad t \in J,\ n \in Z^+,\ q \in (0,1), \\ x(0) = g(x) + x_0. \end{cases}$$

Obviously, $q = \frac{19}{20} > \frac{1}{10} = \frac{1}{p}$. Furthermore, if T and $h(t,y)$ satisfy

$$T(\operatorname{mes}(\Omega))^{\frac{7}{4}} \max_{t\in J, y\in\bar{\Omega}} |h(t,y)| < 2$$

then all the assumptions given in Theorem 2 are verified. Therefore, the problem (4.1) has at least one mild solution.

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